Resonances for Shear Flows and Complex Deformations

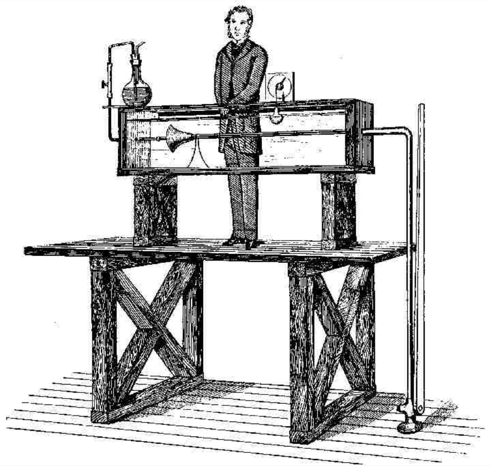
Jian Wang

Institut des Hautes Études Scientifiques

Joint work with Malo Jézéquel (CNRS)

Reynolds experiment

Reynolds experiment, 1883



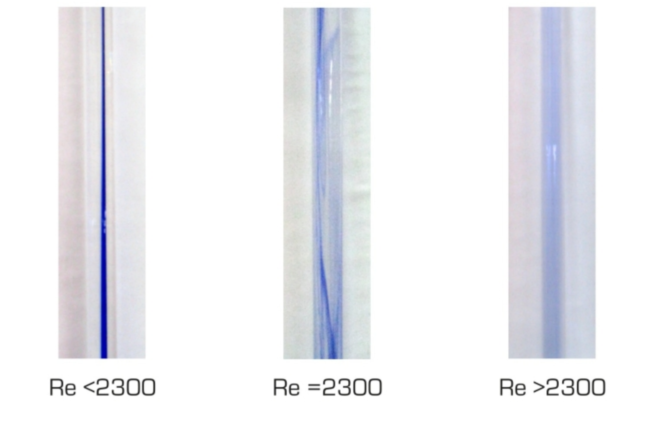
Reynolds experiment

Reynolds experiment

https://youtu.be/BBiR6FWmyv4

Reynolds Experiment

laminar flow | transition | turbulent flow



https://www.gunt.de/en/

Orr–Sommerfeld equation

Shear flows: stationary solutions $(0, U(x))$ to the Navier–Stokes equation (with forcing) in $[a, b] \times \mathbb{R}$.

Orr–Sommerfeld equation

Shear flows: stationary solutions $(0, U(x))$ to the Navier–Stokes equation (with forcing) in $[a, b] \times \mathbb{R}$.

Linearizing Navier–Stokes equation around a shear flow, following Kelvin, Reynolds, Rayleigh, Orr, Sommerfeld, Heisenberg, C. C. Lin, ..., one obtains the Orr–Sommerfeld equation on $[a, b]$

$$\left(\frac{i}{\alpha R}(\partial_x^2 - \alpha^2)^2 + (U(x) - c)(\partial_x^2 - \alpha^2) - U''(x)\right)\psi = 0,$$

$$\psi(a) = \psi(b) = \psi'(a) = \psi'(b) = 0.$$

Orr–Sommerfeld equation

Shear flows: stationary solutions $(0, U(x))$ to the Navier–Stokes equation (with forcing) in $[a, b] \times \mathbb{R}$.

Linearizing Navier–Stokes equation around a shear flow, following Kelvin, Reynolds, Rayleigh, Orr, Sommerfeld, Heisenberg, C. C. Lin, ..., one obtains the Orr–Sommerfeld equation on $[a, b]$

$$\left(\frac{i}{\alpha R}(\partial_x^2 - \alpha^2)^2 + (U(x) - c)(\partial_x^2 - \alpha^2) - U''(x)\right)\psi = 0,$$

$$\psi(a) = \psi(b) = \psi'(a) = \psi'(b) = 0.$$

- $R > 0$ — Reynolds number; $1/R$ — viscosity;
- $\alpha > 0$ — frequency in y direction;
- $c \in \mathbb{C}$ — spectral parameter

Orr–Sommerfeld equation

Shear flows: stationary solutions $(0, U(x))$ to the Navier–Stokes equation (with forcing) in $[a, b] \times \mathbb{R}$.

