School of Mathematics

Interior Point Methods for Linear Programming: Motivation & Theory

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Outline

- **IPM for LP: Motivation**
  - complementarity conditions
  - first order optimality conditions
  - central trajectory
  - primal-dual framework
- **Polynomial Complexity of IPM**
  - Newton method
  - short step path-following method
  - polynomial complexity proof

Building Blocks of the IPM

What do we need to derive the **Interior Point Method**?

- duality theory:
  Lagrangian function;
  first order optimality conditions.
- logarithmic barriers.
- Newton method.

Primal-Dual Pair of Linear Programs

Primal

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b, \\ & x \geq 0; \end{array}$$

Dual

$$\begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y + s = c, \\ & s \geq 0. \end{array}$$

Lagrangian

$$L(x, y) = c^T x - y^T (Ax - b) - s^T x.$$

Optimality Conditions

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ XSe &= 0, \quad (\text{ i.e., } x_j \cdot s_j = 0 \quad \forall j), \\ (x, s) &\geq 0, \end{aligned}$$

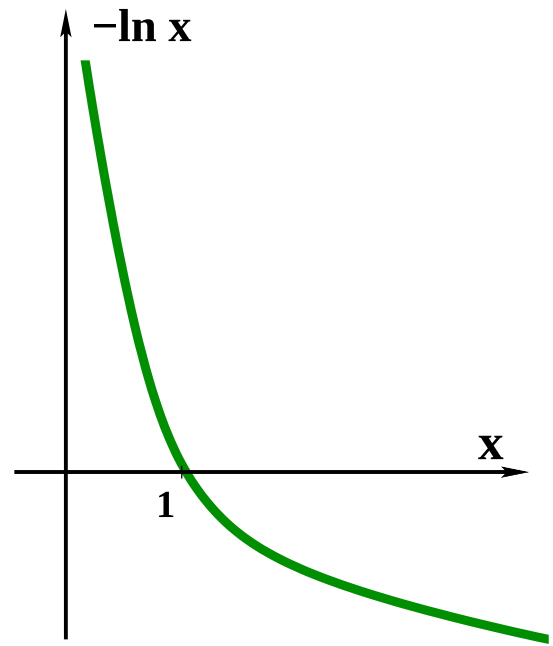
$X = diag\{x_1, \cdots, x_n\}$, $S = diag\{s_1, \cdots, s_n\}$, $e = (1, \cdots, 1) \in \mathcal{R}^n$.

Logarithmic barrier

$-\ln x_j$

"replaces" the inequality

$x_j \geq 0$.



Observe that

$$\min \, e^{-\sum_{j=1}^{n} \ln x_j} \quad \Longleftrightarrow \quad \max \prod_{j=1}^{n} x_j$$

The minimization of $-\sum_{j=1}^{n} \ln x_j$ is equivalent to the maximization of the product of distances from all hyperplanes defining the positive orthant: it prevents all x_j from approaching zero.

Logarithmic barrier

Replace the **primal** LP

$$\begin{array}{rll} \min & c^T x & \\ \text{s.t.} & Ax & = b, \\ & x \geq 0, & \end{array}$$

with the **primal barrier program**

$$\begin{array}{ll} \min & c^T x - \mu \sum_{j=1}^{n} \ln x_j \\ \text{s.t.} & Ax = b. \end{array}$$

Lagrangian: $$L(x, y, \mu) = c^T x - y^T (Ax - b) - \mu \sum_{j=1}^{n} \ln x_j.$$

Conditions for a stationary point of the Lagrangian

$$\begin{aligned} \nabla_x L(x,y,\mu) &= c - A^T y - \mu X^{-1} e = 0 \\ \nabla_y L(x,y,\mu) &= \qquad\qquad\quad Ax - b = 0, \end{aligned}$$

where $X^{-1} = diag\{x_1^{-1}, x_2^{-1}, \cdots, x_n^{-1}\}$.

Let us denote

$$s = \mu X^{-1} e, \qquad \text{i.e.} \qquad XSe = \mu e.$$

The **First Order Optimality Conditions** are:

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ XSe &= \mu e, \\ (x,s) &> 0. \end{aligned}$$

The pronunciation of Greek letter μ [mi]

Robert De Niro, Taxi Driver (1976)

Central Trajectory

The first order optimality conditions for the barrier problem

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ XSe &= \mu e, \\ (x, s) &\geq 0 \end{aligned}$$

approximate the first order optimality conditions for the LP

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ XSe &= 0, \\ (x, s) &\geq 0 \end{aligned}$$

more and more closely as μ goes to zero.

