# Lecture 2

## Non-Atomic Routing Games
## Wardrop Equilibrium

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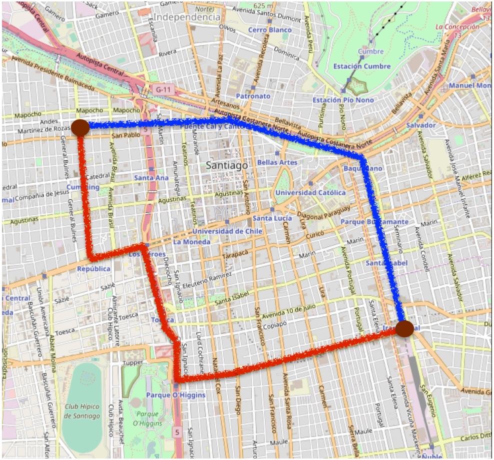
# Lecture 2: Non-Atomic Routing Games

1. Non-Atomic Routing Games
   - Wardrop equilibrium – Definition
   - Wardrop equilibrium – Characterizations
   - Wardrop equilibrium – Existence & Uniqueness

2. Inefficiency of Equilibria
   - Price-of-Anarchy
   - PoA for highly congested networks

# Non-atomic Routing Games

# Urban traffic flows under congestion




2266 nodes / 7636 arcs

| SANTIAGO |
|---|
| 6.000.000 people |
| 11.000.000 daily trips |
| 1.750.000 car trips |

| Morning peak |
|---|
| 500.000 car trips |
| 29.000 OD pairs |

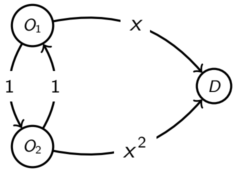
# Non-atomic routing games

Games with many players become computationally hard. Such situations can be idealized by considering players as a continuum and to focus on the fraction of players that use each strategy.

We illustrate this with routing games on transportation networks.

We are given a graph $(V, E)$ with

- a set of *edges* $e \in E$ with continuous non-decreasing costs $c_e : \mathbb{R}_+ \to \mathbb{R}_+$
- a set of *OD pairs* $\kappa \in \mathcal{K}$ with corresponding routes $r \in \mathcal{R}_\kappa \subseteq 2^E$
- a set of aggregate *demands* $d_\kappa \geq 0$ for each $\kappa \in \mathcal{K}$



# Wardrop equilibrium

- Continuum of players / each one has a negligible impact on congestion.
- Perfectly divisible / aggregate demands $d_\kappa \geq 0$ for each OD pair $\kappa \in \mathcal{K}$.

Let $\mathcal{F}$ be the set of splittings $(y, x)$ of the demands $d_\kappa$ into *route-flows* $y_r \geq 0$, together with their induced *edge-loads* $x_e$ :

$$d_\kappa = \sum_{r \in \mathcal{R}_\kappa} y_r \quad (\forall \kappa \in \mathcal{K}),$$

$$x_e = \sum_{r \ni e} y_r \qquad (\forall e \in E).$$

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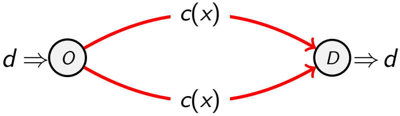
The analog of Nash equilibria for a continuum of players is:

**Definition (Wardrop, 1952)**

*A* Wardrop equilibrium *is a pair* $(\hat{y}, \hat{x}) \in \mathcal{F}$ *that uses only shortest routes:*

$$(\forall \kappa \in \mathcal{K})(\forall r, r' \in \mathcal{R}_\kappa) \quad \hat{y}_r > 0 \Rightarrow \sum_{e \in r} c_e(\hat{x}_e) \leq \sum_{e \in r'} c_e(\hat{x}_e).$$

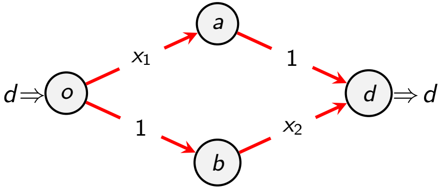
# Example: Single OD with 2 identical parallel links




At equilibrium the demand splits 50%-50% : $(\frac{d}{2}, \frac{d}{2})$.

# Example (Braess paradox):

Total demand is $d = 1$.




The upper and lower routes have cost $T_u = x_1 + 1$ and $T_l = 1 + x_2$.
Wardrop equilibrium sends $\frac{1}{2}$ on each route with travel time $T_{eq} = 1.5$.