Linearizing Navier–Stokes equation around a shear flow, following Kelvin, Reynolds, Rayleigh, Orr, Sommerfeld, Heisenberg, C. C. Lin, ..., one obtains the Orr–Sommerfeld equation on $[a, b]$

$$\left(\frac{i}{\alpha R}(\partial_x^2 - \alpha^2)^2 + (U(x) - c)(\partial_x^2 - \alpha^2) - U''(x) \right) \psi = 0,$$
$$\psi(a) = \psi(b) = \psi'(a) = \psi'(b) = 0.$$

- $R > 0$ — Reynolds number; $1/R$ — viscosity;
- $\alpha > 0$ — frequency in y direction;
- $c \in \mathbb{C}$ — spectral parameter

$\mathrm{Im} c > 0 \Rightarrow$ *unstable solutions to the linearized Navier–Stokes equation*

Spectral instability

Spectral instability can occur as Reynolds number grows to $+\infty$.

Spectral instability

Spectral instability can occur as Reynolds number grows to $+\infty$.

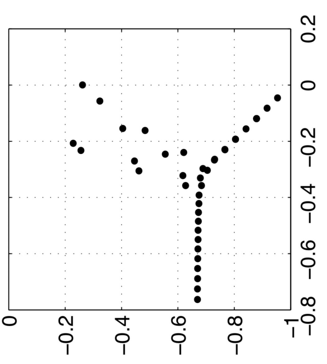
- Couette flow $U(x) = x$, $x \in [-1, 1]$: spectrally stable

Spectral instability

Spectral instability can occur as Reynolds number grows to $+\infty$.

- Couette flow $U(x) = x$, $x \in [-1, 1]$: spectrally stable
- Poiseuille flow $U(x) = 1 - x^2$, $x \in [-1, 1]$: spectrally unstable.

  Orszag '71, Trefethen '00, Grenier–Guo–Nguyen '16, Almog–Helffer '22



Poiseuille flow, $R = 5772$

Inviscid limits

$R < +\infty \Rightarrow$ the Orr–Sommerfeld equation is *elliptic*

$\Rightarrow$ discrete eigenvalues Σ_R

Inviscid limits

$R < +\infty \Rightarrow$ the Orr–Sommerfeld equation is *elliptic*
$\Rightarrow$ discrete eigenvalues Σ_R

Questions (Inviscid limits):

- What is the limit of Σ_R as $R \to +\infty$?
- Can Σ_R contain unstable eigenvalues?

Inviscid limits

$R < +\infty \Rightarrow$ the Orr–Sommerfeld equation is *elliptic*
$\Rightarrow$ discrete eigenvalues Σ_R

Questions (Inviscid limits):

- What is the limit of Σ_R as $R \to +\infty$?
- Can Σ_R contain unstable eigenvalues?

This is a question of long history:

- Kelvin, Reynolds, Rayleigh, Orr, Sommerfeld, Heisenberg, C. C. Lin, ...
- Tatsumi–Gotoh–Ayukawa '64: $U(x) = \tanh(x)$, $x \in \mathbb{R}$ — $\lim_{R\to+\infty} \Sigma_R$ is NOT the set of eigenvalues for $R = +\infty$.
- Grenier–Guo–Nguyen '16: characteristic boundary layers, symmetric analytic shear flows — Σ_R must have unstable eigenvalues for R large and $\alpha = \alpha(R)$ small

Inviscid limits

Question (Inviscid limits): *What is the limit of Σ_R as $R \to +\infty$?*

- $R < +\infty$, Orr–Sommerfeld equation

$$\left(\frac{i}{\alpha R}(\partial_x^2 - \alpha^2)^2 + (U(x) - c)(\partial_x^2 - \alpha^2) - U''(x) \right) \psi = 0,$$

$$\psi(a) = \psi(b) = \psi'(a) = \psi'(b) = 0.$$

Inviscid limits

Question (Inviscid limits): *What is the limit of Σ_R as $R \to +\infty$?*

- $R < +\infty$, Orr–Sommerfeld equation

$$\left(\frac{i}{\alpha R}(\partial_x^2 - \alpha^2)^2 + (U(x) - c)(\partial_x^2 - \alpha^2) - U''(x)\right)\psi = 0,$$
$$\psi(a) = \psi(b) = \psi'(a) = \psi'(b) = 0.$$

