The polyhedral structure of multistage stochastic linear problem with general cost distribution

Maël Forcier, Stéphane Gaubert, Vincent Leclère

December 1st, 2020

mael.forcier@enpc.fr

Multistage stochastic linear programming (MSLP)

$$\begin{aligned}
\min_{(\mathbf{x}_t)_{t\in[t_{\max}]}} \quad & \mathbb{E}\Big[\sum_{t=1}^{t_{\max}} \mathbf{c}_t^\top \mathbf{x}_t\Big] & \\
\text{s.t.} \quad & \mathbf{T}_t\mathbf{x}_{t-1} + \mathbf{W}_t\mathbf{x}_t \leqslant \mathbf{h}_t & \forall t \in [t_{\max}] \\
& \mathbf{x}_t \preccurlyeq \sigma(\mathbf{c}_k, \mathbf{T}_k, \mathbf{W}_k, \mathbf{h}_k)_{k\leqslant t} & \forall t \in [t_{\max}] \\
& \mathbf{x}_0 \equiv x_0 \text{ given} &
\end{aligned}$$

where $(\mathbf{c}_t, \mathbf{T}_t, \mathbf{W}_t, \mathbf{h}_t)_{t\in[t_{\max}]}$ is a sequence of independent random variables.

We set $V_{t_{\max}+1} \equiv 0$ and:

$$V_t(x_{t-1}) := \mathbb{E}\left[\begin{aligned} \min_{x_t\in\mathbb{R}^{n_t}} \quad & \mathbf{c}_t^\top x_t + V_{t+1}(x_t) \\ \text{s.t.} \quad & \mathbf{T}_t x_{t-1} + \mathbf{W}_t x_t \leqslant \mathbf{h}_t \end{aligned}\right]$$

Multistage stochastic linear programming (MSLP)

$$\begin{aligned}
\min_{(\mathbf{x}_t)_{t\in[t_{\max}]}} \quad & \mathbb{E}\Big[\sum_{t=1}^{t_{\max}} \mathbf{c}_t^\top \mathbf{x}_t\Big] & \\
\text{s.t.} \quad & \mathbf{T}_t \mathbf{x}_{t-1} + \mathbf{W}_t \mathbf{x}_t \leqslant \mathbf{h}_t & \forall t \in [t_{\max}] \\
& \mathbf{x}_t \preccurlyeq \sigma(\mathbf{c}_k, \mathbf{T}_k, \mathbf{W}_k, \mathbf{h}_k)_{k\leqslant t} & \forall t \in [t_{\max}] \\
& \mathbf{x}_0 \equiv x_0 \text{ given} &
\end{aligned}$$

where $(\mathbf{c}_t, \mathbf{T}_t, \mathbf{W}_t, \mathbf{h}_t)_{t\in[t_{\max}]}$ is a sequence of independent random variables.

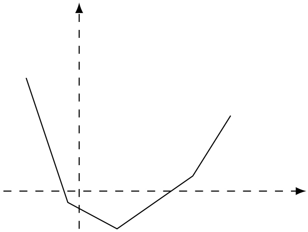
We set $V_{t_{\max}+1} \equiv 0$ and:

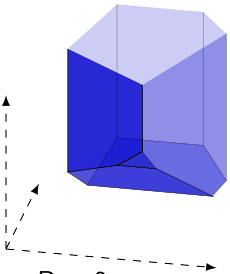
$$V_t(x_{t-1}) := \mathbb{E}\left[\begin{array}{l} \min_{x_t\in\mathbb{R}^{n_t}} \mathbf{c}_t^\top x_t + V_{t+1}(x_t) \\ \quad \text{s.t.} \;\; \mathbf{T}_t x_{t-1} + \mathbf{W}_t x_t \leqslant \mathbf{h}_t \end{array}\right]$$

Is V polyhedral ?

$$V(x) = \mathbb{E}\left[\begin{array}{ll}\min\limits_{y\in\mathbb{R}^m} & \mathbf{c}^\top y + R(y) \\ \quad \text{s.t.} & \mathbf{T}x + \mathbf{W}y \leqslant \mathbf{h}\end{array}\right] = \mathbb{E}\Big[\min_{y\in\mathbb{R}^m}\big(\mathbf{c}^\top y + R(y) + \mathbb{I}_{\mathbf{T}x+\mathbf{W}y\leqslant\mathbf{h}}\big)\Big]$$

Question : On which conditions on the random variable $\mathbf{c}$, $\mathbf{T}$, $\mathbf{W}$ and $\mathbf{h}$, is V polyhedral ?





For simplicity, by use of lift variable, we assume $R = 0$.

$\mathbf{c}$, $\mathbf{T}$, $\mathbf{W}$, $\mathbf{h}$ with finite support $\Rightarrow V$ polyhedral

Theorem (see e.g. Shapiro, Dentcheva, Ruszczyński)

If $\mathbf{c}$, $\mathbf{T}$, $\mathbf{W}$, $\mathbf{h}$ *have a finite support, then* V *is polyhedral*

Proof:

$$
\begin{aligned}
V(x) &= \sum_{k=1}^{N} p_k V_k(x) \\
&= \sum_{k=1}^{N} p_k \min_{y \in \mathbb{R}^m} \left(c_k^\top y + \mathbb{I}_{T_k x + W_k y \leqslant h_k} \right)
\end{aligned}
$$

where $p_k := \mathbb{P}\big[(\mathbf{c}, \mathbf{T}, \mathbf{W}, \mathbf{h}) = (c_k, T_k, W_k, h_k)\big]$.
Each V_k is polyhedral and $p_k \geqslant 0$.