Central Trajectory

Parameter μ controls the distance to optimality.

$$c^T x - b^T y = c^T x - x^T A^T y = x^T (c - A^T y) = x^T s = n\mu.$$

Analytic centre (μ-centre): a (unique) point

$$(x(\mu), y(\mu), s(\mu)), \quad x(\mu) > 0, \; s(\mu) > 0$$

that satisfies FOC.

The path

$$\{(x(\mu), y(\mu), s(\mu)) : \mu > 0\}$$

is called the **primal-dual central trajectory**.

Newton Method

is used to find a stationary point of the barrier problem.

Recall how to use Newton Method to find a root of a nonlinear equation

$$f(x) = 0.$$

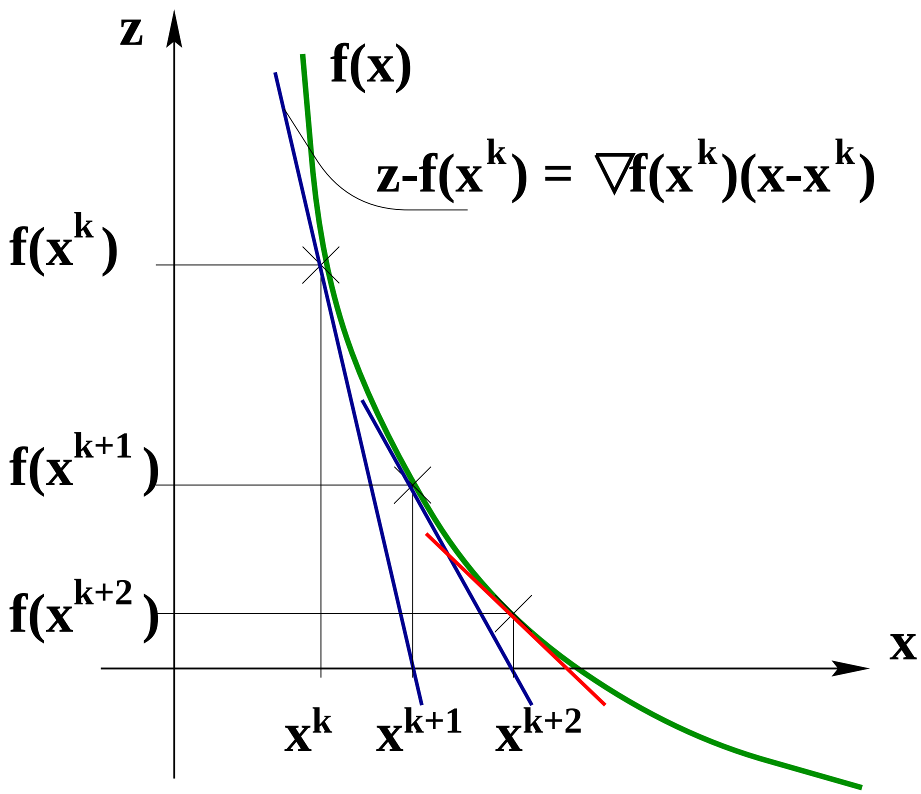
A tangent line

$$z - f(x^k) = \nabla f(x^k) \cdot (x - x^k)$$

is a local approximation of the graph of the function $f(x)$.
Substituting $z = 0$ defines a new point

$$x^{k+1} = x^k - (\nabla f(x^k))^{-1} f(x^k).$$

Newton Method



$$x^{k+1} = x^k - (\nabla f(x^k))^{-1} f(x^k).$$

Apply Newton Method to the FOC

The first order optimality conditions for the barrier problem form a large system of nonlinear equations

$$f(x, y, s) = 0,$$

where $f : \mathcal{R}^{2n+m} \mapsto \mathcal{R}^{2n+m}$ is a mapping defined as follows:

$$f(x, y, s) = \left[\begin{array}{rcl} Ax & - & b \\ A^T y + s & - & c \\ XSe & - & \mu e \end{array} \right].$$

Actually, the first two terms of it are **linear**; only the last one, corresponding to the complementarity condition, is **nonlinear**.

Newton Method (cont'd)

Note that

$$\nabla f(x,y,s) = \begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix}.$$

Thus, for a given point (x,y,s) we find the Newton direction $(\Delta x, \Delta y, \Delta s)$ by solving the system of linear equations:

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} b - Ax \\ c - A^T y - s \\ \mu e - XSe \end{bmatrix}.$$

Interior-Point Framework

The **logarithmic barrier**

$$-\ln x_j$$

"replaces" the inequality $x_j \geq 0.$

We derive the **first order optimality conditions** for the primal barrier problem:

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ XSe &= \mu e, \end{aligned}$$

and apply **Newton method** to solve this system of (nonlinear) equations.

