Sequential guaranteed estimation methods for the Cox-Ingersoll-Ross (CIR) models

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Greater Paris Region Mathematics Postdocs Day

* This research is supported by the French RIN project MaSyComB
and the Jacques Hadamard Mathematics Foundation

21 October 2025

Plan

Plan

General model and sampling Method

$$\mathrm{d}X_t = b(\theta_0, X_t)\mathrm{d}t + \sigma(\theta_1, X_t)\mathrm{d}W_t.$$

- **Problem**: Using data observed from a single trajectory of X to estimate parameters $\theta_0 \in \Theta_0 \subset \mathbb{R}^{d_0}$ and $\theta_1 \in \Theta_1 \subset \mathbb{R}^{d_1}$
- Continuous sampling: $(X_t)_{t\in[0,T]}$
- Discrete sampling: $(X_{k\Delta_n})_{0\leq k\leq n}$
  - High frequency data in a fixed period: $\Delta_n \to 0$ and $n\Delta_n = T$ fixed
  - Low frequency data in a long period: $\Delta_n = \Delta$ fixed, and $n\Delta_n \to \infty$
  - High frequency data in a long period: $\Delta_n \to 0$ and $n\Delta_n \to \infty$
- The estimation of θ_1 which is called volatility in mathematical finance, is based on the fact that the quadratic variation $\sum_{i=1}^{n}(X_{t_i} - X_{t_{i-1}})^2 \to \int_0^T \sigma(\theta_1, X_s)^2\mathrm{d}s.$
- For any fixed T, θ_1 can be consistently estimated if $\Delta_n \to 0$.
- In the following, we suppose that θ_1 is known and try to estimate θ_0

Maximum likelihood estimation

In Liptser & Shiryaev'01 and Kutoyants'04, Radon-Nikodym derivative of the probability measure induced by

$$\mathrm{d}X_t = b(\theta, X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0 = x \in \mathbb{R}$$

with $\theta \in \Theta \subset \mathbb{R}^d$ is given and the likelihood function is

$$L_T(\theta) = \exp\left(\int_0^T \frac{b(\theta, X_t) - b(\theta_0, X_t)}{\sigma^2(X_t)}\mathrm{d}X_t - \frac{1}{2}\int_0^T \frac{b^2(\theta, X_t) - b^2(\theta_0, X_t)}{\sigma^2(X_t)}\mathrm{d}t\right)$$

for any fixed $\theta_0 \in \Theta \subset \mathbb{R}^d$. The MLE $\widehat{\theta}_T$ of θ is defined by

$$\widehat{\theta}_T = \mathrm{argmax}_{\theta \in \Theta} L_T(\theta)$$

The consistency and asymptotic normality of $\widehat{\theta}_T$ have been shown under some regularity conditions of b and σ, generally in ergodic case.

The sequential analysis approaches

- The first approaches: in Novikov'72 and Liptser & Shiryaev'01 for **the scalar Ornstein-Uhlenbeck process in continuous time**

$$\mathrm{d}X_t = \theta X_t \mathrm{d}t + \mathrm{d}W_t. \tag{1}$$

The Maximum Likelihood Estimator (MLE) for the parameter θ defined as

$$\widehat{\theta}_T = (\int_0^T X_s^2 \mathrm{d}s)^{-1} \int_0^T X_s \mathrm{d}X_s \,.$$

Then the Sequential Maximum Likelihood Estimator (SMLE) for the parameter θ defined as

$$\widehat{\theta}_{\tau_H^*} = \frac{1}{H} \int_0^{\tau_H^*} X_s \mathrm{d}X_s \,,$$

with the stopping time

$$\tau_H^* = \inf \left\{ t \geq 0 \,:\, \int_0^t X_s^2 \mathrm{d}s \geq H \right\} \,,$$

where $H > 0$ is some fixed non random arbitrary constant. One can check that $\widehat{\theta}_{\tau_H^*}$ is $\mathcal{N}(\theta, H^{-1})$ for any $H > 0$.