- $R = +\infty$, Rayleigh equation

$$\big((U(x) - c)(\partial_x^2 - \alpha^2) - U''(x)\big)\psi = 0,$$

Inviscid limits

Question (Inviscid limits): *What is the limit of Σ_R as $R \to +\infty$?*

- $R < +\infty$, Orr–Sommerfeld equation

$$\left(\frac{i}{\alpha R}(\partial_x^2 - \alpha^2)^2 + (U(x) - c)(\partial_x^2 - \alpha^2) - U''(x)\right)\psi = 0,$$
$$\psi(a) = \psi(b) = \psi'(a) = \psi'(b) = 0.$$

- $R = +\infty$, Rayleigh equation

$$\big((U(x) - c)(\partial_x^2 - \alpha^2) - U''(x)\big)\psi = 0,$$
$$\psi(a) = \psi(b) = 0.$$

Inviscid limits

Question (Inviscid limits): *What is the limit of Σ_R as $R \to +\infty$?*

- $R < +\infty$, Orr–Sommerfeld equation

$$\left(\frac{i}{\alpha R}(\partial_x^2 - \alpha^2)^2 + (U(x) - c)(\partial_x^2 - \alpha^2) - U''(x)\right)\psi = 0,$$
$$\psi(a) = \psi(b) = \psi'(a) = \psi'(b) = 0.$$

- $R = +\infty$, Rayleigh equation

$$\big((U(x) - c)(\partial_x^2 - \alpha^2) - U''(x)\big)\psi = 0,$$
$$\psi(a) = \psi(b) = 0.$$

Challenges:

- *Degeneracy at $U^{-1}(c)$*

Inviscid limits

Question (Inviscid limits): *What is the limit of Σ_R as $R \to +\infty$?*

- $R < +\infty$, Orr–Sommerfeld equation

$$\left(\frac{i}{\alpha R}(\partial_x^2 - \alpha^2)^2 + (U(x) - c)(\partial_x^2 - \alpha^2) - U''(x)\right)\psi = 0,$$
$$\psi(a) = \psi(b) = \psi'(a) = \psi'(b) = 0.$$

- $R = +\infty$, Rayleigh equation

$$\big((U(x) - c)(\partial_x^2 - \alpha^2) - U''(x)\big)\psi = 0,$$
$$\psi(a) = \psi(b) = 0.$$

Challenges:

- *Degeneracy at $U^{-1}(c)$*
- *Neumann boundary conditions disappear*

Complex deformation

Assume U analytic, $c_0 \in U((a,b))$, $U'|_{U^{-1}(c_0)} \neq 0$, $c_0 \neq U(a), U(b)$.

Complex deformation

Assume U analytic, $c_0 \in U((a,b))$, $U'|_{U^{-1}(c_0)} \neq 0$, $c_0 \neq U(a), U(b)$.

Idea: Deform $[a,b]$ on the complex plane to avoid $U^{-1}(c_0)$.

Complete deformation

Assume U analytic, $c_0 \in U((a,b))$, $U'|_{U^{-1}(c_0)} \neq 0$, $c_0 \neq U(a), U(b)$.

Idea: Deform $[a,b]$ on the complex plane to avoid $U^{-1}(c_0)$.

Criteria for a "good" complex deformation

- Fix the boundary points a & b — to keep the boundary conditions

Complex deformation

Assume U analytic, $c_0 \in U((a,b))$, $U'|_{U^{-1}(c_0)} \neq 0$, $c_0 \neq U(a), U(b)$.

Idea: Deform $[a,b]$ on the complex plane to avoid $U^{-1}(c_0)$.