Actually, we fix the barrier parameter μ and make only **one** (damped) Newton step towards the solution of FOC. We do not solve the current FOC exactly. Instead, we immediately reduce the barrier parameter μ (to ensure progress towards optimality) and repeat the process.

Interior Point Algorithm

Initialize

$$k = 0 \qquad\qquad (x^0, y^0, s^0) \in \mathcal{F}^0$$
$$\mu_0 = \frac{1}{n} \cdot (x^0)^T s^0 \qquad\qquad \alpha_0 = 0.9995$$

Repeat until optimality

$k = k + 1$

$\mu_k = \sigma \mu_{k-1}$, where $\sigma \in (0, 1)$

$\Delta = (\Delta x, \Delta y, \Delta s) =$ Newton direction towards μ-centre

Ratio test:

$$\begin{aligned} \alpha_P &:= \max \{\alpha > 0 : \; x + \alpha \Delta x \geq 0\}, \\ \alpha_D &:= \max \{\alpha > 0 : \; s + \alpha \Delta s \geq 0\}. \end{aligned}$$

Make step:

$$\begin{aligned} x^{k+1} &= x^k + \alpha_0 \alpha_P \Delta x, \\ y^{k+1} &= y^k + \alpha_0 \alpha_D \Delta y, \\ s^{k+1} &= s^k + \alpha_0 \alpha_D \Delta s. \end{aligned}$$

Notations

$$X = diag\{x_1, x_2, \cdots, x_n\} = \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{bmatrix}.$$

$$e = (1, 1, \cdots, 1) \in \mathcal{R}^n, \;\; X^{-1} = diag\{x_1^{-1}, x_2^{-1}, \cdots, x_n^{-1}\}.$$

An equation $\quad XSe = \mu e,$

is equivalent to $\quad x_j s_j = \mu, \quad \forall j = 1, 2, \cdots, n.$

Notations(cont'd)

Primal feasible set $\mathcal{P} = \{x \in \mathcal{R}^n \,|\, Ax = b,\ x \geq 0\}$.
Primal strictly feasible set $\mathcal{P}^0 = \{x \in \mathcal{R}^n \,|\, Ax = b,\ x > 0\}$.

Dual feasible set $\mathcal{D} = \{y \in \mathcal{R}^m, s \in \mathcal{R}^n \,|\, A^T y + s = c,\ s \geq 0\}$.
Dual strictly feasible set $\mathcal{D}^0 = \{y \in \mathcal{R}^m, s \in \mathcal{R}^n \,|\, A^T y + s = c,\ s > 0\}$.

Primal-dual feasible set
$\mathcal{F} \ = \{(x, y, s) \,|\, Ax = b,\ A^T y + s = c,\ (x, s) \geq 0\}$.
Primal-dual strictly feasible set
$\mathcal{F}^0 = \{(x, y, s) \,|\, Ax = b,\ A^T y + s = c,\ (x, s) > 0\}$.

Path-Following Algorithm

The analysis given in this lecture comes from the book of **Steve Wright**: *Primal-Dual Interior-Point Methods*, SIAM Philadelphia, 1997.

We analyze a **feasible** interior-point algorithm with the following properties:

- all its iterates are feasible and stay in a close neighbourhood of the central path;
- the iterates follow the central path towards optimality;
- systematic (though slow) reduction of duality gap is ensured.

This algorithm is called
the **short-step path-following method**.
Indeed, it makes very slow progress (short-steps) to optimality.

Central Path Neighbourhood

Assume a primal-dual strictly feasible solution $(x, y, s) \in \mathcal{F}^0$ lying in a neighbourhood of the central path is given; namely (x, y, s) satisfies:

$$\begin{aligned} Ax &= b, \\ A^T y + s &= c, \\ XSe &\approx \mu e. \end{aligned}$$

We define a θ**-neighbourhood** of the central path $N_2(\theta)$, a set of primal-dual strictly feasible solutions $(x, y, s) \in \mathcal{F}^0$ that satisfy:

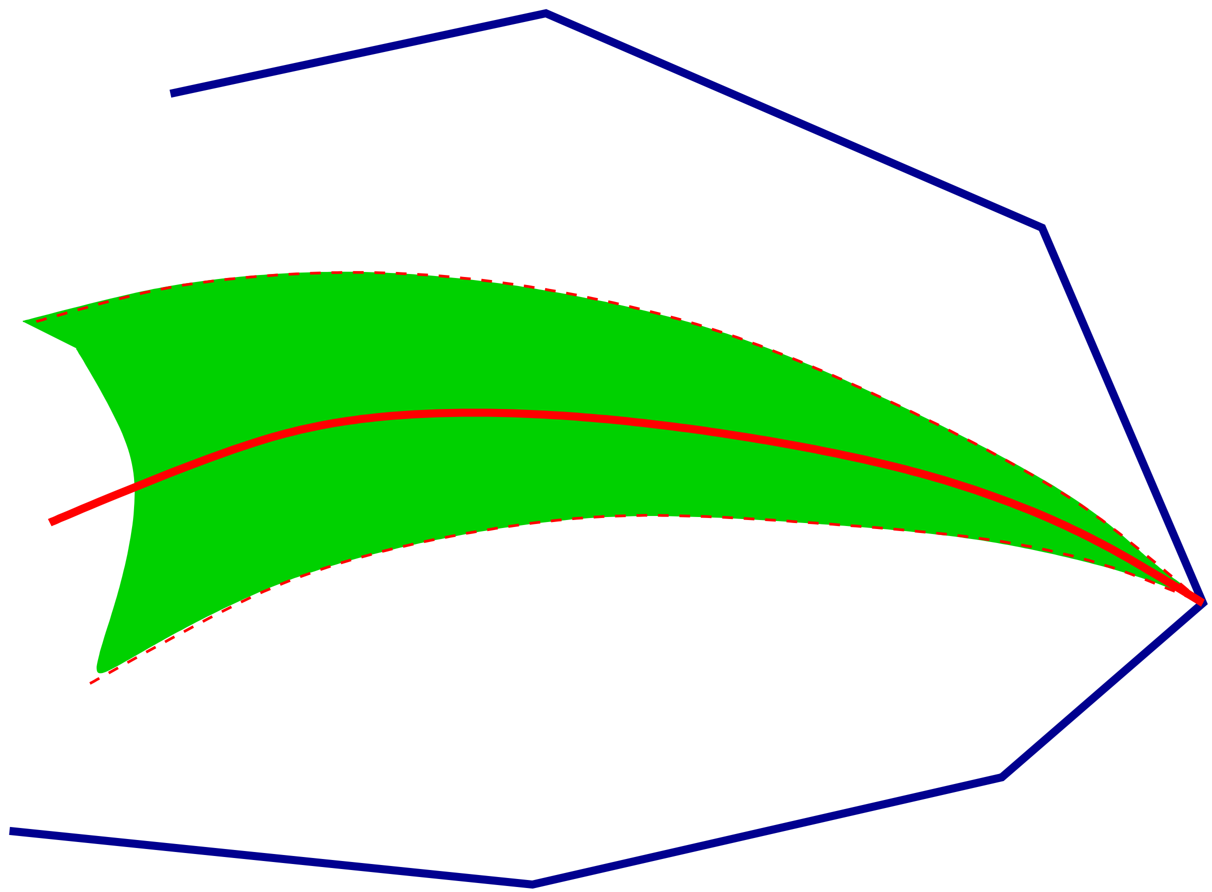
$$\|XSe - \mu e\| \leq \theta \mu,$$

where $\theta \in (0, 1)$ and the barrier μ satisfies:

$$x^T s = n\mu.$$

Hence $N_2(\theta) = \{(x, y, s) \in \mathcal{F}^0 \mid \|XSe - \mu e\| \leq \theta\mu\}$.

Central Path Neighbourhood



$N_2(\theta)$ neighbourhood of the central path

Progress towards optimality

Assume a primal-dual strictly feasible solution $(x, y, s) \in N_2(\theta)$ for some $\theta \in (0, 1)$ is given.

Interior point algorithm tries to move from this point to another one that also belongs to a θ-neighbourhood of the central path but corresponds to a smaller μ. The required reduction of μ is small:

$$\mu^{k+1} = \sigma \mu^k, \qquad \text{where} \qquad \sigma = 1 - \beta/\sqrt{n},$$

for some $\beta \in (0, 1)$.

This is a **short-step** method:
It makes short steps to optimality.

Progress towards optimality

Given a new μ-centre, interior point algorithm computes Newton direction:

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sigma \mu e - XSe \end{bmatrix},$$

and makes step in this direction.