The sequential analysis approaches

- Discrete time: Borisov & Konev'77 (non-asymptotic), Lai & Siegmund'83 (asymptotic).
- *Guaranteed* two-step estimation method for multidimensional parameter case: Konev & Pergamenshchikov'81(1)(2), Konev & Pergamenshchikov'88, Galtchouk & Konev'01.
- Guaranteed estimation property in the non-asymptotic setting with dependent observations: Konev & Pergamenshchikov'97.
- For the proposed sequential estimation methods, the asymptotic properties, as $H \to \infty$, were studied (see, e.g., in Konev & Pergamenshchikov'84'85'86, Pergamenshchikov'85).

Parameter estimation problems for the CIR processes

We consider the stochastic differential equation

$$\mathrm{d}X_t = (a - bX_t)\mathrm{d}t + \sqrt{\sigma X_t}\mathrm{d}W_t, \quad X_0 = x > 0\,, \tag{2}$$

where $a > 0$, $b \in \mathbb{R}$ and $\sigma > 0$ and $(W_t)_{t\geq 0}$ is a standard Brownian motion.

- In finance, the CIR is used to describes the evolution of interest rates and stochastic volatility stock markets modeling.

  $\hookrightarrow$ Rfs : [Cox, Ingersoll & Ross'85], [Lamberton & Lapeyre'97], [Heston'93], [Berdjane and Pergamenshchikov'13] and [Nguyen and Pergamenshchikov'17], etc.

- In biology, the CIR can be used to model population dynamics and in the epidemic analysis.

  $\hookrightarrow$ Rfs: [Bansaye & Méléard'15], [Pergamenchtchikov, Tartakovsky & Spivak'22], etc.

Parameter estimation problems for the CIR processes

$$\mathrm{d}X_t = (a - bX_t)\mathrm{d}t + \sqrt{\sigma X_t}\mathrm{d}W_t, \quad X_0 = x > 0\,, \tag{3}$$

- For $\theta = b$, the MLE (see, e.g., in Ben Alaya & Kebaier'12) is defined as $\widehat{\theta}_T = \frac{aT - X_T + x}{\int_0^T X_s \mathrm{d}s}$.
- For $\theta = a$, the MLE is given as $\widehat{\theta}_T = \frac{bT + \int_0^T X_t^{-1}\mathrm{d}X_t}{\int_0^T X_t^{-1}\mathrm{d}t}$.
- For $\theta = (a, b)$, the MLE (see, e.g., in Ben Alaya & Kebaier'13) is defined as

$$\widehat{\theta}_T = \begin{cases} \widehat{a}_T &= \frac{\int_0^T X_t \mathrm{d}t \int_0^T X_t^{-1}\mathrm{d}X_t - T(X_T - x)}{\int_0^T X_t \mathrm{d}t \int_0^T X_t^{-1}\mathrm{d}t - T^2} \\ \widehat{b}_T &= \frac{T\int_0^T X_t^{-1}\mathrm{d}X_t - (X_T - x)\int_0^T X_t^{-1}\mathrm{d}t}{\int_0^T X_t \mathrm{d}t \int_0^T X_t^{-1}\mathrm{d}t - T^2} \end{cases}$$

- These papers show the asymptotic behavior of the error $(\widehat{\theta}_T - \theta)$ for ergodic and non-ergodic cases.
- There are still no results on a guaranteed estimation for the CIR process.

Plan

Plan

Case $\theta = b$

We define the sequential procedure $\delta_H^* = (\tau_H^*, \theta_H^*)$ with $H > 0$ for the parameter b as

$$\tau_H^* = \inf\left\{t : \int_0^t X_s \mathrm{d}s \geq H\right\} \quad \text{and} \quad \theta_H^* = \frac{a\tau_H^* - X_{\tau_H^*} + x}{H},$$

First we study non asymptotic properties of this procedure, i.e. for any fixed threshold $H > 0$.

Theorem 1 (Ben Alaya, N. and Pergamenchtchikov'25 (1))

For any $\boxed{a > 0, \ b \in \mathbb{R}}$ and for any fixed $H > 0$, δ_H^* possesses the following properties:

1. $\mathbf{P}_\theta(\tau_H^* < \infty) = 1$;
2. the sequential estimator θ_H^* is normally distributed with parameters

$$\mathbf{E}_\theta \theta_H^* = b \quad \text{and} \quad \mathbf{E}_\theta(\theta_H^* - b)^2 = \frac{\sigma}{H}.$$

Case $\theta = a$

We define the sequential estimation procedure $\delta_H^* = (\tau_H^*, \theta_H^*)$ with $H > 0$ for the parameter a as

$$\tau_H^* = \inf\left\{t : \int_0^t X_s^{-1}\mathrm{d}s \geq H\right\} \quad \text{and} \quad \theta_H^* = \frac{b\tau_H^* + \int_0^{\tau_H^*} X_s^{-1}\mathrm{d}X_s}{H}.$$

First we study non asymptotic properties of this procedure, i.e. for any fixed threshold $H > 0$.

