Loop Models on Causal Triangulations

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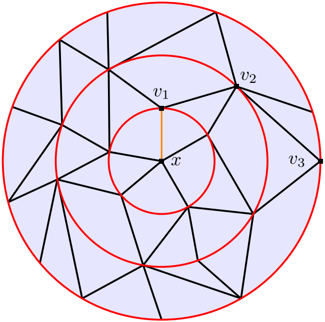
Joint work with B. Durhuus, J. Rasmussen and X. Poncini, arXiv:2104.14176 .

Overview

1. Loop Models
   - Causal triangulations
   - Dense loop model
   - Dilute loop model
2. Tree Correspondences
   - Pure CDT
   - Loop models
   - Partition functions
3. Critical behavior
   - Pure CDT and dense model: trees
   - Dilute model: transfer matrix

- A causal triangulation of a disk: a central vertex x and a sequence of cycles $S_0 \equiv \{x\}, S_1, \ldots, S_m$.
- For each $k = 0, 1, \ldots, m-1$, the annulus A_k is triangulated.

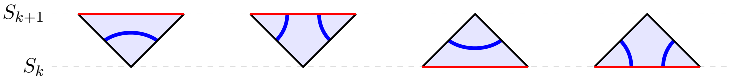


- A vertex v_1 in S_1 distinguished.

$$Z_m(g) := \sum_{C \in \mathcal{C}_m} g^{|C|}$$

$$Z(g) := \sum_{m=0}^{\infty} Z_m(g)$$

Dense loop model



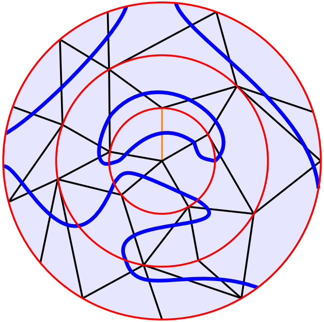
The set $\mathcal{L}_m^{de}$ obtained by replacing elementary triangles with the ones above.

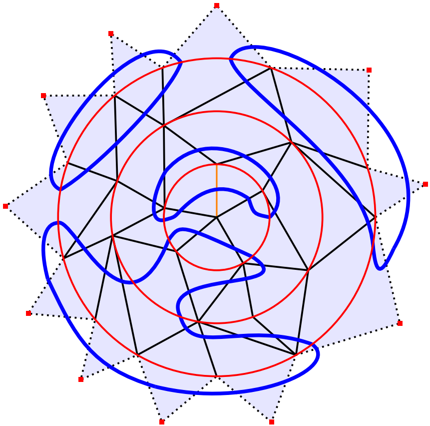
Now set

$$Z_m^{de}(g,\alpha) := \sum_{L\in\mathcal{L}_m^{de}} g^{|L|}\alpha^{s(L)}, \qquad Z^{de}(g,\alpha) = \sum_{m=0}^{\infty} Z_m^{de}(g,\alpha)$$

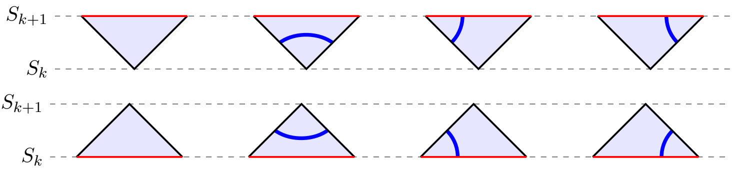
where $s(L)$ is the # of intersections, $\alpha \in [0,1]$ and $Z_0^{de}(g,\alpha) = 1$.