Counter examples with stochastic constraints

Stochastic left hand side constraint $\mathbf{T}$

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y \in \mathbb{R}^m} & y \\ \text{s.t.} & \mathbf{u}x \leqslant y \\ & 1 \leqslant y \end{bmatrix}$$

$$= \mathbb{E}\big[\max(\mathbf{u}x, 1)\big]$$

$$= \begin{cases} 1 & \text{if } x \leqslant 1 \\ \frac{x}{2} + \frac{1}{2x} & \text{if } x \geqslant 1 \end{cases}$$

Stochastic right hand side constraint $\mathbf{h}$

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y \in \mathbb{R}^m} & y \\ \text{s.t.} & \mathbf{u} \leqslant y \\ & x \leqslant y \end{bmatrix}$$

$$= \mathbb{E}\big[\max(x, \mathbf{u})\big]$$

$$= \begin{cases} \frac{1}{2} & \text{if } x \leqslant 0 \\ \frac{x^2+1}{2} & \text{if } x \in [0,1] \\ x & \text{if } x \geqslant 1 \end{cases}$$

where $\mathbf{u}$ is uniform on $[0, 1]$.

Remaining case: only $\mathbf{c}$ stochastic

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y\in\mathbb{R}^m} & \mathbf{c}^\top y \\ \text{s.t.} & Tx + Wy \leqslant h \end{bmatrix} = \mathbb{E}\big[\min_{y\in\mathbb{R}^m}\big(\mathbf{c}^\top y + \mathbb{I}_{Tx+Wy\leqslant h}\big)\big]$$

Theorem (FGL 2020)

If T, W and h are finitely supported, then for all distributions of $\mathbf{c}$ such that V is well defined, V is polyhedral.

Remaining case: only $\mathbf{c}$ stochastic

$$V(x) = \mathbb{E}\begin{bmatrix} \min\limits_{y \in \mathbb{R}^m} & \mathbf{c}^\top y \\ \text{s.t.} & Tx + Wy \leqslant h \end{bmatrix} = \mathbb{E}\Big[\min_{y \in \mathbb{R}^m} \big(\mathbf{c}^\top y + \mathbb{I}_{Tx+Wy \leqslant h} \big) \Big]$$

Theorem (FGL 2020)

If T, W and h are finitely supported, then for all distributions of $\mathbf{c}$ such that V is well defined, V is polyhedral.

Reduction to finite sum through normal fan

For a given x, we have

$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y\Big], \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Tx + Wy \leqslant h\}$$

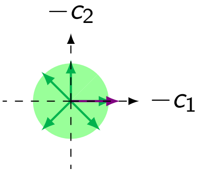
Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$

P_x for $x = 0.3$

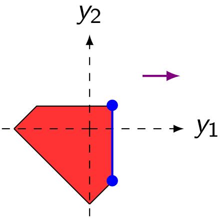
Reduction to finite sum through normal fan

For a given x, we have

$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y \Big], \quad \text{where} \quad P_x := \{ y \in \mathbb{R}^m \mid Tx + Wy \leqslant h \}$$



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$

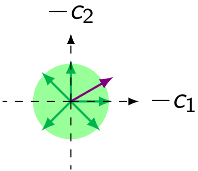


P_x for $x = 0.3$

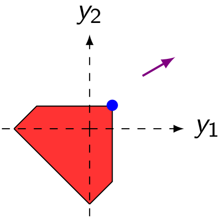
Reduction to finite sum through normal fan

For a given x, we have

$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y\Big], \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Tx + Wy \leqslant h\}$$



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$

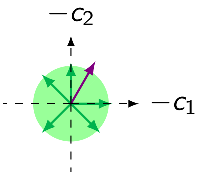


P_x for $x = 0.3$

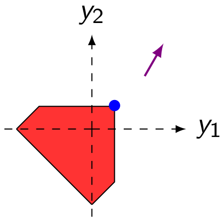
Reduction to finite sum through normal fan

For a given x, we have

$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y\Big], \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Tx + Wy \leqslant h\}$$



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$

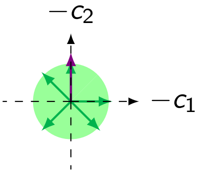


P_x for $x = 0.3$

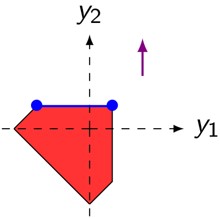
Reduction to finite sum through normal fan

For a given x, we have

$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y\Big], \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Tx + Wy \leqslant h\}$$



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$

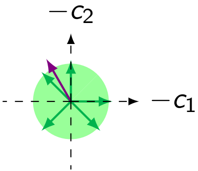


P_x for $x = 0.3$

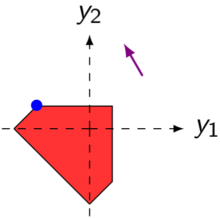
Reduction to finite sum through normal fan

For a given x, we have

$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y \Big], \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Tx + Wy \leqslant h\}$$



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$

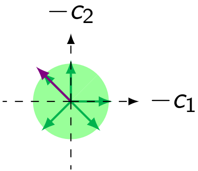


P_x for $x = 0.3$

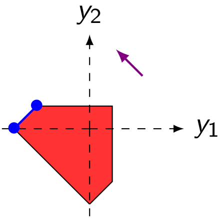
Reduction to finite sum through normal fan