Theorem 2 (Ben Alaya, N. and Pergamenchtchikov'25 (1))

For any $\boxed{b \geq 0,\ a > 0}$ and for any fixed $H > 0$ the sequential procedure δ_H^* possesses the following properties:

1. $\mathbf{P}_\theta(\tau_H^* < \infty) = 1$;
2. the sequential estimator θ_H^* is normally distributed with parameters

$$\mathbf{E}_\theta \theta_H^* = a \quad \text{and} \quad \mathbf{E}_\theta(\theta_H^* - a)^2 = \frac{\sigma}{H}.$$

+ When $a < \sigma/2$, $\int_0^t X_s^{-1/2}\mathrm{d}W_s$ is not defined for any fixed non random $t > 0$. Therefore, the non sequential MLE can not be calculated for this case, but the sequential procedure is well defined.

Plan

Case $\theta = (a, b)$

We rewrite the CIR model as

$$\mathrm{d}X_t = \mathbf{g}_t^\top \theta \mathrm{d}t + \sqrt{\sigma X_t}\mathrm{d}W_t \,,$$

where $\mathbf{g}_t = (1\,,\,-X_t)^\top$. In this section we assume, that $b > 0$ and $a > \sigma/2$. Then, in view of the results from Ben Alaya & Kebaier'13, the random matrix

$$G_t = \int_0^t X_s^{-1} \mathbf{g}_s \mathbf{g}_s^\top \mathrm{d}s = \begin{pmatrix} \int_0^t X_s^{-1} \mathrm{d}s & -t \\ -t & \int_0^t X_s ds \end{pmatrix}$$

possesses the following asymptotic property

$$\lim_{t\to\infty} \frac{1}{t} G_t = F = \begin{pmatrix} f_1 & -1 \\ -1 & f_2 \end{pmatrix} \qquad \mathbf{P}_\theta - \quad \text{a.s.}\,,$$

where $f_1 = 2b/(2a - \sigma)$ and $f_2 = a/b$. Here, F is positively definite matrix.

We use the two-step sequential fixed accuracy estimation method developed in Konev & Pergamenshchikov'81'85

- $\boxed{\text{First step}}$: we construct the sequence of the sequential procedures $\left(\delta_n = (\mathbf{t}_n, \widehat{\theta}_{\mathbf{t}_n})\right)_{n\geq 1}$. We fix a non random sequence of non-decreasing positive numbers $(\kappa_n)_{n\geq 1}$ for which

$$\rho = \sum_{n\geq 1} \frac{1}{\kappa_n} < \infty\,. \tag{4}$$

Now for any $z > 0$ we set

$$\mathbf{t}_z = \inf\left\{t \geq 0 : \int_0^t X_s^{-1}\,|\mathbf{g}_s|^2 \mathrm{d}s \geq z\right\}\,, \tag{5}$$

Let $\mathbf{t}_n = \mathbf{t}_{\kappa_n}$ and define the sequential MLE as

$$\widehat{\theta}_{\mathbf{t}_n} = G_{\mathbf{t}_n}^{-1} \int_0^{\mathbf{t}_n} X_s^{-1} \mathbf{g}_s \mathrm{d}X_s \tag{6}$$