Dense loop model





Dilute loop model

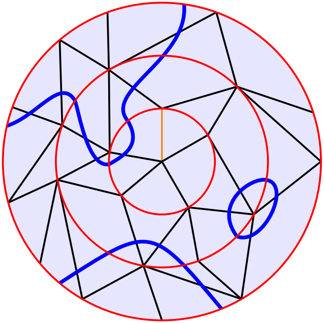


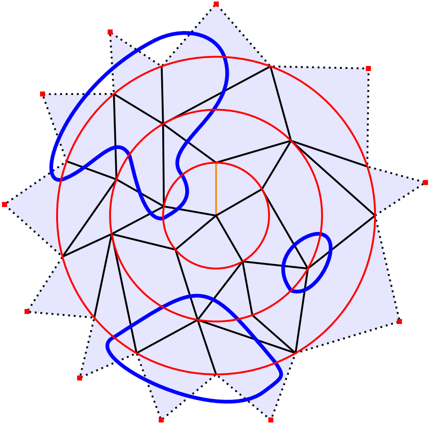
The set $\mathcal{L}_m^{di}$ obtained by replacing elementary triangles with the ones above.

$$Z_m^{di}(g,\alpha) := \sum_{L \in \mathcal{L}_m^{di}} g^{|L|}\alpha^{s(L)}, \qquad Z^{di}(g,\alpha) := \sum_{m=0}^{\infty} Z_m^{di}(g,\alpha)\,.$$

Compatibility condition: $s_k(L) \equiv s_{k'}(L) \pmod 2, \quad \forall k, k'.$

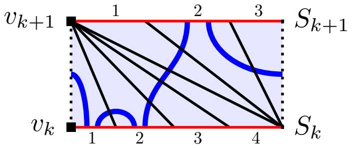
Dilute loop model

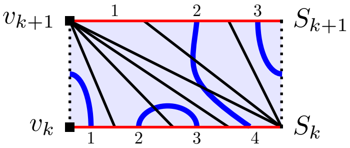


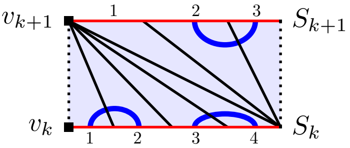


Lemma (3.1, 3.2):

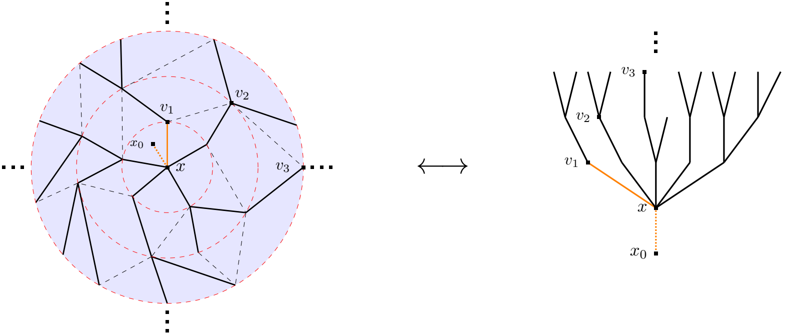
1. The number of possible dense, resp. dilute, loop configurations on a triangulated annulus A_k only depends on the boundary lengths l_k and l_{k+1}, not the details of the triangulation.
2. Given arbitrary (resp. even size) subsets of its spacelike edges, a triangulation of A_k admits one dense (resp. 2 dilute) loop configuration.







Tree correspondences



Bijective correspondence, $\psi \ : \ \mathcal{C}_m(N) \to \mathcal{T}_m(N)$.

$$V_k(T) := \{v \in T \mid d_T(x_0, v) = k+1\} = \{v_{k,i} \mid i = 1, \ldots, |V_k(T)|\}\,.$$

$$V(T) := \bigcup_{k=1}^{m} V_k(T)$$

Loop models on trees

$$\widetilde{\mathcal{T}}_m := \{(T,\delta) \mid T \in \mathcal{T}_m,\ \delta : V(T) \to \{0,1\}\}, \quad |\delta| := \sum_{v \in V(T)} \delta(v)$$

$$\widetilde{\mathcal{T}}_m^{ev}(N) := \left\{(T,\delta) \in \widetilde{\mathcal{T}}_m \,|\, T \in \mathcal{T}_m(N),\, \delta_k \in 2\mathbb{N}_0,\, k = 1,\dots,m\right\}$$