For a given x, we have

$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y\Big], \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Tx + Wy \leqslant h\}$$



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$

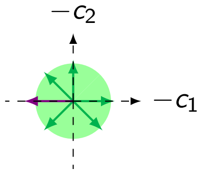


P_x for $x = 0.3$

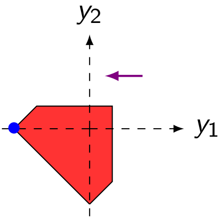
Reduction to finite sum through normal fan

For a given x, we have

$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y\Big], \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Tx + Wy \leqslant h\}$$



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$

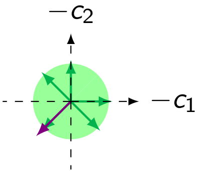


P_x for $x = 0.3$

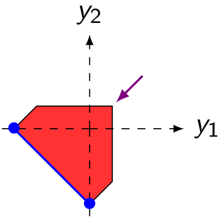
Reduction to finite sum through normal fan

For a given x, we have

$$V(x) = \mathbb{E}\Big[\min_{y \in P_x} \mathbf{c}^\top y\Big], \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Tx + Wy \leqslant h\}$$



Cost $-c$ and $\mathcal{N}(P_x)$ for $x = 0.3$

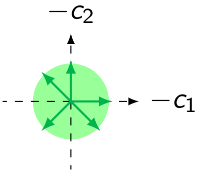


P_x for $x = 0.3$

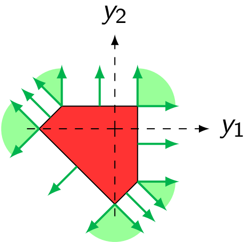
Reduction to finite sum through normal fan

For a given x, we have

$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big], \quad \text{where} \quad P_x := \{y \in \mathbb{R}^m \mid Tx + Wy \leqslant h\}$$



$\mathcal{N}(P_x)$ for $x = 0.3$



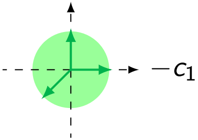
P_x for $x = 0.3$

$$V(x) = \mathbb{E}\big[\min_{y \in P_x} \mathbf{c}^\top y\big] = \sum_{N \in \mathcal{N}(P_x)} \mathbb{P}\big(\mathbf{c} \in -\operatorname{ri} N\big) \min_{y \in P_x} c_N^\top y,$$

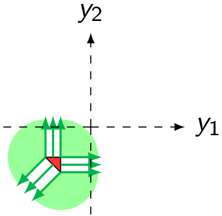
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

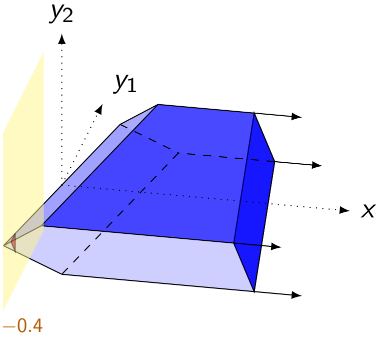
$x = -0.4$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

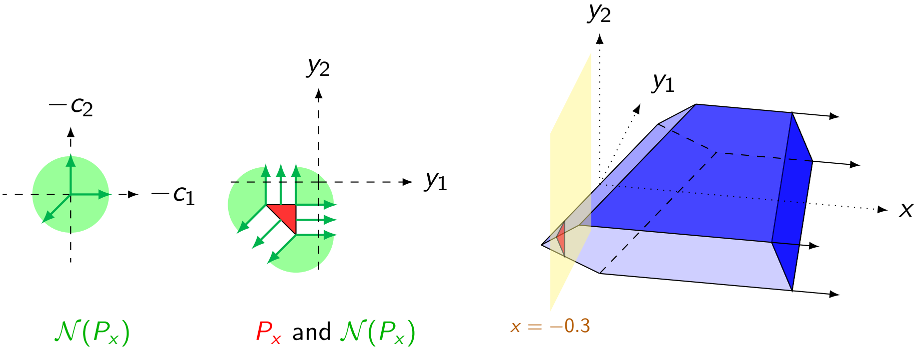


P and P_x

$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$$x = -0.3$$



$\mathcal{N}(P_x)$

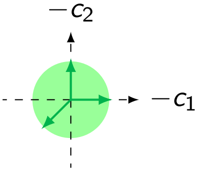
P_x and $\mathcal{N}(P_x)$

P and P_x

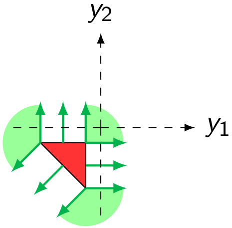
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

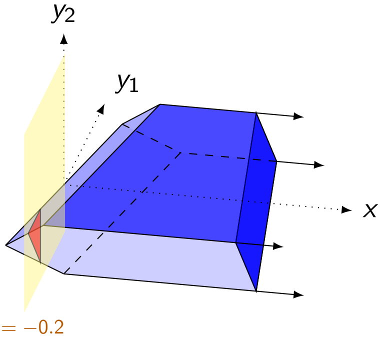
$$x = -0.2$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

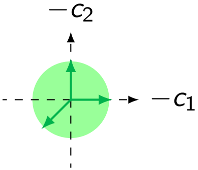


P and P_x

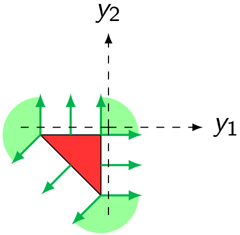
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