- $\boxed{\text{Second step}}$: we construct a sequential aggregation estimation procedure which is defined as weighted sum of the estimators (6). First we set

$$\mathbf{b}_n = \frac{1}{|G_{\mathbf{t}_n}^{-1}|\,\kappa_n}\mathbf{1}_{\{\lambda_{min}(G_{\mathbf{t}_n})>0\}}$$

where $|G|^2 = \operatorname{tr} GG^\top$, and we define the stopping time as

$$\upsilon_H^* = \inf\left\{k \geq 1 : \sum_{n=1}^{k} \mathbf{b}_n^2 \geq H\right\}, \tag{7}$$

for a positive non random threshold $H > 0$. We define the sequential estimator as

$$\theta_H^* = \left(\sum_{n=1}^{\upsilon_H^*} \mathbf{b}_n^2\right)^{-1} \sum_{n=1}^{\upsilon_H^*} \mathbf{b}_n^2\,\widehat{\theta}_{\mathbf{t}_n}\,. \tag{8}$$

So, we obtain aggregated two-step sequential procedure

$$\boxed{\delta^*_H = \left(\tau^*_H\,,\,\theta^*_H\right) \quad \text{and} \quad \tau^*_H = \mathbf{t}_{\upsilon^*_H}}\,. \tag{9}$$

Theorem 3 (Ben Alaya, N. and Pergamenchtchikov'25 (1))

For any $b > 0$ and $a > \sigma/2$ and for any $H > 0$ the procedure (9) has the following properties

$$\tau^*_H < +\infty \quad \mathbf{P}_\theta - \text{a.s.}$$

and

$$\mathbf{E}_\theta \left|\theta^*_H - \theta\right|^2 \le \rho \frac{\sigma}{H}\,,$$

where the coefficient ρ is defined in (4).

Plan

Minimax inequality

From now, we take the parameter set

$$\Theta \subseteq \{(a,b) \,:\, a > \sigma/2\,,\, b > 0\} =]\sigma/2\,,+\infty[\times]0,+\infty[.$$

Let now $\theta_0 \in \Theta$ and $\gamma > 0$ such that $\{|\theta - \theta_0| \le \gamma\} \subseteq \Theta$. We denote by $\mathcal{H}_T(\theta_0,\gamma)$ the local class of sequential procedures $\delta_T = (\tau\,,\,\widehat{\theta}_\tau)$ such that

$$\boxed{\sup_{|\theta-\theta_0|<\gamma} \mathbf{E}_\theta \tau \le T}.$$

Inspired by the ideas from Corollary 2 in Efroimovich'80, we prove the following proposition.

Proposition (Ben Alaya, N. and Pergamenchtchikov'25 (1))

Assume that, LAN holds for θ_0 from $\Theta \subset \mathbb{R}^k$ with the function $\varphi_T = (I(\theta_0)T)^{-1/2}$ and $I(\theta_0)$ is the Fisher information matrix. Then, for any $\gamma > 0$ for which $\{|\theta - \theta_0| \le \gamma\} \subseteq \Theta$,

$$\lim_{T\to\infty} \inf_{\delta \in \mathcal{H}_T(\theta_0,\gamma)} \sup_{|\theta-\theta_0|<\gamma} \mathbf{E}_\theta \left|\varphi_T^{-1}\,(\widehat{\theta}_\tau - \theta)\right|^2 \ge k\,.$$

Now we need to compare the defined sequential procedure $\delta^*_H = (\tau^*_H, \theta^*_H)$ with other sequential procedure. To this end we set

$$\Xi_H = \left\{ \delta = (\tau, \widehat{\theta}_\tau) \, : \, \sup_{\theta \in \Theta} \frac{\mathbf{E}_\theta \tau}{\mathbf{E}_\theta \tau^*_H} \leq 1 \right\} .$$

Now we obtain a lower bound for this class.

Theorem 4 (Ben Alaya, N. and Pergamenchtchikov'25 (1))

Let θ_0 from $\Theta \subset \mathbb{R}^k$ such that $\{|\theta - \theta_0| < \gamma\} \subset \Theta$ for all sufficiently small $\gamma > 0$. Assume that, LAN holds in θ_0 with the normalizing function $\varphi_T = (I(\theta_0)T)^{-1/2}$ and $I(\theta_0)$ is the Fisher information matrix. Then,

$$\lim_{H \to \infty} \inf_{\delta \in \Xi_H} \sup_{\theta \in \Theta} \mathbf{E}_\theta \, |\upsilon_H(\theta)^{1/2}(\widehat{\theta}_\tau - \theta)|^2 \geq k \, ,$$

where $\upsilon_H(\theta) = I(\theta)\mathbf{E}_\theta \tau^*_H$.

