Geometric optics for quasilinear hyperbolic boundary value problems

Postdoc day

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Ph.D. work with advisor Jean-François Coulombel

Introduction

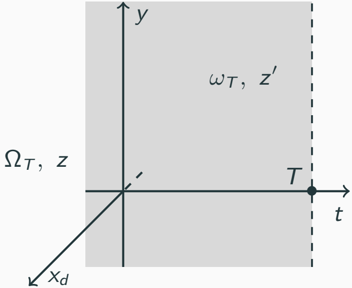
Problem introduction

We consider the following problem

$$\begin{cases} L(u^\varepsilon, \partial_z)\, u^\varepsilon := \partial_t u^\varepsilon + \displaystyle\sum_{j=1}^{d} A_j(u^\varepsilon)\, \partial_{x_j} u^\varepsilon = 0 & \text{in } \Omega_T, \\ B\, u^\varepsilon_{|x_d=0} = \varepsilon\, g^\varepsilon & \text{on } \omega_T, \\ u^\varepsilon_{|t\leqslant 0} = 0, & \end{cases} \tag{1}$$

where

- $\Omega_T := (-\infty, T] \times \mathbb{R}^{d-1} \times \mathbb{R}_+$ and $\omega_T := (-\infty, T] \times \mathbb{R}^{d-1}$, with $T > 0$,
- we denote $z = (t, y, x_d) \in \Omega_T$, and $z' := (t, y) \in \omega_T$,



- the unknown u^ε is a (regular) function from Ω_T to $\mathbb{R}^N$, $N \geqslant 2$,
- for all $j = 1, \ldots, d-1$, A_j is a regular map from $\mathbb{R}^N$ into $\mathcal{M}_N(\mathbb{R})$,
- B belongs to $\mathcal{M}_{M,N}(\mathbb{R})$ for some $1 \leqslant M \leqslant N$ and is of maximal rank.

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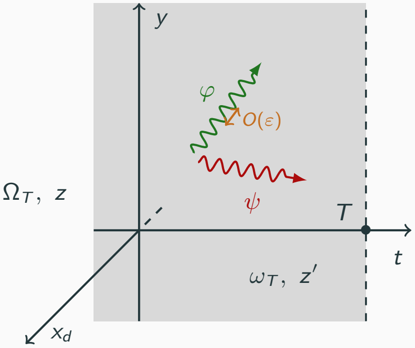
The dependency in ε of the system comes from the boundary term $\varepsilon\, g^\varepsilon$, where g^ε is given by, for $z' \in \omega_T$,

$$g^\varepsilon(z') = G\left(z', \frac{z' \cdot \varphi}{\varepsilon}, \frac{z' \cdot \psi}{\varepsilon}\right),$$

where G belongs to $H^\infty(\omega_T \times \mathbb{T}^2)$, zero for negative times t, and φ, ψ are in $\mathbb{R}^d \setminus \{0\}$.



High frequency regime: geometric optics

→ We are interested here in the qualitative properties of the solution u^ε to (1) when the wavelength ε in (1) is small, that is, in the high frequency regime.

→ Following the analysis of Lax and Hunter-Majda-Rosales, we look for an exact solution to (1) under the form of a formal series, i.e. a WKB expansion reading as

$$\varepsilon\, U_1\Big(z, \frac{\Phi(z)}{\varepsilon}\Big) + \varepsilon^2\, U_2\Big(z, \frac{\Phi(z)}{\varepsilon}\Big) + \varepsilon^3\, U_3\Big(z, \frac{\Phi(z)}{\varepsilon}\Big) + \cdots, \tag{2}$$

where Φ contains the phases of the solution. This is the framework of geometric optics.

→ In the weakly non-linear framework, in the high frequency asymptotic (i.e. when $\varepsilon \to 0$), the leading profile U_1 is proven to satisfy a quasi-linear system.

→ The exact solution to (1) is then to be approximated by a truncated sum of the expansion (2).

Difficulties of the problem

Brief state of the art i

- Same boundary value problem, but with only one phase on the boundary:
  - → Mark Williams. “Singular pseudodifferential operators, symmetrizers, and oscillatory multidimensional shocks”. In: *J. Funct. Anal.* 191.1 (2002), pp. 132–209,
  - → Jean-François Coulombel, Olivier Gues, and Mark Williams. “Resonant leading order geometric optics expansions for quasilinear hyperbolic fixed and free boundary problems”. In: *Comm. Partial Differential Equations* 36.10 (2011), pp. 1797–1859,
  - → Matthew Hernandez. “Resonant leading term geometric optics expansions with boundary layers for quasilinear hyperbolic boundary problems”. In: *Comm. Partial Differential Equations* 40.3 (2015), pp. 387–437.