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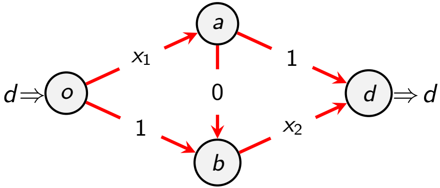


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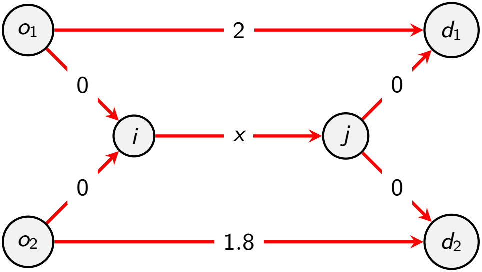
Wardrop equilibrium sends $\frac{1}{2}$ on each route with travel time $T_{eq} = 1.5$.

A central arc $(a, b)$ with cost 0 is added. The new Wardrop equilibrium sends all the flow along the zig-zag path $o$-$a$-$b$-$d$ with travel time $T_{eq} = 2.0$.

# An example with 2 OD pairs

Demands $d_1 = d_2 = 1$




The pair $\kappa_1$ sends all its flow $d_1 = 1$ through the central arc whose cost is always better that the upper route. Given this, $\kappa_2$ sends a traffic 0.8 on the central route until the cost equalizes the lower route which gets a flow of 0.2. The equilibrium cost for both pairs is 1.8.

EXERCISE: Find the equilibrium when $d_1 = d_2 = 2$

# Wardrop equilibrium – Characterizations

# Wardrop equilibrium – Characterizations

Introducing the route costs and minimal times

$$T_r(x) = \sum_{e \in r} c_e(x_e) \qquad ; \qquad \tau_\kappa(x) = \min_{r \in \mathcal{R}_\kappa} T_r(x).$$

the conditions for *Wardrop equilibrium* are

$$(\forall \kappa \in \mathcal{K})(\forall r \in \mathcal{R}_\kappa) \qquad y_r > 0 \Rightarrow T_r(x) = \tau_\kappa(x).$$

**Theorem (Beckman-McGuire-Winsten, 1956)**

*For a feasible flow $(y, x) \in \mathcal{F}$ the following are equivalent:*

a) $(y, x)$ *is a Wardrop equilibrium*

b) $\sum_{r \in \mathcal{R}} T_r(x)(y'_r - y_r) \geq 0 \qquad \forall (y', x') \in \mathcal{F}$

c) $\sum_{e \in E} c_e(x_e)(x'_e - x_e) \geq 0 \qquad \forall (y', x') \in \mathcal{F}$

d) $(y, x)$ *is an optimal solution of* $\displaystyle\min_{(y,x) \in \mathcal{F}} \sum_{e \in E} \int_0^{x_e} c_e(z)\, dz.$

*Proof:* For simplicity we consider the case of a single OD.

# Wardrop equilibrium – Characterization 1

**Proposition**

*A feasible flow $(y, x) \in \mathcal{F}$ is a WE iff*

$$(VI) \qquad \sum_{r\in\mathcal{R}} T_r(x)(y'_r - y_r) \geq 0 \qquad \forall\, (y', x') \in \mathcal{F}.$$

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*Proof:*
$(\Rightarrow)$ If $(y, x)$ is WE then for all $(y', x') \in \mathcal{F}$ we have

$$\sum_r T_r(x)\, y'_r \geq \sum_r \tau(x)\, y'_r = \tau(x)\, d = \sum_r \tau(x)\, y_r = \sum_r T_r(x)\, y_r.$$

$(\Leftarrow)$ Let $(y, x) \in \mathcal{F}$ a solution of (VI). If $y_r > 0$ we may consider the flow $y'$ identical to $y$ except for $y'_r = y_r - \epsilon$ and $y'_p = y_p + \epsilon$ with $p \in \mathcal{R}$ a shortest path

$$\Rightarrow \quad 0 \leq \sum_{q\in\mathcal{R}} T_q(x)(y'_q - y_q) = \epsilon T_p(x) - \epsilon T_r(x)$$

so that $T_r(x) \leq T_p(x) = \tau(x)$. Therefore $y_r > 0 \Rightarrow T_r(x) = \tau(x)$. □

# Wardrop equilibrium – Characterization 2

**Proposition**

*A feasible flow* $(y, x) \in \mathcal{F}$ *is a WE iff*

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# Wardrop equilibrium – Characterization 2