Optimality of SMLE θ^*_H for $\theta = a$, $k = 1$

Theorem 5 (Ben Alaya, N. and Pergamenchtchikov'25 (1))

For any $\boxed{b > 0}$, any compact set $\boxed{\Theta \subset]\sigma/2, +\infty[}$ and for any $r > 0$

$$\lim_{H \to \infty} \sup_{\theta \in \Theta} \mathbf{E}_\theta \left| \frac{\tau^*_H}{H} - I_0^{-1}(\theta) \right|^r = 0\,,$$

where $I_0(\theta) = 2b/(2a - \sigma)$.

In this case:

- $\upsilon_H(\theta) = \sigma^{-1}(2\theta - \sigma)^{-1}\, 2b\, \mathbf{E}_\theta \tau^*_H \approx \sigma^{-1} H$ as $H \to \infty$.
- $\lim_{H \to \infty} \sup_{\theta \in \Theta} \upsilon_H(\theta)\, \mathbf{E}_\theta \left(\theta^*_H - \theta\right)^2 = 1$.

Optimality of SMLE θ^*_H for $\theta = b$, $k = 1$

Theorem 7 (Ben Alaya, N. and Pergamenchtchikov'25 (1))

For any $\boxed{a > \sigma/2}$, any compact set $\boxed{\Theta \subset]0, +\infty[}$ and any $r > 0$

$$\lim_{H\to\infty} \sup_{\theta\in\Theta} \mathbf{E}_\theta \left| \frac{\tau^*_H}{H} - \frac{1}{I_0(\theta)} \right|^r = 0\,,$$

where $I_0(\theta) = a/\theta$.

In this case:

- $\upsilon_H(\theta) = \sigma^{-1}\theta^{-1} a\,\mathbf{E}_\theta \tau^*_H \approx \sigma^{-1} H$ as $H \to \infty$.
- $\lim_{H\to\infty} \sup_{\theta\in\Theta} \upsilon_H(\theta)\,\mathbf{E}_\theta \left(\theta^*_H - \theta\right)^2 = 1.$

Optimality of SMLE θ^*_H for $\theta = (a, b)$, $k = 2$

We chose the sequence $(\kappa_n)_{n\geq 1}$ as follows

$$\kappa_n = \begin{cases} H\,, & \text{for} \quad n \leq \mathbf{n}^*_H\,; \\ \kappa^*_n\,, & \text{for} \quad n > \mathbf{n}^*_H\,, \end{cases}$$

where $\mathbf{n}^*_H = L_H H$ and $L_H \geq 1$ is slowly increasing function, i.e.

$$\lim_{H\to\infty} L_H = +\infty \quad \text{and} \quad \lim_{H\to\infty} \frac{L_H}{H^\delta} = 0 \quad \text{for any} \quad \delta > 0\,.$$

Moreover, $(\kappa^*_n)_{n\geq 1}$ is a sequence of positive increasing numbers such, that for some $\mu > 1$ and $0 < \varrho < 1$,

$$\limsup_{n\to\infty} n^{-\mu}\,\kappa^*_n < \infty \quad \text{and} \quad \limsup_{n\to\infty} n^{-\varrho} \sum_{k=1}^{n} \frac{1}{\sqrt{\kappa^*_k}} < \infty\,.$$

For example, we can take $\mathbf{n}^*_H = H \ln H$ and $\kappa^*_n = n^\mu$ for some $\mu > 1$.

Optimality of SMLE θ^*_H for $\theta = (a, b)$, $k = 2$

Theorem 9 (Ben Alaya, N. and Pergamenchtchikov'25 (1))

For any compact set $\Theta \subset]\sigma/2, +\infty[\times]0, +\infty[$ for the duration time in the sequential procedure (9) we have for any $r > 0$

$$\lim_{H\to\infty} \sup_{\theta\in\Theta} \mathbf{E}_\theta \left| \frac{\tau^*_H}{H} - \frac{1}{\mathrm{tr} F} \right|^r = 0\,,$$

where the matrix F is defined above by $\lim_{t\to\infty} \frac{1}{t} G_t = F$.

