Spectral theory applications in the localization phenomena, scattering, and inverse problems for the wave absorption by rough boundaries

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It is well-known that rough (fractal) boundaries have improved noise absorption properties, and it is conjectured that such boundaries may even be optimal absorbers for certain problems. The main principle has many applications in acoustics (absorbing walls, anechoic chambers) and has already led to patented innovations [Fractal Wall (TM) Brevet Colas-École Polytechnique 03/00881; US patent 7308965B2.]. While this 'global' effect of increased absorption by rough boundaries is understood to some extent, a detailed understanding of 'local' effects is still missing. On the other hand, understanding local effects would be extremely helpful in engineering applications because it would allow us to decompose the noise into parts and predict which of these parts get absorbed with particularly high efficiency at which parts of the boundary. In concrete applications and constructions, one could then focus on the relevant parts and reduce computation, production, or building costs.

First, we wish to achieve three goals: The first goal is to provide a suitable decomposition of the noise signal. The second goal is to express the temporal decay of acoustical energy in terms of this decomposition and to see which configurations might be particularly efficient. The third goal is to analyze where the bulks of the separate parts are located; this is referred to as energy localization. The additional challenge is to go into the inverse rough scattering problems, where the localization in spectral problems must be combined with the layer potential techniques.

For self-adjoint operators, such as Laplacian with homogeneous Dirichlet boundary conditions[1], comparable localization phenomena are well-known. However, for acoustical damping, one needs to study Robin and mixed boundary value problems involving complex dissipation coefficients, and this requires the study of non-self-adjoint operators. In addition, the study of such boundary value problems on rough domains is quite nontrivial because they involve integrals and operators on fractal boundaries. However, the tools we developed recently[2] permit a suitable analysis. We will develop a spectral theory for the non-self-adjoint dissipative operators corresponding to wave absorption (involving complex-valued coefficients) by a rough boundary. This involves the study of operators and their adjoints and related modes, which can then be combined to obtain a Riesz basis. The temporal energy decay can then be expressed in terms of related mode expansions. We will discuss the mechanism of the astride localization[3], i.e., the existence of modes both in lossy and lossless regions, an effect that makes the absorption particularly efficient. The localization phenomena for the second-order boundary conditions is also a completely open problem. Then the spectral problems for the layer potential operators will be studied, allowing to go in the inverse scattering problems.

Our current study of the single and double-layer potentials on non-Lipschitzien boundaries makes it possible to define them using weak formulations, thanks to the generalization of the Neumann-to-Dirichlet operator[4]. These definitions are not related to their integral representation, which is also

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possible for the single-layer potential once we use the corresponding (Borel d-upper regular) boundary measure. However, the differences between this weak approach and the classical approach appear differences in the properties of these operators as the compactness of the resolvent. It is known[5] that even Lipschitz geometries create singular integrals and associated operators have a continuous spectrum. The difference between the two approaches opens the question about the approximation of the layer potential operators defined in the weak sense by a sequence of operators defined in the classical case. It is related to the central question of how to approximate a solution on the irregular domain by solutions on regular domains converging to the irregular one. If the Mosco-type convergence is rather good understood, the convergence of layer potentials has never been considered. The first step in this direction is the convergence of the associated Poincaré-Steklov operators. It is also related to the generalization of the classical theory of layer potentials[6] using the results[7] for pseudo-differential operators on fractals. The aim is to work in a large class of domains with rough boundaries[8]. The understanding of the properties of these operators can be expected to lead to new tools in scattering theory[9] and imaging methods[10] for irregular objects or inclusions.

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