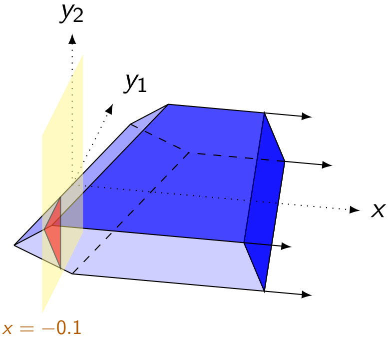
$$x = -0.1$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

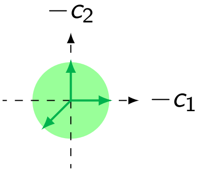


P and P_x

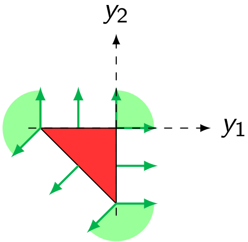
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

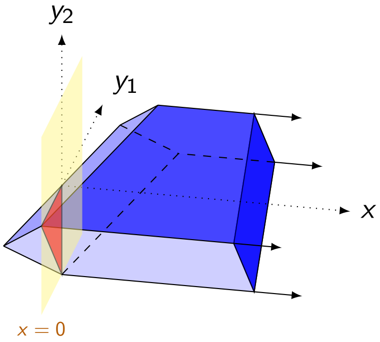
$$x = 0$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

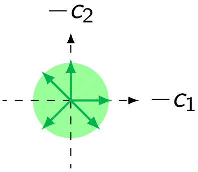


P and P_x

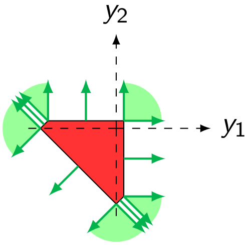
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

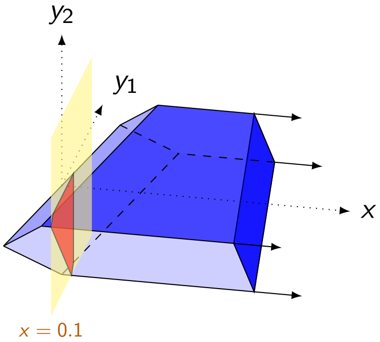
$$x = 0.1$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

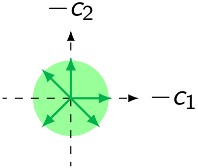


P and P_x

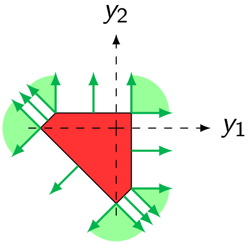
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

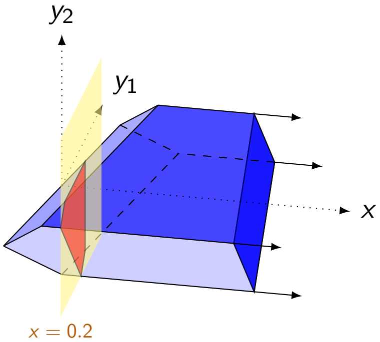
$x = 0.2$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

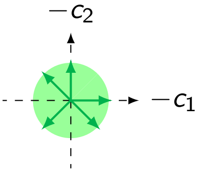


P and P_x

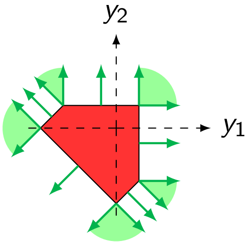
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

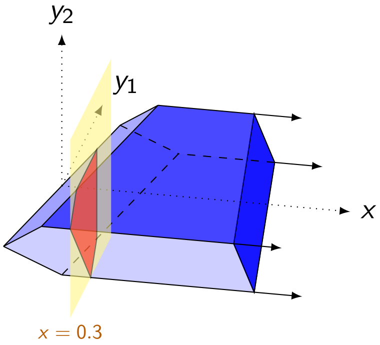
$$x = 0.3$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

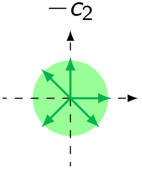


P and P_x

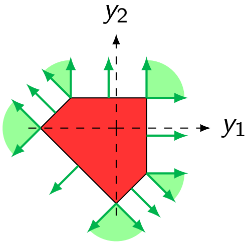
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

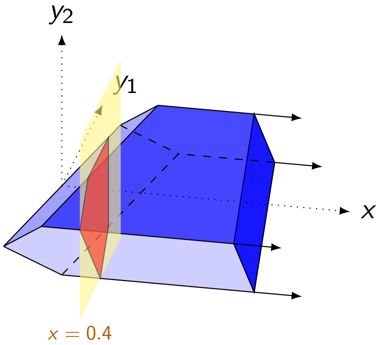
$x = 0.4$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

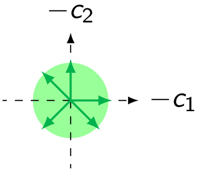


P and P_x

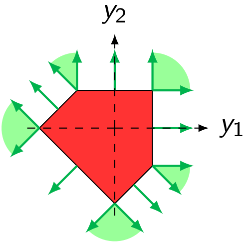
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