Criteria for a "good" complex deformation

- Fix the boundary points a & b — to keep the boundary conditions
- $U - c_0$ stays nonzero on the deformed segment — to gain ellipticity

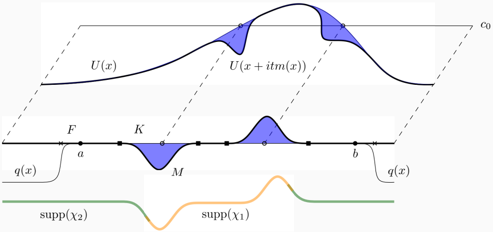
Complex deformation

Assume U analytic, $c_0 \in U((a,b))$, $U'|_{U^{-1}(c_0)} \neq 0$, $c_0 \neq U(a), U(b)$.

Idea: Deform $[a,b]$ on the complex plane to avoid $U^{-1}(c_0)$.

Criteria for a "good" complex deformation

- Fix the boundary points a & b — to keep the boundary conditions
- $U - c_0$ stays nonzero on the deformed segment — to gain ellipticity
- $\mathrm{Im}(U - c_0) \leq 0$ on the deformed segment — same sign as ∂_x^2



Complex deformation

Resonances $c \in \mathcal{R}$ for the shear flow U near c_0

- $c \in (c_0 - \delta, c_0 + \delta) + i(-\delta, +\infty)$

Complex deformation

Resonances $c \in \mathcal{R}$ for the shear flow U near c_0

- $c \in (c_0 - \delta, c_0 + \delta) + i(-\delta, +\infty)$
- $(U - c)(\partial_z^2 - \alpha^2) - U'' : H^2_{\mathrm{D}}(M_{a,b}) \to L^2(M_{a,b})$ is not invertible

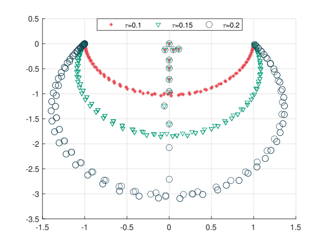
Complex deformation

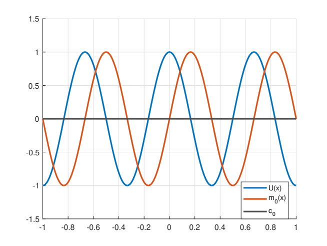
Resonances $c \in \mathcal{R}$ for the shear flow U near c_0

- $c \in (c_0 - \delta, c_0 + \delta) + i(-\delta, +\infty)$
- $(U - c)(\partial_z^2 - \alpha^2) - U'' : H^2_{\mathrm{D}}(M_{a,b}) \to L^2(M_{a,b})$ is not invertible

Example: $U(x) = \cos(3\pi x)$, $x \in [-1, 1]$, $\alpha = \frac{\sqrt{35}\pi}{2}$.

It has only one embedded eigenvalue $c = 0$, but many resonances:





Complex deformation

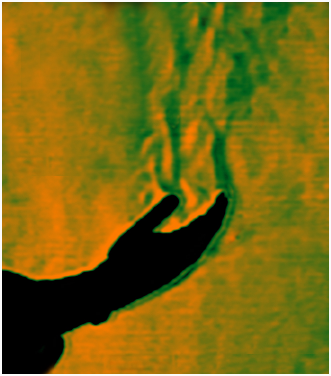
Method of complex deformations

- Scattering resonances: Aguilar–Combes '71, Balslev–Combes '71, Simon '79, Sjöstrand–Zworski '91
- 0th order operators (models of internal waves): Galkowski–Zworski '22
- Anosov flows: Guedes-Bonthonneau–Jézéquel '20
- Metric scattering: Guedes-Bonthonneau–Guillarmou–Jézéquel '24

Complex deformation for Rayleigh/Orr–Sommerfeld

- Rosencrans–Sattinger '66, Stepin '96: analytic continuation of Wronskian determinant
- Tatsumi–Gotoh–Ayukawa '64: $U(x) = \tanh(x)$, $x \in \mathbb{R}$