Proposition (3.5, 3.6): For every $m \in \mathbb{N}_0$ and $N \in \mathbb{N}$

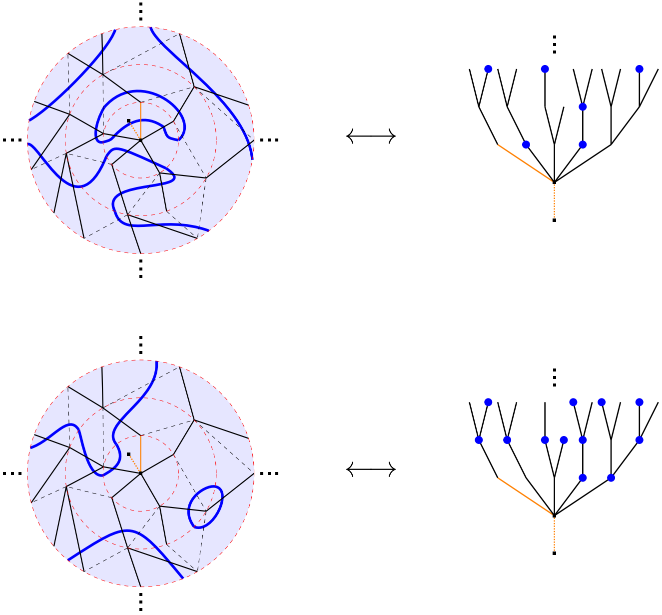
(i) there is a bijective correspondence

$$\tilde{\psi} : \mathcal{L}_m^{de}(N) \to \widetilde{\mathcal{T}}_m(N)$$

(ii) there is a 2^m to 1 correspondence

$$\hat{\psi} : \mathcal{L}_m^{di}(N) \to \widetilde{\mathcal{T}}_m^{ev}(N)$$

such that if $(T,\delta) = \hat{\psi}(L)$ then $T = \psi(C)$.



Partition functions

$$W_m(g,\alpha) := \sum_{(T,\delta)\in\widetilde{\mathcal{T}}_m} g^{|T|-1}\alpha^{|\delta|}, \quad W_m^{ev}(g,\alpha) := \sum_{(T,\delta)\in\widetilde{\mathcal{T}}_m^{ev}} g^{|T|-1}\alpha^{|\delta|}$$

$$\textit{Dense :} \quad Z^{de}(g,\alpha) = W(g,\alpha^2),$$

$$\textit{Dilute :} \quad Z^{di}(g,\alpha) = \sum_{m=0}^{\infty} Z_m^{di}(g,\alpha) = \sum_{m=0}^{\infty} 2^m W_m^{ev}(g,\alpha).$$

The 2^m factor is important!

Critical behavior: labelled trees

Dense case: Critical behavior fully determined via the correspondence.

$$W(g,\alpha) = W(g(1+\alpha)) = \frac{1-\sqrt{1-4g(1+\alpha)}}{2g(1+\alpha)}$$

Dilute case: Not so trivial.

$$W_m^{ev}(g,\alpha) = \sum_{T\in\mathcal{T}_m} g^{|T|-1} \prod_{i=1}^{m} \tfrac{1}{2}\left[(1+\alpha)^{n_i} + (1-\alpha)^{n_i}\right].$$

The best we get from here is an inequality:

$$W^{ev}\big(g(1+\alpha), 0, \tfrac{k}{2}\big) \le W^{ev}(g,\alpha,k) \le W^{ev}\big(g(1+\alpha), 0, \tfrac{k}{1+\alpha}\big).$$

Transfer matrix formalism - Pure CDT

- Encode the degrees of freedom along the space-like boundaries by vectors in $l_2(\mathbb{N})$,