Brief state of the art ii

- Multiple phases for a semilinear problem:
  - → Jean-Luc Joly, Guy Métivier, and Jeffrey Rauch. "Coherent nonlinear waves and the Wiener algebra". In: *Ann. Inst. Fourier (Grenoble)* 44.1 (1994), pp. 167–196,
  - → Mark Williams. "Nonlinear geometric optics for hyperbolic boundary problems". In: *Comm. Partial Differential Equations* 21.11-12 (1996), pp. 1829–1895.
- Multiple phases for the quasilinear Cauchy problem:
  - → Jean-Luc Joly, Guy Métivier, and Jeffrey Rauch. "Coherent and focusing multidimensional nonlinear geometric optics". In: *Ann. Sci. École Norm. Sup. (4)* 28.1 (1995), pp. 51–113.

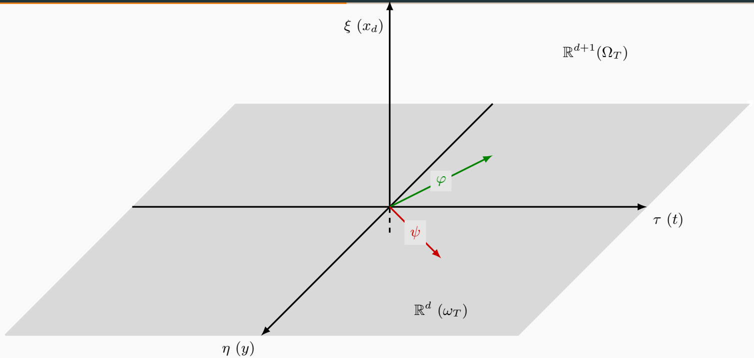
Main difficulties

- Boundary value problems.

- Multiple phases on the boundary.
  - → By nonlinearity, it creates a countable infinite set of frequencies inside the domain, making more complex the functional framework.

1st work: strongly stable systems

Set of frequencies inside the domain



φ, ψ on the boundary.

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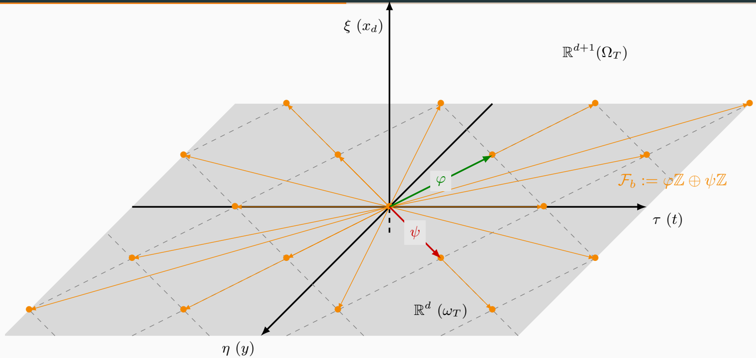

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→ Lattice $\mathcal{F}_b := \varphi\mathbb{Z} \oplus \psi\mathbb{Z}$ on the boundary.

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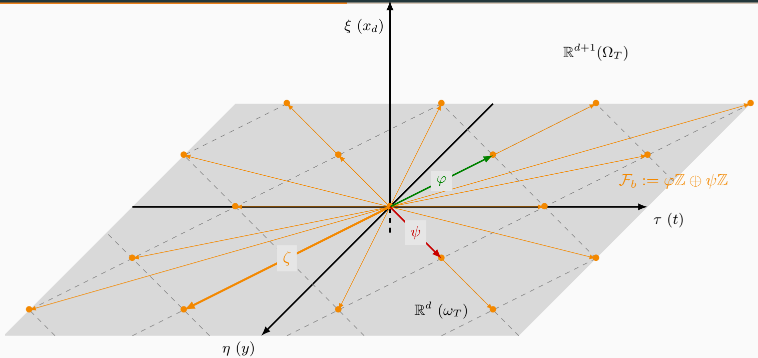

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$\rightarrow$ For each $\zeta \in \mathcal{F}_b$: (ζ, ξ) inside the domain, with