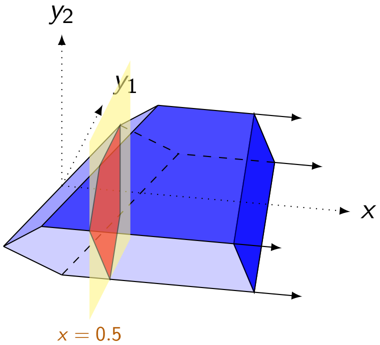
$x = 0.5$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

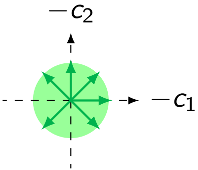


P and P_x

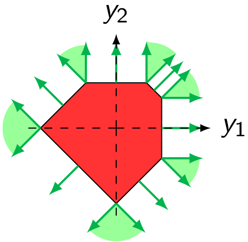
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

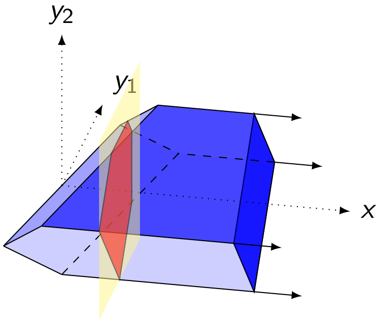
$x = 0.6$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

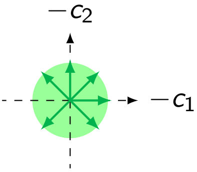


P and P_x

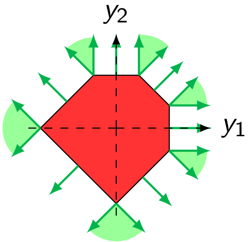
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

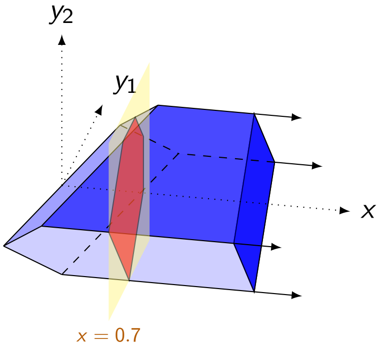
$x = 0.7$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

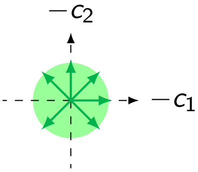


P and P_x

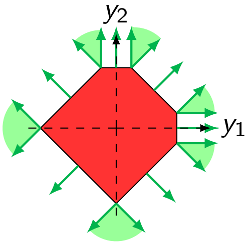
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

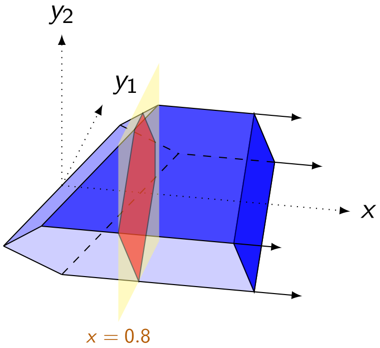
$$x = 0.8$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

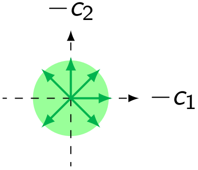


P and P_x

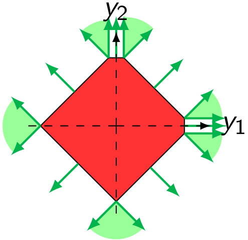
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

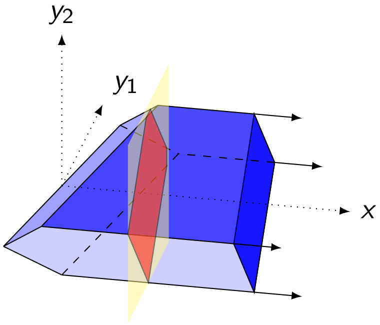
$x = 0.9$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

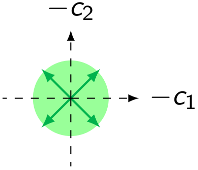


P and P_x

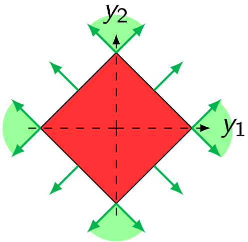
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

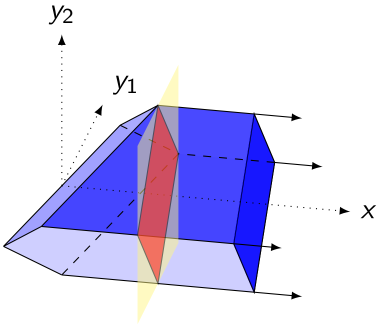
$x = 1$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

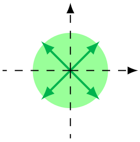


P and P_x

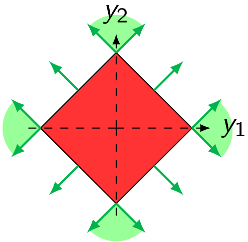
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

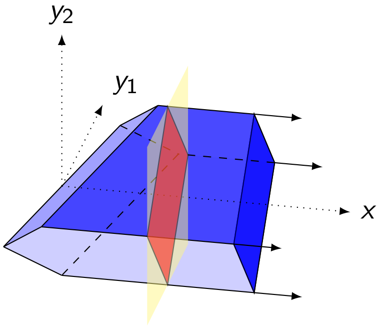
$x = 1.1$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