Magic numbers (will be explained later):

$$\theta = 0.1 \quad \text{and} \quad \beta = 0.1.$$

θ controls the proximity to the central path;
β controls the progress to optimality.

How to prove the $\mathcal{O}(\sqrt{n})$ complexity result

We will prove the following:

- full step in Newton direction is feasible;
- the new iterate
  $(x^{k+1}, y^{k+1}, s^{k+1}) = (x^k, y^k, s^k) + (\Delta x^k, \Delta y^k, \Delta s^k)$
  belongs to the θ-neighbourhood of the new μ-centre
  (with $\mu^{k+1} = \sigma \mu^k$);
- duality gap is reduced $1 - \beta/\sqrt{n}$ times.

$\mathcal{O}(\sqrt{n})$ complexity result

Note that since at one iteration duality gap is reduced $1 - \beta/\sqrt{n}$ times, after $\sqrt{n}$ iterations the reduction achieves:

$$(1 - \beta/\sqrt{n})^{\sqrt{n}} \approx e^{-\beta}.$$

After $C \cdot \sqrt{n}$ iterations, the reduction is $e^{-C\beta}$. For sufficiently large constant C the reduction can thus be arbitrarily large (i.e. the duality gap can become arbitrarily small).

Hence this algorithm has complexity $\mathcal{O}(\sqrt{n})$.

This should be understood as follows:

> *"after the number of iterations proportional to $\sqrt{n}$ the algorithm solves the problem"*.

Worst-Case Complexity Result

Technical Results

Lemma 1

Newton direction $(\Delta x, \Delta y, \Delta s)$ defined by the equation system

$$\begin{bmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S & 0 & X \end{bmatrix} \cdot \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sigma\mu e - XSe \end{bmatrix}, \tag{1}$$

satisfies:

$$\Delta x^T \Delta s = 0.$$

Proof:

From the first two equations in (1) we get

$$A\Delta x = 0 \quad \text{and} \quad \Delta s = -A^T \Delta y.$$

Hence

$$\Delta x^T \Delta s = \Delta x^T \cdot (-A^T \Delta y) = -\Delta y^T \cdot (A\Delta x) = 0.$$

Technical Results (cont'd)

Lemma 2

Let $(\Delta x, \Delta y, \Delta s)$ be the Newton direction that solves the system (1). The new iterate

$$(\bar{x}, \bar{y}, \bar{s}) = (x, y, s) + (\Delta x, \Delta y, \Delta s)$$

satisfies

$$\bar{x}^T \bar{s} = n\bar{\mu},$$

where

$$\bar{\mu} = \sigma \mu.$$

Proof: From the third equation in (1) we get

$$S\Delta x + X\Delta s = -XSe + \sigma\mu e.$$

By summing the n components of this equation we obtain

$$\begin{aligned} e^T(S\Delta x + X\Delta s) = s^T\Delta x + x^T\Delta s &= -e^T XSe + \sigma\mu e^T e \\ &= -x^T s + n\sigma\mu = -x^T s \cdot (1-\sigma). \end{aligned}$$

Thus

$$\begin{aligned} \bar{x}^T \bar{s} &= (x+\Delta x)^T (s + \Delta s) \\ &= x^T s + (s^T \Delta x + x^T \Delta s) + (\Delta x)^T \Delta s \\ &= x^T s + (\sigma - 1) x^T s + 0 = \sigma x^T s, \end{aligned}$$

which is equivalent to:

$$n\bar{\mu} = \sigma n \mu.$$

Reminder: Norms of the vector $x \in \mathcal{R}^n$.

$$\begin{aligned}
\|x\| &= (\sum_{j=1}^{n} x_j^2)^{1/2} \\
\|x\|_\infty &= \max_{j \in \{1..n\}} |x_j| \\
\|x\|_1 &= \sum_{j=1}^{n} |x_j|
\end{aligned}$$

For any $x \in \mathcal{R}^n$:

$$\begin{aligned}
\|x\|_\infty &\leq \|x\|_1 \\
\|x\|_1 &\leq n \cdot \|x\|_\infty \\
\|x\|_\infty &\leq \|x\| \\
\|x\| &\leq \sqrt{n} \cdot \|x\|_\infty \\
\|x\| &\leq \|x\|_1 \\
\|x\|_1 &\leq \sqrt{n} \cdot \|x\|
\end{aligned}$$

Reminder: Triangle Inequality

For any vectors p, q and r and for any norm $\|.\|$

$$\|p - q\| \leq \|p - r\| + \|r - q\|.$$

The relation between *algebraic* and *geometric* means.
For any scalars a and b such that $ab \geq 0$:

$$\sqrt{|ab|} \leq \frac{1}{2} \cdot |a + b|.$$

Technical Result (algebra)

Lemma 3 Let u and v be any two vectors in $\mathcal{R}^n$ such that $u^T v \geq 0$. Then

$$\|UVe\| \leq 2^{-3/2} \|u+v\|^2,$$

where $U = diag\{u_1, \cdots, u_n\}$, $V = diag\{v_1, \cdots, v_n\}$.