**Proposition**

*A feasible flow* $(y, x) \in \mathcal{F}$ *is a WE iff*

$$(VI) \qquad \sum_{e\in E} c_e(x_e)(x'_e - x_e) \geq 0 \qquad \forall\, (y', x') \in \mathcal{F}.$$

*Proof:* The equivalent form of the (VI) follows from an exchange in the sums

$$\begin{aligned}\sum_{r\in\mathcal{R}} T_r(x)(y'_r - y_r) &= \sum_{r\in\mathcal{R}}\sum_{e\in r} c_e(x_e)(y'_r - y_r)\\ &= \sum_{e\in E}\sum_{r\ni e} c_e(x_e)(y'_r - y_r)\\ &= \sum_{e\in E} c_e(x_e)(x'_e - x_e).\end{aligned}$$

□

# Wardrop equilibrium – Characterization 3

**Proposition**

*A feasible flow* $(y,x)\in\mathcal{F}$ *is a WE iff it is an optimal solution of the convex minimization problem*

$$(P)\qquad \min_{(y,x)\in\mathcal{F}} \Phi(y,x)=\sum_{e\in E}\int_0^{x_e} c_e(z)dz.$$

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*Proof:* Since $c_e(\cdot)$ is non-decreasing the function $\Phi(y,x)$ is convex, so that $(y,x)\in\mathcal{F}$ is a minimum iff for all $(y',x')\in\mathcal{F}$ we have

$$0\leq\langle\nabla\Phi(y,x),(y',x')-(y,x)\rangle=\sum_{e\in E}c_e(x_e)(x'_e-x_e).\qquad\square$$

REMARK. $\Phi$ is a continuous analog of Rosenthal's potential for discrete routing games. In the continuous case equilibria *coincide* with the minima of the potential.

# Wardrop equilibrium – Existence & Uniqueness

# Wardrop equilibrium – Existence and uniqueness

**Theorem**

*A non-atomic routing game has a Wardop equilibrium. Moreover, if* $(y, x)$ *and* $(y', x')$ *are two equilibria then* $c_e(x_e) = c_e(x'_e)$*. In particular, if* $c_e(\cdot)$ *is strictly increasing then* $x$ *is unique.*

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*Proof:* $\Phi$ is continuous $\Rightarrow$ its minimum on $\mathcal{F}$ is attained $\Rightarrow$ existence of WE.

If $(y, x)$ and $(y', x')$ are two equilibria, using (VI) we get

$$\begin{array}{r} \sum_{e\in E} c_e(x_e)(x'_e - x_e) \geq 0 \\ \sum_{e\in E} c_e(x'_e)(x_e - x'_e) \geq 0 \\ \hline \sum_{e\in E}(c_e(x_e) - c_e(x'_e))(x'_e - x_e) \geq 0 \end{array}$$

Since $c_e(\cdot)$ is non-decreasing each term in the sum is negative so that $(c_e(x_e) - c_e(x'_e))(x'_e - x_e) = 0$ for all $e \in E$, hence $c_e(x_e) = c_e(x'_e)$. □

# Variational Characterization

Wardrop equilibria are the optimal solutions of the convex program

$$(P) \qquad \min_{(y,x)\in\mathcal{F}} \sum_{e\in E} \int_0^{x_e} c_e(z)dz.$$

- $(P)$ is large scale $\approx 220 \times 10^6$ variables for Santiago
- Objective function different from the social cost

$$SC(x) = \sum_{e\in E} x_e\, c_e(x_e)$$

# Dual Characterization (Fukushima, 1984)

Change of variables: $x_e \leftrightarrow t_e$

(*D*)

$$\underset{t}{\mathrm{Min}} \quad \underbrace{\sum_{e \in E} \int_0^{t_e} c_e^{-1}(z)\, dz - \sum_{\kappa \in \mathcal{K}} d_\kappa\, \tau_\kappa(t)}_{\substack{\Phi(t) \\ \text{strictly convex}}}$$

$$\tau_\kappa(t) = \min_{r \in \mathcal{R}_\kappa} \sum_{e \in r} t_e \quad = \quad \begin{array}{l}\text{ODs minimum travel times} \\ \text{concave, polyhedral}\end{array}$$

Non-smooth but efficiently computable (Bellman, Dijkstra,...)

$$\tau_i^\kappa = \min_{e \in E_i^+} \{t_e + \tau_{j_e}^\kappa\}$$

# Inefficiency of Equilibria – Price-of-Anarchy

# Quantifying Inefficiency: Price-of-Anarchy

For non-atomic routing games

$$\text{Social cost} = \text{Total travel time} = \sum_{e\in E} x_e\, c_e(x_e)$$

$$\text{PoA} = \frac{\text{Social Cost of Equilibrium}}{\text{Minimum Social Cost}} \geq 1$$

**Theorem** (Roughgarden-Tardos, 2002; Roughgarden, 2003)

- $\text{PoA} \leq \frac{4}{3}$ *for non-atomic routing games with affine costs.*
- $\text{PoA} \leq \frac{\sqrt[k]{k+1}}{\sqrt[k]{k+1}-k/(k+1)} \sim O(\frac{k}{\log k})$ *for polynomials of degree k.*

Bounds attained for simple 2-link networks with fine-tuned demands.