- Let $\tilde{F} = F/tr(F)$. In this case $\upsilon_H(\theta) = \sigma^{-1} F\, \mathbf{E}_\theta \tau^*_H \approx \sigma^{-1} H \tilde{F}$ as $H \to \infty$.
- $\lim_{H\to\infty} \sup_{\theta\in\Theta}\ \mathbf{E}_\theta \left| \upsilon_H(\theta)^{1/2} \left(\theta^*_H - \theta\right) \right|^2 \le 2$.

Plan

Truncated sequential approaches

Truncated versions of developed sequential estimators were proposed in [Konev & Pergamenshchikov(92), Konev & Pergamenshchikov(97)], [Ben Alaya, N. and Pergamenchtchikov'25 (2)].

The proposed truncated sequential procedures use essentially fewer observations than classical non-sequential estimators based on the fixed non-random duration of observations.

We still obtain the guaranteed and optimal properties of estimations.

Plan

Case $\theta = b$

We define the truncated sequential procedure $\widetilde{\delta}_{H,T} = (\widetilde{\tau}_{H,T}, \widetilde{\theta}_{H,T})$, in which the alternative stopping time $\widetilde{\tau}_{H,T}$ and the corresponding sequential estimator $\widetilde{\theta}_{H,T}$ are defined as

$$\widetilde{\tau}_{H,T} = \tau_H \wedge T \quad \text{and} \quad \widetilde{\theta}_{H,T} = \theta^*_H \mathbf{1}_{\{\tau_H \le T\}}. \tag{10}$$

For any compact $\Theta \subset]0, +\infty[$, we denote $\mathbf{a}_* = \frac{a}{\mathbf{b}_{max}}$. We choose the value for the parameter H to minimize the estimation accuracy :

$$H^*_T = \mathbf{a}_* T - (2m\mathbf{U}_m \mathbf{a}_*^2/\sigma)^{\frac{1}{2m+1}} T^{\frac{2+m}{2m+1}} (1 + \mathrm{o}(1)) \quad \text{as} \quad T \to \infty\,; \tag{11}$$

We define the optimal truncated procedure

$$(\tau^*_T, \theta^*_T)\,, \quad \boxed{\tau^*_T = \widetilde{\tau}_{H^*_T,T} \quad \text{and} \quad \theta^*_T = \widetilde{\theta}_{H^*_T,T}}. \tag{12}$$

Theorem 10 (Ben Alaya, N. and Pergamenchtchikov'25 (2))

For any integer $m > 1$

$$\sup_{\theta \in \Theta} \mathbf{E}_\theta \left(\theta^*_T - \theta\right)^2 \le \frac{\sigma}{\mathbf{a}_* T} + \mathrm{O}\left(\frac{1}{T^{\frac{3m}{2m+1}}}\right) \quad \text{as} \quad T \to \infty\,. \tag{13}$$

Case $\theta = a$

We define the truncated sequential procedure

$$\widetilde{\tau}_{H,T} = \tau_H \wedge T \quad \text{and} \quad \widetilde{\theta}_{H,T} = \theta^*_H \mathbf{1}_{\{\tau_H \le T\}}. \tag{14}$$

For any compact $\Theta \subset (\sigma/2, +\infty)$, we denote $\mu_{a,\theta} = \int_{\mathbb{R}_+} \min(x^{-1}, \mathbf{r})\, \mathbf{q}_{\theta,b}(z)\mathrm{d}z$ for $\mathbf{r} > 1$ and $\mu_{a,*} = \inf_{\theta\in\Theta} \mu_{a,\theta}$. We choose the value for the parameter H to minimize the estimation accuracy :

$$H^*_T = \mu_{a,*} T - \mathbf{r}^{\frac{2m}{2m+1}} \left(2mV_m \mu^2_{a,*}/\sigma\right)^{\frac{1}{2m+1}} T^{\frac{2+m}{2m+1}} (1 + \mathrm{o}(1)) \quad \text{as} \quad T \to \infty\,; \tag{15}$$

We define the optimal truncated procedure

$$\left(\tau^*_T, \theta^*_T\right), \quad \boxed{\tau^*_T = \widetilde{\tau}_{H^*_T,T} \quad \text{and} \quad \theta^*_T = \widetilde{\theta}_{H^*_T,T}}\,. \tag{16}$$

Theorem 11 (Ben Alaya, N. and Pergamenchtchikov'25 (2))