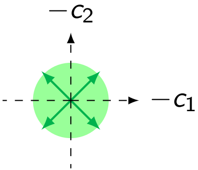


P and P_x

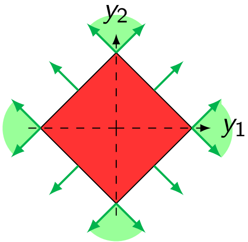
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

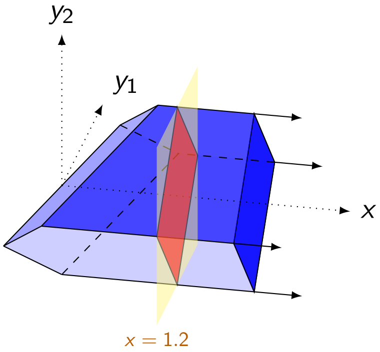
$x = 1.2$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

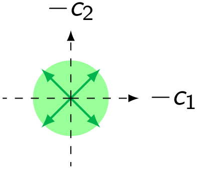


P and P_x

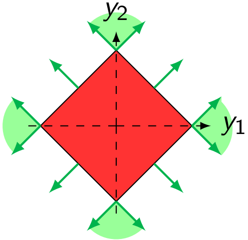
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

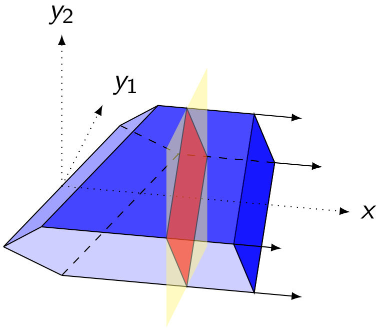
$$x = 1.3$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

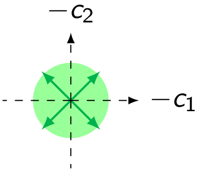


P and P_x

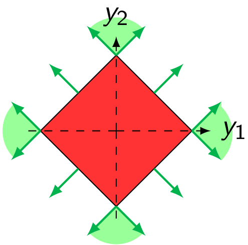
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

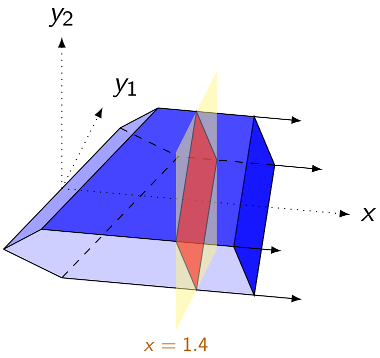
$$x = 1.4$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

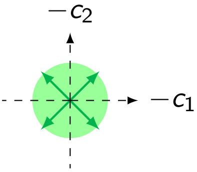


P and P_x

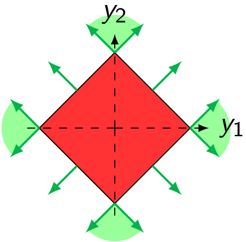
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

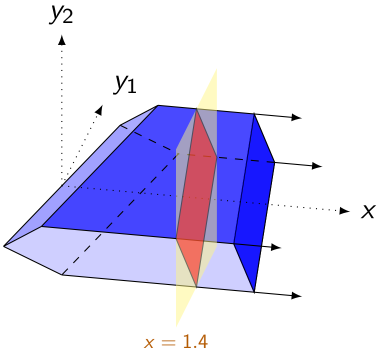
$$x = 1.4$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

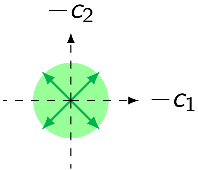


P and P_x

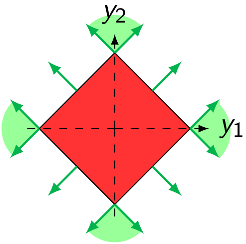
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

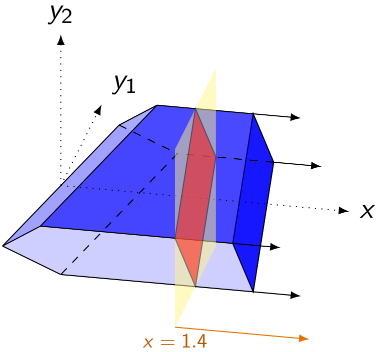
$x = 1.4$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

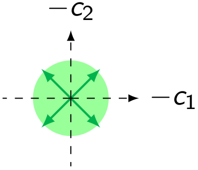


P and P_x

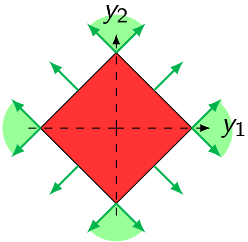
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

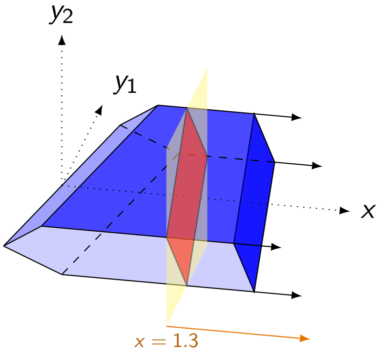
$$x = 1.3$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

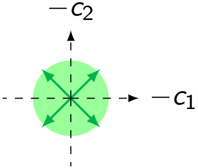


P and P_x

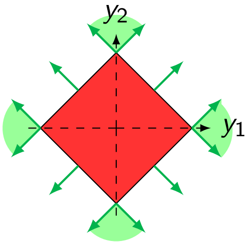
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