# PoA and PoS in non-atomic routing games

Note that

$$\text{Total travel time} = \sum_{e \in E} x_e \, c_e(x_e) = \sum_{r \in \mathcal{R}} y_r \, T_r(x) = \sum_{\kappa \in \mathcal{K}} d_\kappa \, \tau_\kappa(x).$$

All Wardrop equilibria have the same value of $c_e(x_e)$
$\Rightarrow$ the same value of $T_r(x)$
$\Rightarrow$ the same minimal times $\tau_\kappa(x)$
$\Rightarrow$ social cost is constant on the set of Wardrop equilibria
$\Rightarrow$ PoS=PoA.

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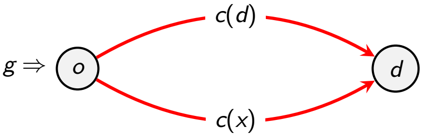
$\Rightarrow$ PoS=PoA.

**Example.** In the Braess paradox, when the central arc is unavailable Wardrop equilibrium splits half and half between with a travel time of 1.5. This coincides with the social optimum that minimizes $x_1(x_1+1)+x_2(x_2+1) \Rightarrow$ PoA=PoS=1.

If we allow the central arc, the new equilibrium sends all the flow on the zig-zag path with travel time 2. The social optimum does not change and the price of anarchy increases to PoA=PoS=$\frac{4}{3}$.

# Example: Pigou network

Let $c : [0, \infty) \to [0, \infty)$ be continuous and increasing and fix $d > 0$.




- Wardrop equilibrium is $x = d$ with social cost $d\,c(d)$
- Minimum cost is $\min_{x \in [0,d]}\ x\,c(x) + (d-x)c(d)$

Hence, PoA on this simple graph can be as large as

$$\alpha(c) = \sup_{d>0} \sup_{x \in [0,d]} \frac{d\,c(d)}{x\,c(x) + (d-x)c(d)} \geq 1.$$

This value allows to bound the PoA on any graph.

# PoA in non-atomic routing games

**Theorem (Correa-Schulz-Stier, 2004)**

*In a non-atomic routing game on a graph* $(N, A)$ *with arc costs* $c_e(\cdot)$ *we have*

$$\text{PoA} = \text{PoS} \leq \alpha \triangleq \max_{e \in E} \alpha(c_e).$$

*Proof:* Let $(y, x)$ be a WE and $(\bar{y}, \bar{x})$ a minimizer of $C(y, x)$. Taking $d = x_e$ and $x = \bar{x}_e$ in the expression for the supremum $\alpha(c_a)$ we get the inequality

$$x_c\, c_e(x_e) \leq \alpha[\bar{x}_e\, c_e(\bar{x}_e) + (x_e - \bar{x}_e)c_e(x_e)]$$

which added together and in view of VI yield

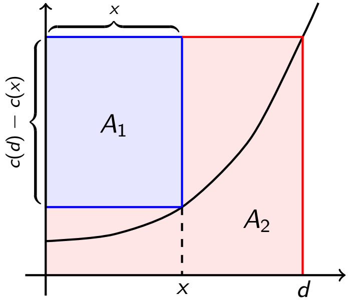
$$C(y, x) \leq \alpha \left[C(\bar{y}, \bar{x}) + \textstyle\sum_{e \in E} c_e(x_e)(x_e - \bar{x}_e)\right] \leq \alpha\, C_{min}.$$

□

# PoA in non-atomic routing games

Note that $\alpha(s)$ can be expressed as $\alpha(s) = 1/[1-\beta(s)]$ where

$$\beta(s) = \sup_{d>0} \sup_{x\in[0,d]} \frac{x[c(d)-c(x)]}{d\,c(d)} = \sup \frac{A_1}{A_2}.$$




If $c(\cdot)$ is affine we have $A_1 \leq \frac{1}{4}A_2$ so that $\beta(c) \leq \frac{1}{4}$. Taking $x = \frac{1}{2}d \to \infty$ we attain asymptotically $\beta(c) = \frac{1}{4}$, and therefore $\alpha(c) = \frac{4}{3}$.