For any integer $m > 1$

$$\sup_{\theta\in\Theta} \mathbf{E}_\theta \left(\theta^*_T - \theta\right)^2 \le \frac{\sigma}{\mu_{a,*} T} + \mathrm{o}\left(\frac{1}{T}\right) \quad \text{as} \quad T \to \infty\,. \tag{17}$$

Plan

Case $\theta = (a, b)$

We define the truncated sequential procedure

$$\widetilde{\tau}_{H,T} = \tau_H \wedge T \quad \text{and} \quad \widetilde{\theta}_{H,T} = \theta^*_H \mathbf{1}_{\{\tau_H \le T\}}. \tag{18}$$

Setting

$$\mathbf{u}_* = \max_{\theta \in \Theta} \left(|F^{-1}| \operatorname{tr} F \right)^2 , \tag{19}$$

we chose the sequence $(\kappa_n)_{n \ge 1}$ as

$$\kappa_n = \begin{cases} H\,, & \text{for} \quad n \le \mathbf{n}^*_H\,; \\ \kappa^*_n\,, & \text{for} \quad n > \mathbf{n}^*_H\,, \end{cases} \tag{20}$$

where $\mathbf{n}^*_H = 2\mathbf{u}_* H$, and $(\kappa^*_n)_{n \ge 1}$ is an increasing sequence such that for all n it is bounded from below as $\kappa^*_n \ge n$ and for some constants $\varpi > 1$ and $0 < \delta^* < 1/2$,

$$\overline{\lim}_{n\to\infty}\, n^{-\varpi}\, \kappa^*_n \,<\, \infty \quad \text{and} \quad \overline{\lim}_{n\to\infty}\, n^{-\delta^*} \sum_{k=1}^{n} \frac{1}{\sqrt{\kappa^*_k}} \,<\, \infty\,. \tag{21}$$

For example, we can take $\kappa^*_n = n^{\varpi}$ and $\delta^* = (2 - \varpi)/2$ for some $1 < \varpi < 2$.

Case $\theta = (a, b)$

For any compact set $\Theta \subset (\sigma/2, +\infty) \times (0, +\infty)$, assume that for some $0 < \delta < 1/2$ the parameter $\mathbf{r}$ is such that

$$\mathbf{r} \to \infty \quad \text{and} \quad \mathbf{r} = \mathrm{O}(T^{\delta}) \quad \text{as} \quad T \to \infty\,. \tag{22}$$

Then, for any $m > (1 - 2\delta)^{-1}$, we choose

$$H_T^* = \bar{\mu}_* T + \mathrm{o}(T) \quad \text{as} \quad T \to \infty\,; \tag{23}$$

where $\bar{\mu}_* = \min_{(a,b)\in\Theta} \operatorname{tr} F$. We define the truncated procedure

$$\left(\tau_T^*, \theta_T^*\right), \quad \boxed{\tau_T^* = \widetilde{\tau}_{H_T^*,T} \quad \text{and} \quad \theta_T^* = \widetilde{\theta}_{H_T^*,T}}\,. \tag{24}$$

Theorem 12 (Ben Alaya, N. and Pergamenchtchikov'25 (2))

$$\sup_{\theta\in\Theta} \mathbf{E}_{\theta} \left(\theta_T^* - \theta\right)^2 \le \frac{2\mathbf{u}_*\sigma}{\bar{\mu}_* T} + \mathrm{o}\left(\frac{1}{T}\right), \tag{25}$$

Plan

Optimality of SMLE θ^*_T for $\theta = b$, $k = 1$

Theorem 13 (Ben Alaya, N. and Pergamenchtchikov'25 (2))

For any compact set $\Theta \subset]0, +\infty[$ the stopping time τ^*_T defined in the procedure (12) for any $r > 0$ satisfies the following asymptotic property

$$\lim_{T\to\infty} \sup_{\theta\in\Theta} \mathbf{E}_\theta \left| \frac{\tau^*_T}{T} - \frac{\theta}{\mathbf{b}_{max}} \right|^r = 0\,. \tag{26}$$

