# Analysis of a two-stage probabilistic programming model

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# A simple two-stage probabilistic program

$x_1 \curvearrowright \xi_1 \curvearrowright x_2 \curvearrowright \xi_2$

$x$ = decisions; $\xi$= random observations

Probabilistic constraint: $\varphi(x_1,x_2) \coloneqq \mathbb{P}\left(\xi_1 \leq x_1, \xi_2 \leq x_2(\xi_1)\right) \geq p$

Probabilistic program: $\min\{f(x_1,x_2) | \varphi(x_1,x_2) \geq p, (x_1,x_2) \in \mathbb{R} \times L^2(\mathbb{R})\}$

For a basic structural analysis of general multistage probabilistic programs with $L^2$ and $W^{1,2}$ decision spaces we refer to:

*T. Gonzalez Grandon, R. Henrion u. P. Pérez-Aros, Dynamic probabilistic constraints under continuous random distributions, to appear in: Mathematical Programming.*

# Properties of the probability function

For simplicity: $\xi = (\xi_1, \xi_2) \sim \mathcal{N}(\mu, \Sigma)$ ( bivariate Gaussian)

Then, $\varphi(x_1, x_2) := \mathbb{P}\left(\xi_1 \leq x_1, \xi_2 \leq x_2(\xi_1)\right)$ is Lipschitz continuous and

$$\nabla_{x_2} \varphi(x_1, x_2)(r) = K \exp\left( -\frac{1}{2} \begin{pmatrix} r - \mu_1 \\ x_2(r) - \mu_2 \end{pmatrix}^T \Sigma^{-1} \begin{pmatrix} r - \mu_1 \\ x_2(r) - \mu_2 \end{pmatrix} \right) \chi_{(-\infty, x_1]}$$

If $x_2(.)$ is continuous, then

$$\frac{\partial \varphi}{\partial x_1}(x_1, x_2) = K \exp\left( -\frac{1}{2\Sigma_{11}} (r - \mu_1)^2 \right) \Phi\left( \frac{x_2(x_1) - \mu_2 - \Sigma_{11}^{-1} \Sigma_{12} (x_1 - \mu_1)}{\sqrt{\Sigma_{22} - \Sigma_{11}^{-1} \Sigma_{12}^2}} \right)$$

# Necessary optimality condition in $L^2$

Probabilistic program: $\min\{c_1 x_1 + c_2 \mathbb{E} x_2(\xi_1)|_{\xi_1 \leq x_1} \; s.t. \; \varphi(x_1, x_2) \geq p\}$

For any fixed $x_1 \in \mathbb{R}$, an optimal policy $x_2(.)$ in the conditional problem

$$\min\{c_2 \mathbb{E} x_2(\xi_1)|_{\xi_1 \leq x_1} \; s.t. \; \varphi(x_1, x_2) \geq p\}$$

has the form $x_2(r) = \frac{\Sigma_{12}}{\Sigma_{11}} r + C \; \left(\xi \sim \mathcal{N}(\mu, \Sigma)\right)$ affine linear $\notin L^2(\mathbb{R})$

No solution in $L^2$!

# Identification of a global solution

Probabilistic program: $\min\{c_1 x_1 + c_2 \mathbb{E} x_2(\xi_1)|_{\xi_1 \le x_1} \;\; s.t. \;\; \varphi(x_1,x_2) \ge \; p\}$

We allow $x_2(.)$ to be an arbitrary Borel measurable function. Global solution:

$$x_1^* = arg \min_{t \epsilon \mathbb{R}} \frac{\Sigma_{12}}{\Sigma_{11}} \beta(t) + \alpha(t) \Phi\left(\frac{t-\mu_1}{\sqrt{\Sigma_{11}}}\right); \quad x_2^*(r) = \frac{\Sigma_{12}}{\Sigma_{11}} r + \alpha(x_1^*)$$

$\alpha(t) = \Phi^{-1}\left(\frac{p}{\Phi\left(\frac{t-\mu_1}{\sqrt{\Sigma_{11}}}\right)}\right)\sqrt{\Sigma_{22} - \frac{\Sigma_{12}{}^2}{\Sigma_{11}}} + \mu_2 - \frac{\Sigma_{12}}{\Sigma_{11}}\mu_1; \; \beta(t) = \mu_1 - \sqrt{\Sigma_{11}} \frac{\varphi\left(\frac{t-\mu_1}{\sqrt{\Sigma_{11}}}\right)}{\Phi\left(\frac{t-\mu_1}{\sqrt{\Sigma_{11}}}\right)}; \;\; \varphi, \Phi$ – density and CDF of $\mathcal{N}(0,1)$