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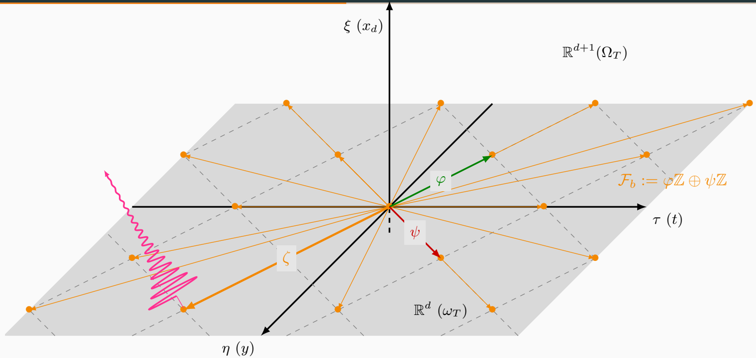

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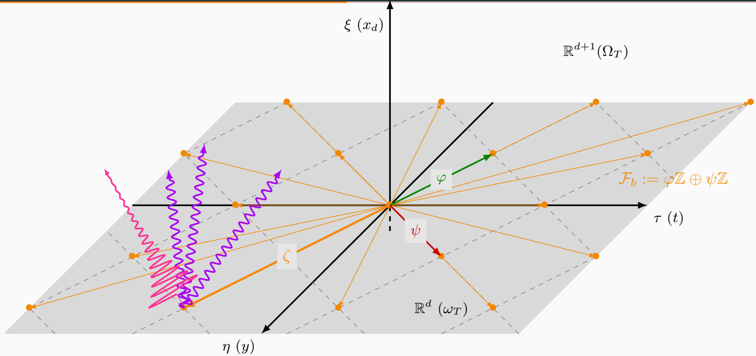

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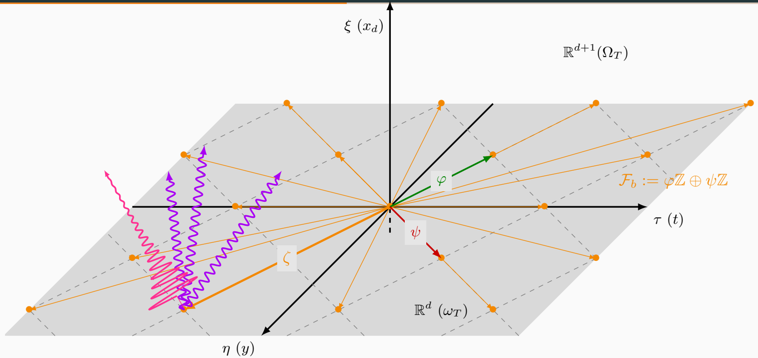
(ii) $\xi \in \mathbb{R}$ and (ζ, ξ) characteristic[a] and incoming

[a] $\alpha = (\tau, \eta, \xi) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}$ is characteristic if $\det\left(\tau I + \sum_{j=1}^{d-1} \eta_j A_j(0) + \xi A_d(0)\right) = 0$.

Set of frequencies inside the domain



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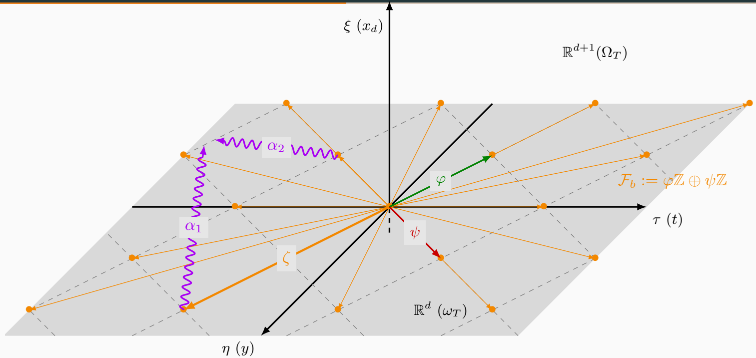

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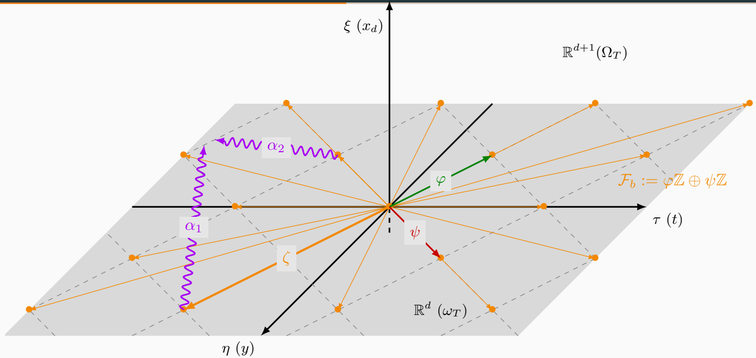