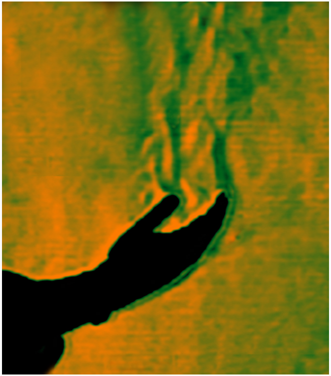
Boundary layers: Vishik–Lyusternik



(https://en.wikipedia.org/wiki/Boundary_layer)

Thin layer of fluid near the boundary, due to the boundary conditions

Boundary layers: Vishik–Lyusternik



(https://en.wikipedia.org/wiki/Boundary_layer)

Thin layer of fluid near the boundary, due to the boundary conditions

- Introduced by Prandtl '04
- Method of Vishik–Lyusternik '62,
- Method of Rayleigh–Airy operators Grenier–Guo–Nguyen '16

Boundary layers: Vishik–Lyusternik

To see how the Neumann boundary conditions disappear, construct approximation solutions $u^{\#}_{c,R}$, $v^{\#}_{c,R}$ concentrating near the boundary

- (Model) Roughly, $u^{\#}_{c,R}$ is like $x \mapsto e^{-\sqrt{R}x}$ for $x \to 0+$

Boundary layers: Vishik–Lyusternik

To see how the Neumann boundary conditions disappear, construct approximation solutions $u^{\#}_{c,R}$, $v^{\#}_{c,R}$ concentrating near the boundary

- (Model) Roughly, $u^{\#}_{c,R}$ is like $x \mapsto e^{-\sqrt{R}x}$ for $x \to 0+$
- (Approximation solutions) $\left(\frac{i}{\alpha R}(\partial_x^2 - \alpha^2)^2 + (U(x) - c)(\partial_x^2 - \alpha^2) - U''(x)\right) u^{\#}_{c,R} = \mathcal{O}(R^{-\infty})$

Boundary layers: Vishik–Lyusternik

To see how the Neumann boundary conditions disappear, construct approximation solutions $u^{\#}_{c,R}$, $v^{\#}_{c,R}$ concentrating near the boundary

- (Model) Roughly, $u^{\#}_{c,R}$ is like $x \mapsto e^{-\sqrt{R}x}$ for $x \to 0+$
- (Approximation solutions)
  $\left(\frac{i}{\alpha R}(\partial_x^2 - \alpha^2)^2 + (U(x) - c)(\partial_x^2 - \alpha^2) - U''(x)\right) u^{\#}_{c,R} = \mathcal{O}(R^{-\infty})$
- (Boundary values)
  $u^{\#}_{c,R}(a) = 1$, $(u^{\#}_{c,R})'(a) = \sqrt{i\alpha(U(a) - c)}R^{\frac{1}{2}} + \mathcal{O}(1)$, $\mathrm{Re}(u^{\#}_{c,R})'(a) < 0$.

Boundary layers: Vishik–Lyusternik

To see how the Neumann boundary conditions disappear, construct approximation solutions $u^{\#}_{c,R}$, $v^{\#}_{c,R}$ concentrating near the boundary

- (Model) Roughly, $u^{\#}_{c,R}$ is like $x \mapsto e^{-\sqrt{R}x}$ for $x \to 0+$
- (Approximation solutions) $\left(\frac{i}{\alpha R}(\partial_x^2 - \alpha^2)^2 + (U(x) - c)(\partial_x^2 - \alpha^2) - U''(x)\right) u^{\#}_{c,R} = \mathcal{O}(R^{-\infty})$
- (Boundary values) $u^{\#}_{c,R}(a) = 1$, $(u^{\#}_{c,R})'(a) = \sqrt{i\alpha(U(a) - c)}R^{\frac{1}{2}} + \mathcal{O}(1)$, $\mathrm{Re}(u^{\#}_{c,R})'(a) < 0$.
- (Smallness) $\|u^{\#}_{c,R}\|_{H^k} \leq R^{-\frac{1-2k}{4}}$, $k \geq 0$.