Proof: Let us partition all products $u_j v_j$ into positive and negative ones:

$$\mathcal{P} = \{j \,|\, u_j v_j \geq 0\} \quad \text{and} \quad \mathcal{M} = \{j \,|\, u_j v_j < 0\} :$$

$$0 \leq u^T v = \sum_{j \in \mathcal{P}} u_j v_j + \sum_{j \in \mathcal{M}} u_j v_j = \sum_{j \in \mathcal{P}} |u_j v_j| - \sum_{j \in \mathcal{M}} |u_j v_j|.$$

Proof (cont'd)

We can now write

$$
\begin{aligned}
\|UVe\| &= (\|[u_j v_j]_{j\in\mathcal{P}}\|^2 + \|[u_j v_j]_{j\in\mathcal{M}}\|^2)^{1/2} \\
&\le (\|[u_j v_j]_{j\in\mathcal{P}}\|_1^2 + \|[u_j v_j]_{j\in\mathcal{M}}\|_1^2)^{1/2} \\
&\le (2\|[u_j v_j]_{j\in\mathcal{P}}\|_1^2)^{1/2} \\
&\le \sqrt{2}\|[\frac{1}{4}(u_j + v_j)^2]_{j\in\mathcal{P}}\|_1 \\
&= 2^{-3/2}\sum_{j\in\mathcal{P}}(u_j + v_j)^2 \\
&\le 2^{-3/2}\sum_{j=1}^{n}(u_j + v_j)^2 \\
&= 2^{-3/2}\|u + v\|^2, \quad \text{as requested.}
\end{aligned}
$$

IPM Technical Results (cont'd)

Lemma 4
If $(x, y, s) \in N_2(\theta)$ for some $\theta \in (0, 1)$, then

$$(1 - \theta)\mu \leq x_j s_j \leq (1 + \theta)\mu \quad \forall j.$$

In other words,

$$\min_{j \in \{1..n\}} x_j s_j \geq (1 - \theta)\mu,$$

$$\max_{j \in \{1..n\}} x_j s_j \leq (1 + \theta)\mu.$$

Proof:

Since $\|x\|_\infty \leq \|x\|$, from the definition of $N_2(\theta)$,

$$N_2(\theta) = \{(x, y, s) \in \mathcal{F}^0 \mid \|XSe - \mu e\| \leq \theta\mu\},$$

we conclude

$$\|XSe - \mu e\|_\infty \leq \|XSe - \mu e\| \leq \theta\mu.$$

Hence

$$|x_j s_j - \mu| \leq \theta\mu \quad \forall j,$$

which is equivalent to

$$-\theta\mu \leq x_j s_j - \mu \leq \theta\mu \quad \forall j.$$

Thus

$$(1 - \theta)\mu \leq x_j s_j \leq (1 + \theta)\mu \quad \forall j.$$

IPM Technical Results (cont'd)

Lemma 5

If $(x, y, s) \in N_2(\theta)$ for some $\theta \in (0,1)$, then

$$\|XSe - \sigma\mu e\|^2 \leq \theta^2\mu^2 + (1-\sigma)^2\mu^2 n.$$

Proof:

Note first that

$$e^T(XSe - \mu e) = x^T s - \mu e^T e = n\mu - n\mu = 0.$$

Therefore

$$\begin{aligned} &\|XSe - \sigma\mu e\|^2 \\ &\quad = \|(XSe - \mu e) + (1-\sigma)\mu e\|^2 \\ &\quad = \|XSe - \mu e\|^2 + 2(1-\sigma)\mu e^T(XSe - \mu e) + (1-\sigma)^2\mu^2 e^T e \\ &\quad \leq \theta^2\mu^2 + (1-\sigma)^2\mu^2 n. \end{aligned}$$

IPM Technical Results (cont'd)

Lemma 6
If $(x, y, s) \in N_2(\theta)$ for some $\theta \in (0,1)$, then

$$\|\Delta X \Delta S e\| \leq \frac{\theta^2 + n(1-\sigma)^2}{2^{3/2}(1-\theta)}\mu.$$