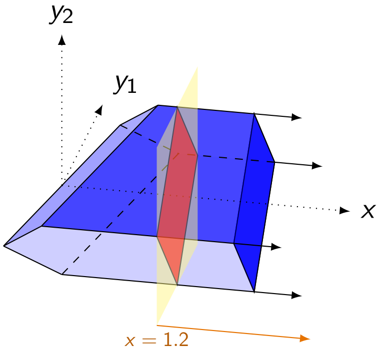
$x = 1.2$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

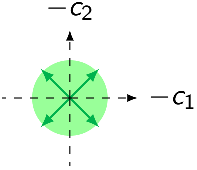


P and P_x

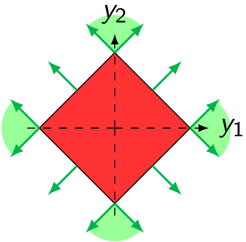
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

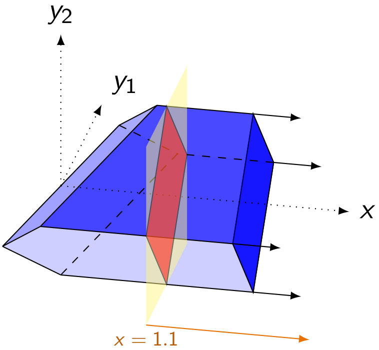
$x = 1.1$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

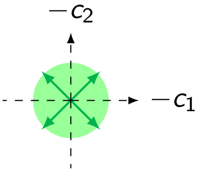


P and P_x

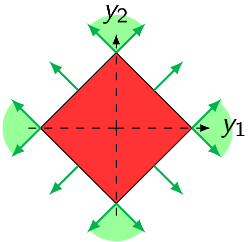
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

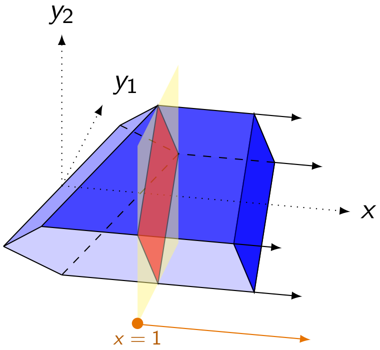
$x = 1$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

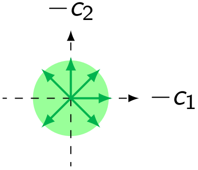


P and P_x

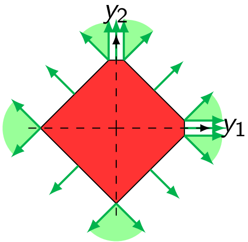
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

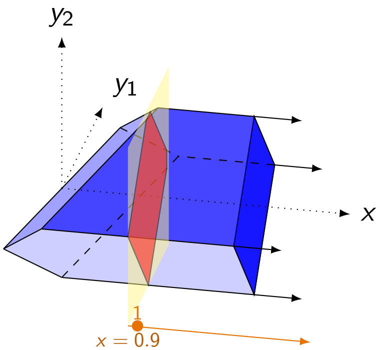
$x = 0.9$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

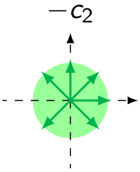


P and P_x

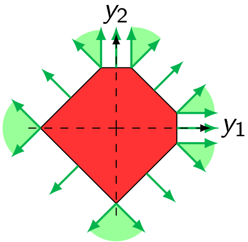
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

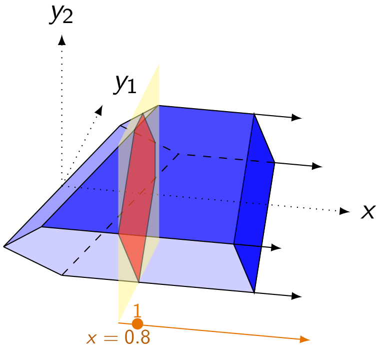
$x = 0.8$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

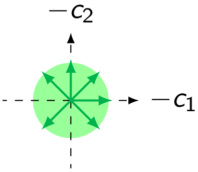


P and P_x

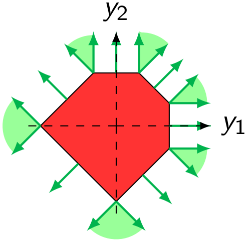
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

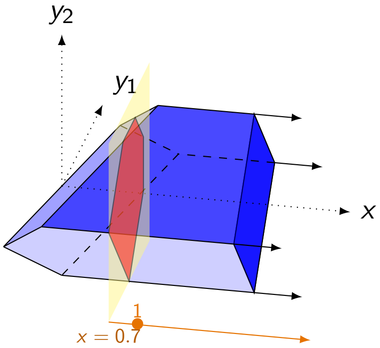
$x = 0.7$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

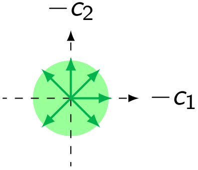


P and P_x

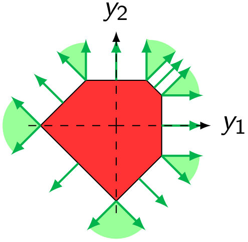
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