Boundary layers: Vishik–Lyusternik

To see how the Neumann boundary conditions disappear, construct approximation solutions $u^{\#}_{c,R}$, $v^{\#}_{c,R}$ concentrating near the boundary

- (Model) Roughly, $u^{\#}_{c,R}$ is like $x \mapsto e^{-\sqrt{R}x}$ for $x \to 0+$
- (Approximation solutions)
  $\left(\frac{i}{\alpha R}(\partial_x^2 - \alpha^2)^2 + (U(x) - c)(\partial_x^2 - \alpha^2) - U''(x)\right) u^{\#}_{c,R} = \mathcal{O}(R^{-\infty})$
- (Boundary values)
  $u^{\#}_{c,R}(a) = 1$, $(u^{\#}_{c,R})'(a) = \sqrt{i\alpha(U(a) - c)}R^{\frac{1}{2}} + \mathcal{O}(1)$, $\mathrm{Re}(u^{\#}_{c,R})'(a) < 0$.
- (Smallness) $\|u^{\#}_{c,R}\|_{H^k} \leq R^{-\frac{1-2k}{4}}$, $k \geq 0$.

Construction of these boundary layers are achieved by the WKB method.

Resonances as inviscid limits

Theorem (Jézéquel–W. '25). *In* $(c_0 - \delta, c_0 + \delta) + i(-\delta, +\infty)$,

- *the set of resonances* $\mathcal{R}$ *is discrete;*

Resonances as inviscid limits

Theorem (Jézéquel–W. '25). *In* $(c_0 - \delta, c_0 + \delta) + i(-\delta, +\infty)$,

- *the set of resonances* $\mathcal{R}$ *is discrete;*
- $c \in \mathcal{R}$, $\mathrm{Im} c > 0 \Rightarrow c$ *is an eigenvalue of Rayleigh equation;*

Resonances as inviscid limits

Theorem (Jézéquel–W. '25). *In* $(c_0 - \delta, c_0 + \delta) + i(-\delta, +\infty)$,

- *the set of resonances* $\mathcal{R}$ *is discrete;*
- $c \in \mathcal{R}$, $\mathrm{Im} c > 0 \Rightarrow c$ *is an eigenvalue of Rayleigh equation;*
- $c \in \mathcal{R}$, $\mathrm{Im} c = 0 \Rightarrow c$ *is a "generalized embedded eigenvalue"*

  (Wei–Zhang–Zhao '19)

Resonances as inviscid limits

Theorem (Jézéquel–W. '25). *In* $(c_0 - \delta, c_0 + \delta) + i(-\delta, +\infty)$,

- *the set of resonances* $\mathcal{R}$ *is discrete;*
- $c \in \mathcal{R}$, $\mathrm{Im} c > 0 \Rightarrow c$ *is an eigenvalue of Rayleigh equation;*
- $c \in \mathcal{R}$, $\mathrm{Im} c = 0 \Rightarrow c$ *is a "generalized embedded eigenvalue"* *(Wei–Zhang–Zhao '19)*
- Σ_R *converges to* $\mathcal{R}$ *as* $R \to +\infty$*;*

Resonances as inviscid limits

Theorem (Jézéquel–W. '25). *In* $(c_0 - \delta, c_0 + \delta) + i(-\delta, +\infty)$,

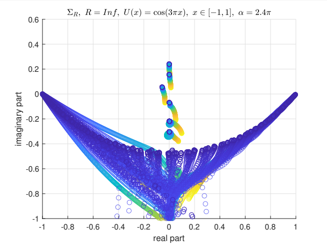
- *the set of resonances* $\mathcal{R}$ *is discrete;*
- $c \in \mathcal{R}$, $\mathrm{Im} c > 0 \Rightarrow c$ *is an eigenvalue of Rayleigh equation;*
- $c \in \mathcal{R}$, $\mathrm{Im} c = 0 \Rightarrow c$ *is a "generalized embedded eigenvalue"* *(Wei–Zhang–Zhao '19)*
- Σ_R *converges to* $\mathcal{R}$ *as* $R \to +\infty$*;*
- *for "simple" resonances, the rate of convergence is* $R^{-\frac{1}{2}}$.