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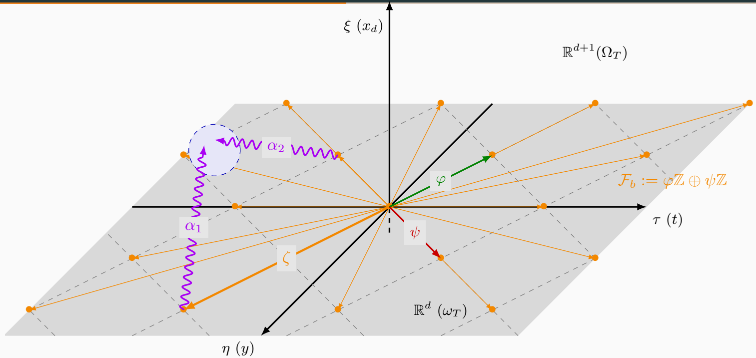

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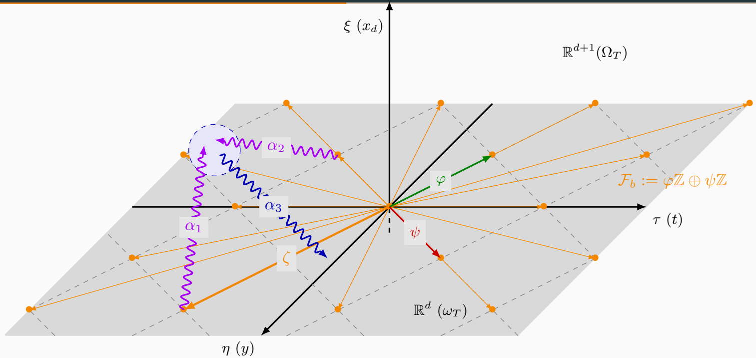

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If $\alpha_3 := \alpha_1 + \alpha_2$ is outgoing, a new frequency is created.

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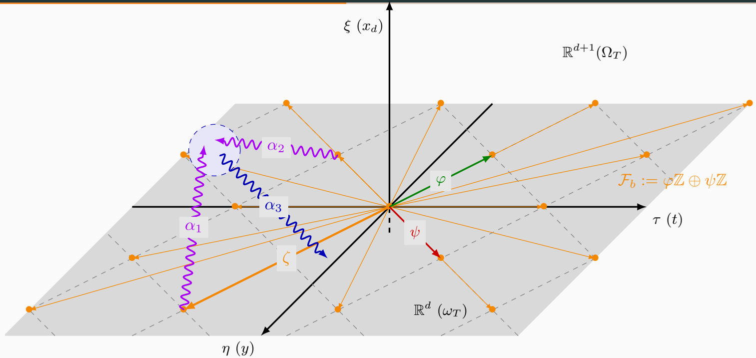

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Functional framework : almost-periodic functions

We need a functional framework allowing to consider functions of the form

$$\sum_{\alpha \in \mathcal{F}} U_\alpha(z)\, e^{iz\cdot\alpha/\varepsilon}.$$

We introduce new fast variables $\theta = (\theta_1, \theta_2) = (z' \cdot \varphi/\varepsilon, z' \cdot \psi/\varepsilon) \in \mathbb{T}^2$ and $\chi_d := x_d/\varepsilon \in \mathbb{R}_+$ so that, if $\alpha = (\zeta, \xi) = \big(n_1\varphi + n_2\psi, \xi\big)$,

$$U_\alpha(z)\, e^{iz\cdot\alpha/\varepsilon} = U_\alpha(z)\, e^{in_1\,\theta_1}\, e^{in_2\,\theta_2}\, e^{i\xi\,\chi_d}.$$

We use the framework of almost-periodic functions in the sense of Bohr.
Roughly, these are series of the form

$$\sum_{\alpha} U_\alpha(z)\, e^{in_1\theta_1}\, e^{in_2\theta_2} e^{i\xi\chi_d}$$

with uniform convergence and norm for (x_d, χ_d) and of Sobolev type for (z', θ).

We look for an approximate solution of (1) under the form of a formal series $u^{\varepsilon,\mathrm{app}}(z) = v(z, z' \cdot \varphi/\varepsilon, z' \cdot \psi/\varepsilon, x_d/\varepsilon)$, where v is given by

$$v(z, \theta, \chi_d) := \sum_{k \geqslant 1} \varepsilon^k \, U_k(z, \theta, \chi_d),$$

with U_1 an almost periodic function in the sense of Bohr.

Theorem (K. 2021)

Under the uniform Kreiss-Lopatinskii condition and with assumptions on the set of resonances, for $s \geqslant 0$ large enough, there exists a time $T > 0$ and a leading profile U_1 solution to the problem (3) given below, that governs the evolution of the leading profile.