For some family of sequential procedures $\left(\tau^*_T, \theta^*_T\right)_{T>0}$ such that for any parameter $\theta \in \Theta$ the expectation $\mathbf{E}_\theta\, \tau^*_T \to +\infty$ as $T \to \infty$ we use the following class

$$\Xi^*_T = \left\{ (\tau, \widehat{\theta}_\tau) \,:\, \sup_{\theta\in\Theta} \frac{\mathbf{E}_\theta \tau}{\mathbf{E}_\theta \tau^*_T} \le 1 \right\}. \tag{27}$$

Theorem 14 (Ben Alaya, N. and Pergamenchtchikov'25 (2))

For any compact set $\Theta \subset]0, +\infty[$,

$$\lim_{T\to\infty} \frac{\inf_{(\tau,\widehat{\theta}_\tau)\in\Xi^*_T}\, \sup_{\theta\in\Theta}\, \mathbf{E}_\theta\, (\widehat{\theta}_\tau - \theta)^2}{\sup_{\theta\in\Theta}\, \mathbf{E}_\theta\, (\theta^*_T - \theta)^2} = 1\,. \tag{28}$$

Optimality of SMLE θ^*_T for $\theta = a$, $k = 1$

Theorem 15 (Ben Alaya, N. and Pergamenchtchikov'25 (2))

For any fixed $b > 0$ any compact set $\Theta \subset]\sigma/2, +\infty[$ the stopping time τ^*_T defined in the procedure (16) for any $r > 0$ satisfies the following asymptotic property

$$\lim_{T\to\infty} \sup_{\theta\in\Theta} \mathbf{E}_\theta \left| \frac{\tau^*_T}{T} - \frac{2\theta - \sigma}{2\mathbf{a}_{max} - \sigma} \right|^r = 0 \,. \tag{29}$$

Theorem 16 (Ben Alaya, N. and Pergamenchtchikov'25 (2))

For any $b > 0$ and any compact set $\Theta \subset]\sigma/2, +\infty[$,

$$\lim_{T\to\infty} \frac{\inf_{(\tau,\widehat{\theta}_\tau)\in\Xi^*_T} \sup_{\theta\in\Theta} \mathbf{E}_\theta \, (\widehat{\theta}_\tau - \theta)^2}{\sup_{\theta\in\Theta} \mathbf{E}_\theta \, (\theta^*_T - \theta)^2} = 1 \,. \tag{30}$$

Optimality of SMLE θ^*_T for $\theta = (a, b)$, $k = 2$

Theorem 17 (Ben Alaya, N. and Pergamenchtchikov'25 (2))

For any compact set $\Theta \subset (\sigma/2, +\infty) \times (0, +\infty)$ and for any $r > 0$,

$$\lim_{H\to\infty} \sup_{\theta\in\Theta} \mathbf{E}_\theta \left| \frac{\tau^*_T}{T} - \frac{\bar{\mu}_*}{\operatorname{tr} F} \right|^r = 0\,. \tag{31}$$

Theorem 18 (Ben Alaya, N. and Pergamenchtchikov'25 (2))

For any compact set $\Theta \subset (\sigma/2\,, +\infty) \times (0\,, +\infty)$ the sequential procedure (24) is asymptotically optimal in the minimax sense, i.e.

$$\lim_{T\to\infty} \frac{\inf_{(\tau,\widehat{\theta}_\tau)\in\Xi^*_T} \sup_{\theta\in\Theta} \mathbf{E}_\theta \left| \widetilde{F}^{1/2}\,(\widehat{\theta}_\tau - \theta)\right|^2}{\sup_{\theta\in\Theta} \mathbf{E}_\theta \left| \widetilde{F}^{1/2}\,(\theta^*_T - \theta)\right|^2} = 1\,. \tag{32}$$

Plan

What's new

- The (truncated) sequential estimation procedures are constructed for CIR processes and non asymptotic mean square accuracy are obtained.

- It should be emphasized, that in the estimation problem for the parameter a, the sequential estimator is well defined and possess the fixed accuracy estimation property in the cases when the classical maximum likelihood estimator is not defined for CIR model.

- Based on the LAN property, the minimax estimation theory for the sequential estimation procedures in the continuous time was developed.

- For the first time, the minimax properties for the sequential procedures in the continuous time are obtained in the class of all possible sequential procedures with the same mean observation duration.

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THANK YOU!