Resonances as inviscid limits

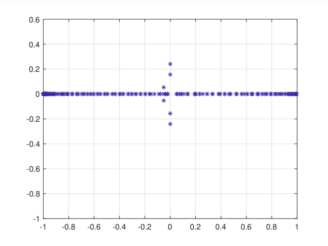
Theorem (Jézéquel–W. '25). *In* $(c_0 - \delta, c_0 + \delta) + i(-\delta, +\infty)$,

- *the set of resonances* $\mathcal{R}$ *is discrete;*
- $c \in \mathcal{R}$, $\mathrm{Im} c > 0 \Rightarrow c$ *is an eigenvalue of Rayleigh equation;*
- $c \in \mathcal{R}$, $\mathrm{Im} c = 0 \Rightarrow c$ *is a "generalized embedded eigenvalue"*

  (Wei–Zhang–Zhao '19)
- Σ_R *converges to* $\mathcal{R}$ *as* $R \to +\infty$*;*
- *for "simple" resonances, the rate of convergence is* $R^{-\frac{1}{2}}$.



Σ_R as $R \to +\infty$



spectrum of Rayleigh equation

Resonances as inviscid limits

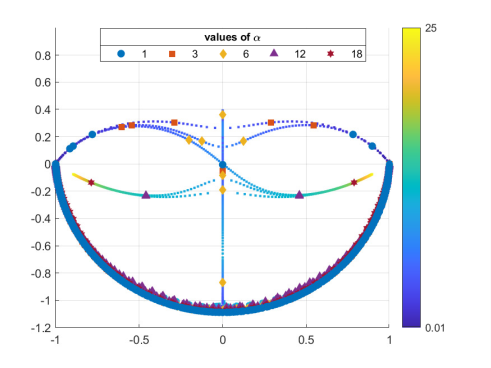
Resonances as inviscid limits

- Pollicott–Ruelle resonances:
  Dyatlov–Zworski '15 Anosov flows;
  Drouot '17 geodesic flows;
  Dang–Rivière '18 gradient flows for Morse–Smale functions
- Scattering resonances for Schrödinger:
  Zworski '15, '18: $V \in L^\infty_{\mathrm{comp}}$
  Kameoka–Nakamura '20: Wigner–von Neumann '20
  Xiong '20, '21, '22: black box etc.
- 0th order operators (models of internal waves):
  Galkowski–Zworski '22