# PoA with polynomial costs

**Proposition**

*For polynomials* $c(x) = a_0 + a_1 x + \cdots + a_k x^k$ *with* $a_i \geq 0$ *and* $a_k > 0$ *we have*

$$\alpha(c) = \alpha_k \triangleq \left[1 - k(k+1)^{-(k+1)/k}\right]^{-1} \sim \tfrac{k}{\ln k}.$$

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $\alpha_k$ | 1.3333 | 1.6258 | 1.8956 | 2.1505 | 2.3944 | 2.6297 |

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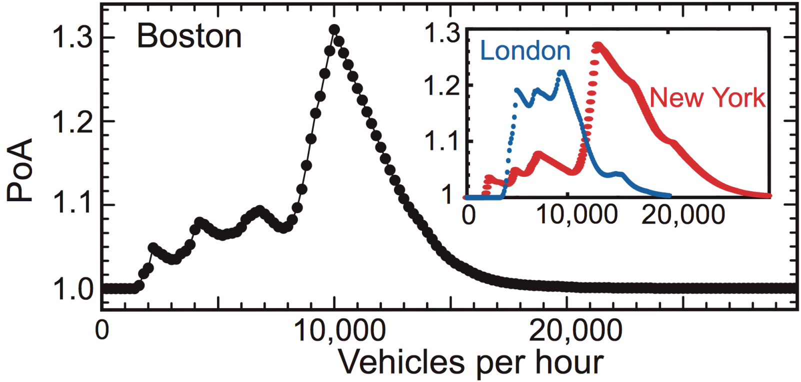
*Proof:* Note that $\beta(c) = \sup_{d>0} \sup_{x \in [0,d]} \frac{x}{d}[1 - \frac{c(x)}{c(d)}]$. From $a_i \geq 0$ we have that $c(x)/x^k$ is decreasing so that $c(x)/x^k \geq c(d)/d^k$ and then

$$\beta(c) \leq \sup_{d>0} \sup_{x \in [0,d]} \tfrac{x}{d}[1 - (\tfrac{x}{d})^k] = \sup_{z \in [0,1]} z(1 - z^k)$$

which is attained at $z^* = (k+1)^{-1/k}$. Hence $\beta(c) \leq k(k+1)^{-(k+1)/k}$ and therefore $\alpha(c) \leq \alpha_k$. This bound is tight: take $x = z^* d$ with $d \to \infty$. □

# Empirical observation (Youn et al. 2008, O'Hare et al. 2016,...)

In practice PoA is usually close to 1 both under high and low traffic, with fluctuations in the intermediate regime.

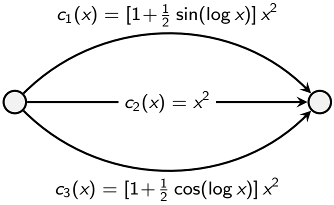


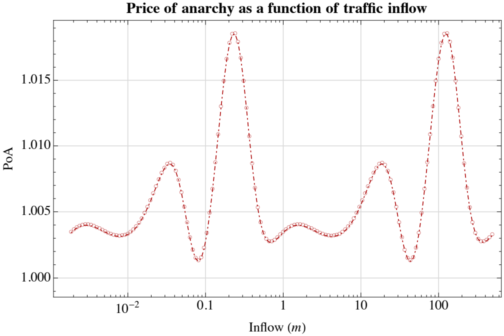
# Is it always true?

- Is it always the case that PoA=1 when the demand is small, and it goes back to one as the demand grows to $\infty$ ?
- Is it at least true for single OD networks ?
- Is it at least true for parallel networks ?
- Is it true for convex and smooth costs ?

# No, no, no, no...

PoA may oscillate and remain bounded away from 1 even for simple networks with smooth strongly convex costs:



# ...but eventually yes !

**Definition (Karamata, 1930)**

A function $c : [0, \infty) \to (0, \infty)$ is called regularly varying if for all $x > 0$ the limit $\lim_{t\to\infty} \frac{c(t x)}{c(t)}$ is finite and nonzero

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**Theorem** (Colini-C-Mertikopoulos-Scarsini, 2016, 2017)

- *Regularly varying costs:* PoA $\to 1$ *in the high congestion regime.*
- *Polynomial costs:* PoA $\to 1$ *plus sharp convergence rates.*