- Transfer matrix T, an operator on $l_2(\mathbb{N})$:

$$\mathsf{T}_{r,s}(g) := \binom{r+s-1}{r} g^{\frac{r+s}{2}} .$$

- $$Z_m(g) = \left\langle v(g) \middle| \big(\mathsf{T}(g)\big)^{m-1} \middle| v(g) \right\rangle , \; v_n(g) := g^{\frac{n}{2}} \in l_2(\mathbb{N}) .$$

- $\mathsf{T}(g)$ is not symmetric but admits a factorization $\mathsf{T}(g) = \mathsf{D}\mathsf{K}(g)$ where

$$\mathsf{D}_{r,s} := \frac{\delta_{r,s}}{r}, \qquad \mathsf{K}_{r,s}(g) := \frac{(r+s-1)!}{(r-1)!(s-1)!} g^{\frac{r+s}{2}} .$$

Transfer matrix

Proposition (5.1): The operator $\mathsf{K}(g)$ is trace-class for $g \in \mathbb{D}$, it is positive definite for $g \in (0, \frac{1}{4})$, and the function $h \mapsto \mathsf{K}(h^2)$ is analytic on $\{h \in \mathbb{C} \,|\, |h| < \frac{1}{2}\}$.

$$\|\mathsf{K}(g)\|_1 = \mathrm{tr}\,|\mathsf{K}(g)| = \|\mathsf{K}(|g|)\|_1 = \frac{|g|}{(1-4|g|)^{\frac{3}{2}}}$$

$$\mathrm{tr}\big(\mathsf{T}(g)\big) = \sum_{s=1}^{\infty} \mathsf{T}_{s,s}(g) = \frac{1-\sqrt{1-4g}}{2\sqrt{1-4g}}, \quad g \in (0, \frac{1}{4})\,.$$

Proposition (5.3): The largest eigenvalue $\lambda_1(g)$ of $\mathsf{D}^{\frac{1}{2}}\mathsf{K}(g)\mathsf{D}^{\frac{1}{2}}$ satisfies

$$\lambda_1(g) \nearrow 1 \qquad \text{as} \quad g \nearrow \tfrac{1}{4}.$$

Transfer matrix: loops

- **Dense:** Just scale $g \to g(1+\alpha^2)$, the rest works out the same.
- **Dilute:** $\mathsf{T}^{di}(g,\alpha) = 2\mathsf{D}\mathsf{K}^{di}(g,\alpha)$ where

$$\mathsf{K}^{di}_{r,s}(g,\alpha) = \frac{1}{2}\frac{(r+s-1)!}{(r-1)!(s-1)!}\left[(1+\alpha)^r + (1-\alpha)^r\right]^{\frac{1}{2}}\left[(1+\alpha)^s + (1-\alpha)^s\right]^{\frac{1}{2}} g^{\frac{r+s}{2}}.$$

- Similar to the pure case:

$$Z^{di}(g,\alpha) = 1 + \sum_{m=1}^{\infty} \left\langle v(g,\alpha)\middle|\left(\mathsf{T}^{di}(g,\alpha)\right)^{m-1}\middle|v(g,\alpha)\right\rangle.,$$

$$v_n(g,\alpha) := \left[(1+\alpha)^n + (1-\alpha)^n\right]^{\frac{1}{2}} g^{\frac{n}{2}}, \qquad n \in \mathbb{N}.$$

Transfer matrix: dilute

- $\mathsf{K}^{di}(g,\alpha)$ and $\mathsf{D}^{\frac{1}{2}}\mathsf{K}^{di}(g,\alpha)\mathsf{D}^{\frac{1}{2}}$ are positive definite trace-class operators on $l^2(\mathbb{N})$ for $g \in (0, \frac{1}{4(1+\alpha)})$, $\alpha \in [0,1]$, and the Perron-Frobenius...