Proof: 3rd equation in the Newton system gives

$$S\Delta x + X\Delta s = -XSe + \sigma\mu e.$$

Having multiplied it with $(XS)^{-1/2}$, we obtain

$$X^{-1/2}S^{1/2}\Delta x + X^{1/2}S^{-1/2}\Delta s = (XS)^{-1/2}(-XSe + \sigma\mu e).$$

Proof (cont'd)

Define $u = X^{-1/2} S^{1/2} \Delta x$ and $v = X^{1/2} S^{-1/2} \Delta s$ and observe that (by Lemma 1) $u^T v = \Delta x^T \Delta s = 0$. Now apply Lemma 3:

$$
\begin{aligned}
\|\Delta X \Delta S e\| &= \|(X^{-1/2} S^{1/2} \Delta X)(X^{1/2} S^{-1/2} \Delta S) e\| \\
&\leq 2^{-3/2} \|X^{-1/2} S^{1/2} \Delta x + X^{1/2} S^{-1/2} \Delta s\|^2 \\
&= 2^{-3/2} \|X^{-1/2} S^{-1/2} (-XSe + \sigma \mu e)\|^2 \\
&= 2^{-3/2} \sum_{j=1}^{n} \frac{(-x_j s_j + \sigma \mu)^2}{x_j s_j} \\
&\leq 2^{-3/2} \frac{\|XSe - \sigma \mu e\|^2}{\min_j x_j s_j} \\
&\leq \frac{\theta^2 + n(1-\sigma)^2}{2^{3/2}(1-\theta)} \mu \qquad \text{(by Lemmas 4 and 5).}
\end{aligned}
$$

Magic Numbers

We have previously set two parameters for the short-step path-following method:

$$\theta \in [0.05,\, 0.1] \quad \text{and} \quad \beta \in [0.05,\, 0.1].$$

Now it's time to justify this particular choice.

Both θ and β have to be small to make sure that a full step in the Newton direction does not take the new iterate outside the neighbourhood $N_2(\theta)$.

θ controls the proximity to the central path;
β controls the progress to optimality.

Magic numbers choice lemma

Lemma 7 If $\theta \in [0.05,\ 0.1]$ and $\beta \in [0.05,\ 0.1]$, then

$$\frac{\theta^2 + n(1-\sigma)^2}{2^{3/2}(1-\theta)} \leq \sigma\theta.$$

Proof:
Recall that

$$\sigma = 1 - \beta/\sqrt{n}.$$

Hence

$$n(1-\sigma)^2 = \beta^2$$

and for any $\beta \in [0.05,\ 0.1]$ (for any $n \geq 1$)

$$\sigma \geq 0.9.$$

Substituting $\theta \in [0.05,\ 0.1]$ and $\beta \in [0.05,\ 0.1]$, we obtain

$$\frac{\theta^2 + n(1-\sigma)^2}{2^{3/2}(1-\theta)} = \frac{0.1^2 + 0.1^2}{2^{3/2} \cdot 0.9} \leq 0.02 \leq 0.9 \cdot 0.1 \leq \sigma\theta.$$

Magic Numbers

Set $\theta = 0.07$ and $\beta = 0.07$

0.07 is a **Super Number**.

The name is Bond, James Bond.

Full Newton step in $N_2(\theta)$

Lemma 8 Suppose $(x, y, s) \in N_2(\theta)$ and $(\Delta x, \Delta y, \Delta s)$ is the Newton direction computed from the system (1). Then the new iterate

$$(\bar{x}, \bar{y}, \bar{s}) = (x, y, s) + (\Delta x, \Delta y, \Delta s)$$

satisfies $(\bar{x}, \bar{y}, \bar{s}) \in N_2(\theta)$, i.e. $\|\bar{X}\bar{S}e - \bar{\mu}e\| \leq \theta\bar{\mu}$.

Proof: From Lemma 2, the new iterate $(\bar{x}, \bar{y}, \bar{s})$ satisfies

$$\bar{x}^T\bar{s} = n\bar{\mu} = n\sigma\mu,$$

so we have to prove that $\|\bar{X}\bar{S}e - \bar{\mu}e\| \leq \theta\bar{\mu}$.

