Defect reconstruction in waveguides using resonant frequencies

Angèle Niclas

Collaboration avec
Éric Bonnetier, Claire Prada, Laurent Seppecher, Grégory Vial

Overview of my academic background

- ENS Lyon, Mathematics department
- L3 Fondamental Mathematics (ENS Lyon)
- Internship at Ecole Centrale de Lyon
- M1 Advanced Mathematics - PDE, Statistics, Probability (ENS Lyon)
- Internship at UNSW Sydney
- Agrégation de Mathématiques
- M2 Advanced Mathematics - PDE (ENS Lyon)
- M2 Maths in Action: From Concept to Innovation (University of Lyon)
- Internship before PhD thesis at Ecole Centrale de Lyon

PhD thesis and Post-doc

PhD Thesis: Inverse problems and local resonances in irregular mechanical waveguides.
École Centrale de Lyon - Institut Camille Jordan

- Inverse problems
- Wave propagation
- Theoretical questions
- Numerical validations

\+ Interactions with physicists

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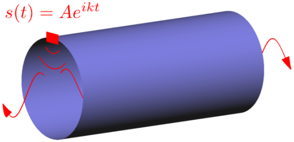
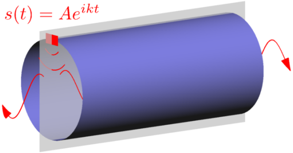
Post-doc: Imaging of randomly perturbed elastic media.
École polytechnique - Centre de Mathématiques Appliquées

- Inverse problems
- Stochastic process

PhD thesis: context



PhD thesis: context



Most frequencies k:

Figure: $k = 10$, $|u| = 0.06$

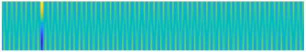
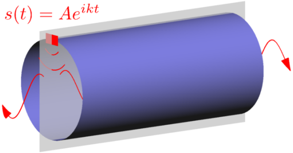


Figure: $k = 41$, $|u| = 0.05$

Simulated data generated using Matlab and Freefem++ with finite element methods and perfectly matched layers

PhD thesis: context



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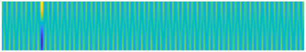

Figure: $k = 41$, $|u| = 0.05$

Resonant frequencies k:

Figure: $k = 31.428$, $|u| = 10$

Figure: $k = 31.429$, $|u| = 17$

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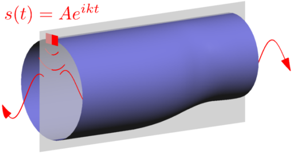
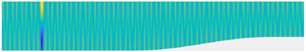
PhD thesis: context



Most frequencies k:

Figure: $k = 10$, $|u| = 0.08$



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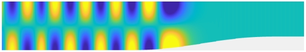

Figure: $k = 31.2$, $|u| = 0.8$

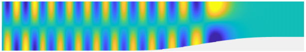

Figure: $k = 31.7$, $|u| = 0.6$

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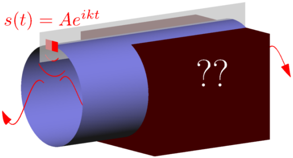
PhD thesis: context



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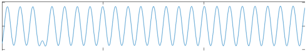

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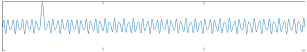

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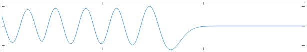

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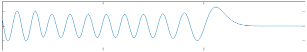

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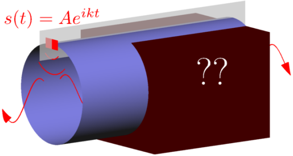

Most frequencies k:

- ✓ Known mathematical solution
- ✓ Existing reconstructions [1]
- ✗ Not sensitive to defects
- ✗ Not robust to noised data

Resonant frequencies k:

- ✗ No mathematical study
- ✗ No link between defects and waves
- ✓ Sensitive to defects
- ✓ High amplitudes, not very sensitive to noised data

[1] Ammari, Iakovleva, Kang. Reconstruction of a small inclusion in a two- dimensional open waveguide. SIAM Journal on Applied Mathematics, 2005.

Forward problem: study of waves with defect

Let us denote $h(x)$ the width of the waveguide.
Assumption: h is slowly varying and h', h'' are small.

Theorem: forward problem approximation [2]

- There exists a unique wave u propagating at resonant frequencies.

[2] Bonnetier, Niclas, Seppecher, Vial. The Helmholtz problem in slowly varying waveguides at locally resonant frequencies, submitted, 2022

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- This wave can be approach with u^{app} defined by

$$u^{\mathrm{app}}(x,y) = C \times \mathrm{Airy}^{(1)} \left(- \left(\frac{3}{2} \int_{x^\star}^{x} \sqrt{k^2 - \frac{n^2\pi^2}{h(x)^2}} \right)^{2/3} \right) \cos\left(\frac{n\pi y}{h(x)} \right).$$

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- We control the approximation error by

$$\|u - u^{\text{app}}\| \leq D \|h'\|. \tag{1}$$

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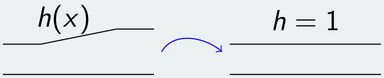
Sketch of the proof

Change of variable



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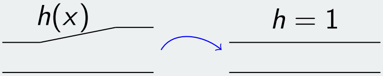

→

Modal decomposition

$$u(x,y) = \sum_{n \in \mathbb{N}} u_n(x) \cos(n\pi y)$$

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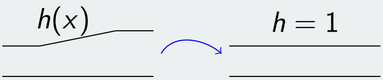

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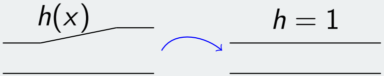

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$$u_n''(x) + \left(k^2 - \frac{n^2\pi^2}{h(x)^2}\right) u_n(x) = -\delta_s$$

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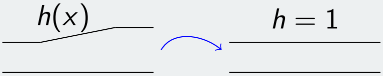

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Adaptation of results from quantum mechanics [3]

$$G_n^{\text{app}} = C_1(s)\text{Airy}^{(1)}\left(-\xi(x)^{2/3}\right) + C_2(s)\text{Airy}^{(2)}\left(-\xi(x)^{2/3}\right).$$

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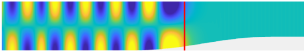
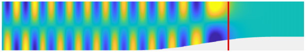


Figure: u^{app} for $k = 31.2$ and $x^\star$



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The coordinate $x^\star$ contains information about h:

$$h(x^\star) = \frac{n\pi}{k}$$

- We choose different resonant frequencies $(k_i)_{i=1,\cdots,m}$
- For each frequency k_i, we find the location of $x_i^\star$ using measurements of the wave
- We find a good approximation of the width h in each coordinate $x_i^\star$

Defects reconstruction

Theorem: Stability of the inverse problem [4]

- The previous algorithm converges and produces a good approximation of h denoted h^{app}.
- We control the reconstruction error by

$$\|h - h^{\text{app}}\| \leq C\|h'\| + D \times (\text{measurement noise}).$$

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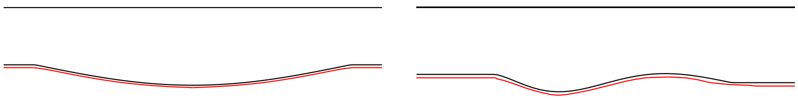

Figure: In black, the initial shape of h. In red, the reconstruction h^{app} slowly shifted to improve comparison.

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Figure: Relative reconstruction errors

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k resonant	0.2 %	4%	9%

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Conclusion

Extensions:

- Elastic plates (boat hulls, aircraft parts)
- Other obstacles in waveguides (inhomogeneities, torsions...)

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