- $\mathsf{D}^{\frac{1}{2}}\mathsf{K}^{di}(g,\alpha)\mathsf{D}^{\frac{1}{2}}$ is analytic in $(\sqrt{g},\alpha)$ for $|\alpha| < 1$ and $|g| < \frac{1}{4(1+|\alpha|)}$.

Proposition (5.4) For every $\alpha \in [0,1]$, the largest eigenvalue $\lambda_1^{di}(g,\alpha)$ of $\mathsf{D}^{\frac{1}{2}}\mathsf{K}^{di}(g,\alpha)\mathsf{D}^{\frac{1}{2}}$ is a strictly increasing function of g. As g approaches $\frac{1}{4(1+\alpha)}$ from below, its limit $\bar{\lambda}_1^{di}(\alpha)$ satisfies

$$\bar{\lambda}_1^{di}(\alpha) \leq 1.$$

Final result

Observe:

$\mathsf{K}^{di}(g,0) = \mathsf{K}(g), \;\; \mathsf{K}^{di}(g,1) = \mathsf{K}(2g)/2 \quad \Rightarrow \quad \bar{\lambda}_1^{di}(0) = 1, \;\; \bar{\lambda}_1^{di}(1) = \frac{1}{2}.$

Theorem [Durhuus, Poncini, Rasmussen, U.]: For α real and sufficiently small, the critical coupling $g_c^{di}(\alpha)$ for $Z^{di}(g,\alpha)$ is determined by the equation

$$\lambda_1^{di}\big(g_c^{di}(\alpha), \alpha\big) = \tfrac{1}{2},$$

and there exist $C_1(\alpha), C_2(\alpha) > 0$ such that

$$\frac{C_1(\alpha)}{g_c^{di}(\alpha) - g} \leq Z^{di}(g,\alpha) \leq \frac{C_2(\alpha)}{g_c^{di}(\alpha) - g}$$

for g close to $g_c^{di}(\alpha)$.

Conclusions / Questions

- For small α, we have height coupled trees. Exponent characterising the singularity changes form $\frac{1}{2}$ to -1.

- Strong evidence that the background geometry is affected.

- Is there a phase transition for some value of α or is the height coupling effective all the way to 1?

- Correspondence with Ising clusters: new information?

  Encoding the length of the loops?

Thank you for your attention!

Transfer matrix formalism - Pure CDT

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$$\mathsf{T}_{r,s}(g) := \binom{r+s-1}{r} g^{\frac{r+s}{2}} .$$

- $\mathsf{T}(g)$ is not symmetric but admits a factorization $\mathsf{T}(g) = \mathsf{D}\mathsf{K}(g)$ where

$$\mathsf{D}_{r,s} := \frac{\delta_{r,s}}{r}, \qquad \mathsf{K}_{r,s}(g) := \frac{(r+s-1)!}{(r-1)!(s-1)!} g^{\frac{r+s}{2}} .$$

Proposition (5.1): The operator $\mathsf{K}(g)$ is trace-class for $g \in \mathbb{D}$, it is positive definite for $g \in (0, \frac{1}{4})$, and the function $h \mapsto \mathsf{K}(h^2)$ is analytic on $\{h \in \mathbb{C} \,|\, |h| < \frac{1}{2}\}$.

$$\|\mathsf{K}(g)\|_1 = \operatorname{tr}|\mathsf{K}(g)| = \|\mathsf{K}(|g|)\|_1 = \frac{|g|}{(1-4|g|)^{\frac{3}{2}}}$$

$$\operatorname{tr}\big(\mathsf{T}(g)\big) = \sum_{s=1}^{\infty} \mathsf{T}_{s,s}(g) = \frac{1-\sqrt{1-4g}}{2\sqrt{1-4g}}, \quad g \in (0, \frac{1}{4}) .$$