WKB cascade

For $u^{\varepsilon,\text{app}}$ to formally satisfy the system (1), a WKB study and a decoupling of the cascade obtained shows that the leading profile U_1 has to satisfy the following system

$$\mathbf{E}\, U_1 = U_1 \tag{3a}$$

$$\mathbf{E}\Big[L(0,\partial_z)\, U_1 + \mathcal{M}(U_1, U_1)\Big] = 0 \tag{3b}$$

$$B\, U_1|_{x_d=0,\chi_d=0} = G \tag{3c}$$

$$U_1|_{t\leqslant 0} = 0. \tag{3d}$$

with $\mathbf{E}$ a projector.

Existence of a solution to (3) is obtained using energy estimates without loss of derivative. Two terms have to be treated.

Transport part

If U_1 reads as

$$U_1(z,\theta,\chi_d) = \sum_{\alpha} U^1_\alpha(z)\, e^{in_1\theta_1}\, e^{in_2\theta_2} e^{i\xi\chi_d},$$

then the transport part $\mathbf{E}\Big[L(0,\partial_z)\,U_1\Big]$ reads as a sum of transport terms

$$\mathbf{E}\Big[L(0,\partial_z)\,U_1\Big] = \sum_{\alpha}\Big(\partial_t + v_\alpha\cdot\nabla_x\Big)\, U^1_\alpha(z)\, e^{in_1\theta_1}\, e^{in_2\theta_2} e^{i\xi\chi_d},$$

which are easy to treat in energy estimates.

Remark. The sign of the x_d-component of v_α determines if the frequency α is incoming or outgoing.

Quadratic part

As for the quadratic term $\mathbf{E}\Big[\mathcal{M}(U_1, U_1)\Big]$, we have

$$\mathbf{E}\Big[\mathcal{M}(U_1, U_1)\Big] = \sum_{\alpha,\alpha'} \pi_{\alpha+\alpha'}\, L_1\Big(U^1_\alpha, n'_1\,\varphi + n'_2\,\psi\Big)\, U^1_{\alpha'}\, e^{i(n_1+n'_1)\theta_1}\, e^{i(n_2+n'_2)\theta_2} e^{i(\xi+\xi')\chi_d}.$$

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Remarks.

- This term is quadratic.

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  - → If not, this is a real resonance, and generates terms of convolution type, that are more difficult to handle.
- The main additional difficulty compared to [Joly-Métivier-Rauch 1995] is the lack of symmetry in the resonance terms.

Proof of stability ?

- To prove stability, one solution consists in studying the difference

$$u^\varepsilon - \sum_{k=1}^{N} \varepsilon^k \, U_k\Big(., \frac{\Phi_k(.)}{\varepsilon}\Big),$$

but we do not know if the exact solution u^ε exists on a time interval independent of ε.

- One could also use a large number of correctors U_k of the expansion

$$u^{\varepsilon,\text{app}} \sim \sum_{k\geqslant 1} \varepsilon^k \, U_k(z, \theta, \chi_d).$$

This leads to questions about the functional framework.

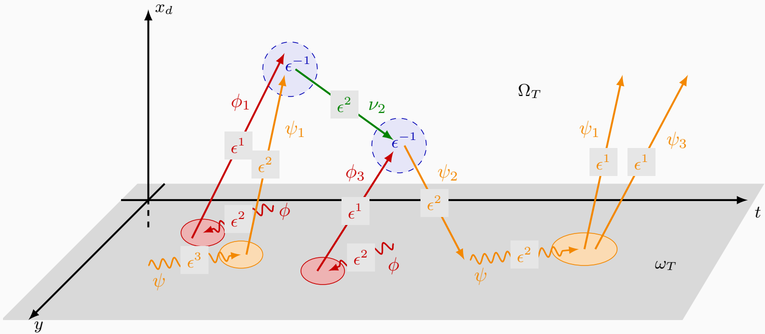
2nd work: instability of the expansion

Weakening the assumption on the boundary allows amplification to happen on the boundary.

Considering a perturbation H of small amplitude $O(\varepsilon^M)$ ($M \geqslant 3$) of a periodic forcing boundary term G of amplitude $O(\varepsilon^2)$,

$$\varepsilon\, g^\varepsilon(z') = \varepsilon^2\, G\left(z', \frac{z' \cdot \varphi}{\varepsilon}\right) + \varepsilon^M\, H\left(z', \frac{z' \cdot \psi}{\varepsilon}\right),$$

with a particular configuration of boundary frequencies φ and ψ, we prove (K. 2022), on a study model, that an instability may be created.



Thank you for your attention !