For a given component $j \in \{1..n\}$, we have

$$\begin{aligned}
\bar{x}_j\bar{s}_j - \bar{\mu} &= (x_j + \Delta x_j)(s_j + \Delta s_j) - \bar{\mu} \\
&= x_j s_j + (s_j \Delta x_j + x_j \Delta s_j) + \Delta x_j \Delta s_j - \bar{\mu} \\
&= x_j s_j + (-x_j s_j + \sigma\mu) + \Delta x_j \Delta s_j - \sigma\mu \\
&= \Delta x_j \Delta s_j.
\end{aligned}$$

Proof (cont'd)

Thus, from Lemmas 6 and 7, we get

$$\begin{aligned} \|\bar{X}\bar{S}e - \bar{\mu}e\| &= \|\Delta X \Delta S e\| \\ &\leq \frac{\theta^2 + n(1-\sigma)^2}{2^{3/2}(1-\theta)}\mu \\ &\leq \sigma\theta\mu \\ &= \theta\bar{\mu}. \end{aligned}$$

A property of log function

Lemma 9 For all $\delta > -1$:

$$\ln(1+\delta) \leq \delta.$$

Proof:
Consider the function

$$f(\delta) = \delta - \ln(1+\delta)$$

and its derivative

$$f^{'}(\delta) = 1 - \frac{1}{1+\delta} = \frac{\delta}{1+\delta}.$$

Obviously $f^{'}(\delta) < 0$ for $\delta \in (-1, 0)$ and $f^{'}(\delta) > 0$ for $\delta \in (0, \infty)$. Hence $f(.)$ has a *minimum* at $\delta = 0$. We find that $f(\delta = 0) = 0$. Consequently, for any $\delta \in (-1, \infty)$, $f(\delta) \geq 0$, i.e.

$$\delta - \ln(1+\delta) \geq 0.$$

$\mathcal{O}(\sqrt{n})$ Complexity Result

Theorem 10

Given $\epsilon > 0$, suppose that a feasible starting point $(x^0, y^0, s^0) \in N_2(0.1)$ satisfies

$$(x^0)^T s^0 = n\mu^0, \text{ where } \mu^0 \leq 1/\epsilon^\kappa,$$

for some positive constant κ. Then there exists an index K with $K = \mathcal{O}(\sqrt{n} \ln(1/\epsilon))$ such that

$$\mu^k \leq \epsilon, \quad \forall k \geq K.$$

$\mathcal{O}(\sqrt{n})$ Complexity Result

Proof: From Lemma 2, $\mu^{k+1} = \sigma\mu^k$. Having taken logarithms of both sides of this equality we obtain

$$\ln \mu^{k+1} = \ln \sigma + \ln \mu^k.$$

By repeatedly applying this formula and using $\mu^0 \leq 1/\epsilon^\kappa$, we get

$$\ln \mu^k = k \ln \sigma + \ln \mu^0 \leq k \ln(1 - \beta/\sqrt{n}) + \kappa \ln(1/\epsilon).$$

From Lemma 9 we have $\ln(1-\beta/\sqrt{n}) \leq -\beta/\sqrt{n}$. Thus

$$\ln \mu^k \leq k(-\beta/\sqrt{n}) + \kappa \ln(1/\epsilon).$$

To satisfy $\mu^k \leq \epsilon$, we need:

$$k(-\beta/\sqrt{n}) + \kappa \ln(1/\epsilon) \leq \ln \epsilon.$$

This inequality holds for any $k \geq K$, where

$$K = \frac{\kappa + 1}{\beta} \cdot \sqrt{n} \cdot \ln(1/\epsilon).$$

Polynomial Complexity Result

Main ingredients of the polynomial complexity result for the short-step path-following algorithm:

Stay close to the central path:
all iterates stay in the $N_2(\theta)$ neighbourhood of the central path.

Make (slow) progress towards optimality:
reduce systematically duality gap

$$\mu^{k+1} = \sigma \mu^k,$$

where

$$\sigma = 1 - \beta/\sqrt{n},$$

for some $\beta \in (0, 1)$.

Reading about IPMs

S. Wright
Primal-Dual Interior-Point Methods, SIAM Philadelphia, 1997.

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Interior point methods 25 years later,
European J. of Operational Research 218 (2012) 587–601.
`http://www.maths.ed.ac.uk/~gondzio/reports/ipmXXV.html`

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Direct solution of linear systems of size 10^9 arising in optimization with interior point methods, in: *Parallel Processing and Applied Mathematics PPAM 2005*, R. Wyrzykowski, J. Dongarra, N. Meyer and J. Wasniewski (eds.), *Lecture Notes in Computer Science*, 3911, Springer-Verlag, Berlin, 2006, pp 513–525.

OOPS: Object-Oriented Parallel Solver

`http://www.maths.ed.ac.uk/~gondzio/parallel/solver.html`