- Perron-Frobenius: For $g \in (0, \frac{1}{4})$, $\mathsf{K}(g)$ has a simple positive largest eigenvalue which equals $\|\mathsf{K}(g)\|$, corresponding normalized eigenvector has positive entries.
- Kato-Rellich: This eigenvalue is analytic on $(0, \frac{1}{4})$ as well as the components of the corresponding eigenvector (up to a phase).
- Same hold for $\mathsf{D}^{\frac{1}{2}}\mathsf{K}(g)\mathsf{D}^{\frac{1}{2}}$.
- Now, express the partition function as a matrix element:

$$Z_m(g) = \big\langle v(g) \big| \mathsf{T}^{m-1}(g) \big| v(g) \big\rangle, \quad v_n(g) := g^{\frac{n}{2}}, \qquad m, n \in \mathbb{N}\,.$$

- Define $Z_m^{per}(g) := \mathrm{tr}\big(\mathsf{T}^{m-1}(g)\big) = \mathrm{tr}\Big((\mathsf{D}^{\frac{1}{2}}\mathsf{K}(g)\mathsf{D}^{\frac{1}{2}})^{m-1}\Big),\ m \geq 2.$
- A simple observation: $Z_m(g) \leq Z_{m+1}^{per}(g)$ for $m \geq 1$ and $g \in [0, \frac{1}{4})$.

Proposition (5.3): The largest eigenvalue $\lambda_1(g)$ of $\mathsf{D}^{\frac{1}{2}}\mathsf{K}(g)\mathsf{D}^{\frac{1}{2}}$ satisfies

$$\lambda_1(g) \nearrow 1 \qquad \text{as} \quad g \nearrow \frac{1}{4}.$$

Transfer matrix - dilute model

- Observe! $\mathsf{T}^{di}(g,\alpha) = 2\mathsf{D}\mathsf{K}^{di}(g,\alpha)$ where

$$\mathsf{K}^{di}_{r,s}(g,\alpha) = \frac{1}{2}\frac{(r+s-1)!}{(r-1)!(s-1)!}\left[(1+\alpha)^r + (1-\alpha)^r\right]^{\frac{1}{2}}\left[(1+\alpha)^s + (1-\alpha)^s\right]^{\frac{1}{2}} g^{\frac{r+s}{2}}.$$

- Similar to the pure case:

$$Z^{di}(g,\alpha) = 1 + \sum_{m=1}^{\infty} \langle v(g,\alpha) | \left(\mathsf{T}^{di}(g,\alpha)\right)^{m-1} | v(g,\alpha)\rangle.,$$

$$v_n(g,\alpha) := \left[(1+\alpha)^n + (1-\alpha)^n\right]^{\frac{1}{2}} g^{\frac{n}{2}}, \qquad n \in \mathbb{N}.$$

- $\mathsf{K}^{di}(g,\alpha)$ and $\mathsf{D}^{\frac{1}{2}}\mathsf{K}^{di}(g,\alpha)\mathsf{D}^{\frac{1}{2}}$ are positive definite trace-class operators on $l^2(\mathbb{N})$ for $g \in (0, \frac{1}{4(1+\alpha)})$, $\alpha \in [0,1]$, and the Perron-Frobenius...
- $\mathsf{D}^{\frac{1}{2}}\mathsf{K}^{di}(g,\alpha)\mathsf{D}^{\frac{1}{2}}$ is analytic in $(\sqrt{g},\alpha)$ for $|\alpha| < 1$ and $|g| < \frac{1}{4(1+|\alpha|)}$. **Proposition (5.4)** For every $\alpha \in [0,1]$, the largest eigenvalue $\lambda_1^{di}(g,\alpha)$ of $\mathsf{D}^{\frac{1}{2}}\mathsf{K}^{di}(g,\alpha)\mathsf{D}^{\frac{1}{2}}$ is a strictly increasing function of g. As g approaches $\frac{1}{4(1+\alpha)}$ from below, its limit $\bar{\lambda}_1^{di}(\alpha)$ satisfies

$$\bar{\lambda}_1^{di}(\alpha) \leq 1.$$