Gallery: Resonances

$$U(x) = \cos(3\pi x),\ x \in [-1, 1]$$

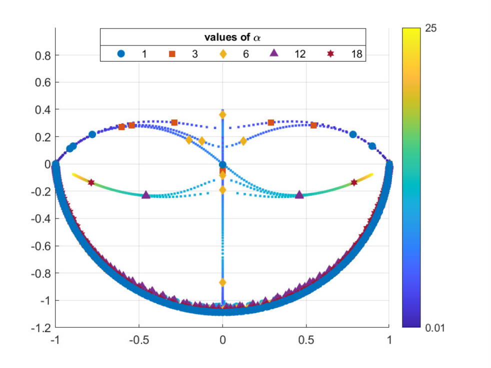
Plot of resonances, color $= \alpha$



Gallery: Resonances

$$U(x) = \cos(3\pi x),\ x \in [-1, 1]$$

Plot of resonances, color $= \alpha$

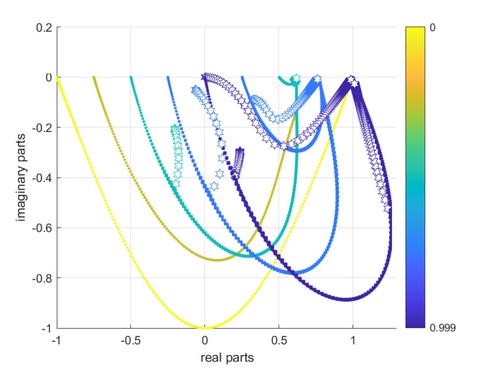


Proposition. *No resonances near c_0 for large α.*

Gallery: Resonances

Couette–Poiseuille $U(x) = (1-\theta)x + \theta(1-x^2)$, $x \in [-1, 1]$

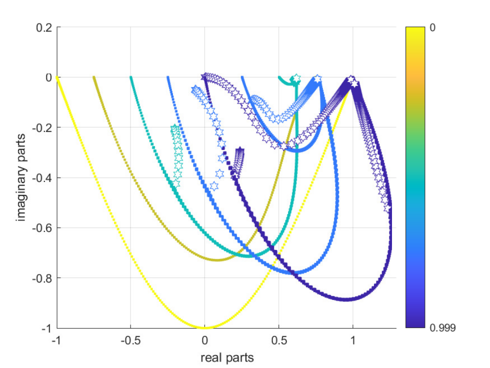
Plot of resonances, color $= \theta$



Gallery: Resonances

Couette–Poiseuille $U(x) = (1-\theta)x + \theta(1-x^2)$, $x \in [-1, 1]$

Plot of resonances, color $= \theta$

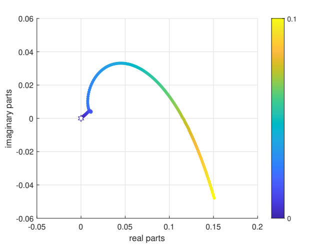


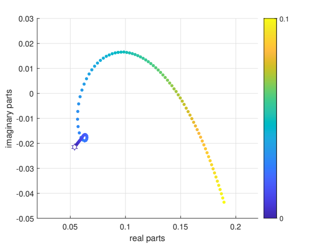
Proposition. *Couette flow has no resonances.*

Gallery: Viscous perturbations

$$U(x) = \cos(0.7\pi x),\ x \in [-1, 1]$$

Plot of Σ_R, color $= R^{-\frac{1}{2}}$





$$\alpha = \sqrt{0.7^2 - 0.5^2}\pi$$

unstable embedded eigenvalue 0

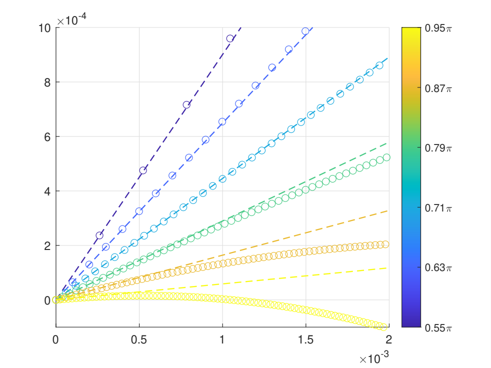
$$\alpha = \sqrt{0.7^2 - 0.45^2}\pi$$

unstable complex resonance

Gallery: Viscous perturbations

$$U(x) = \cos(\omega x),\ x \in [-1,1],\ \omega \in (0.5\pi, \pi),\ \alpha = \sqrt{\omega^2 - 0.25\pi^2}$$

Plot of Σ_R, color $= \omega$



Proposition. *First order approximation* $c(R) = \dot{c}(0)R^{-\frac{1}{2}} + \mathcal{O}(R^{-1})$

$$\dot{c}(0) = \frac{\pi^2 e^{\frac{\pi i}{4}}}{2\omega^2\sqrt{\alpha|\cos(\omega)|}}\left(\mathrm{p.v.}\int_{-1}^{1}\frac{(\cos(\frac{\pi x}{2}))^2}{\cos(\omega x)}\mathrm{d}x + \frac{2\pi i}{\omega}\left(\cos\left(\frac{\pi^2}{4\omega}\right)\right)^2\right)^{-1},\quad \mathrm{Im}\dot{c}(0) > 0.$$

Perspectives

- Inviscid limits of eigenvalues near critical/boundary values of U?
- Applications to evolution problems of shear flows?
- Higher dimensional models in fluid mechanics: baroclinic flows; secondary instability of Görtler vortices
  (More complicated boundary conditions/systems)
- Inviscid limits of internal waves Jézéquel–W. '25+

Perspectives

- Inviscid limits of eigenvalues near critical/boundary values of U?
- Applications to evolution problems of shear flows?
- Higher dimensional models in fluid mechanics: baroclinic flows; secondary instability of Görtler vortices
  (More complicated boundary conditions/systems)
- Inviscid limits of internal waves Jézéquel–W. '25+

Thank you!