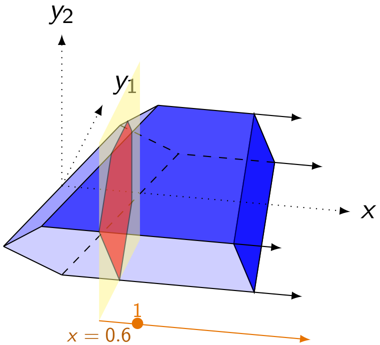
$x = 0.6$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

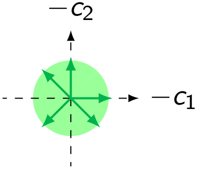


P and P_x

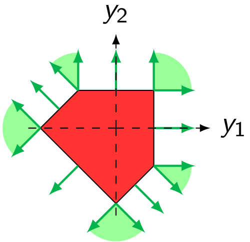
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

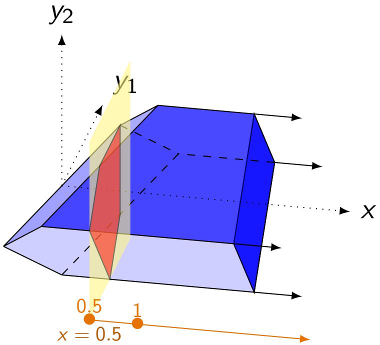
$$x = 0.5$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

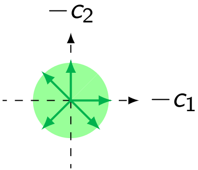


P and P_x

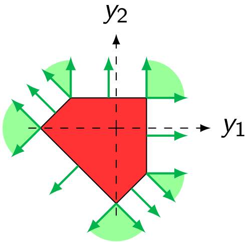
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

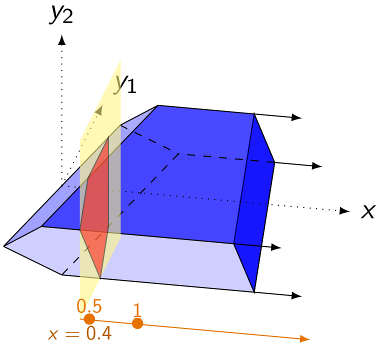
$x = 0.4$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

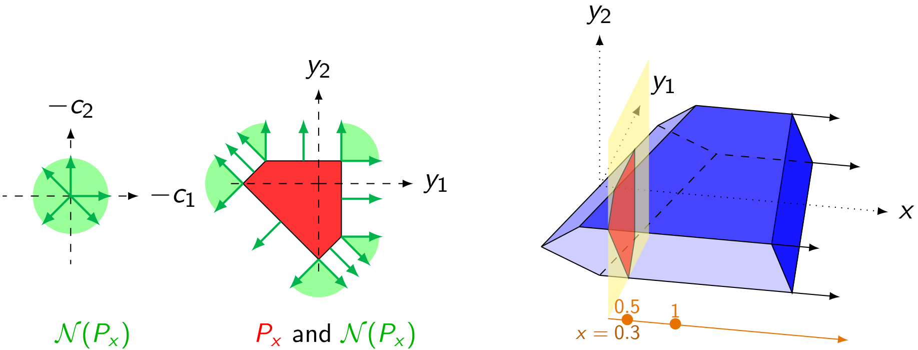


P and P_x

$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$$x = 0.3$$

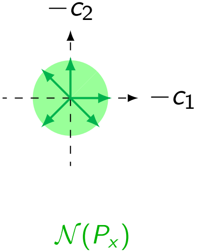


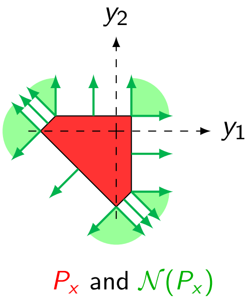
$\mathcal{N}(P_x)$ — P_x and $\mathcal{N}(P_x)$ — P and P_x

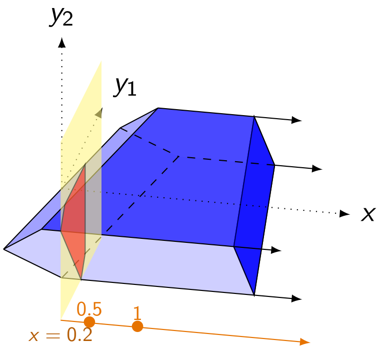
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

$x = 0.2$



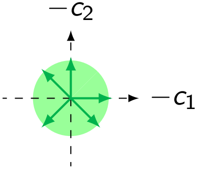




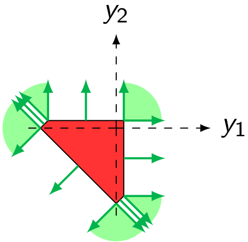
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

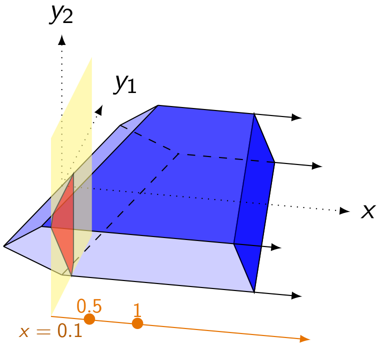
$$x = 0.1$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

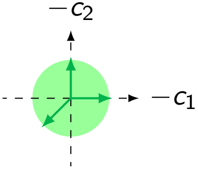


P and P_x

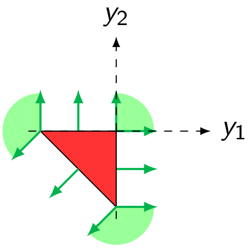
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

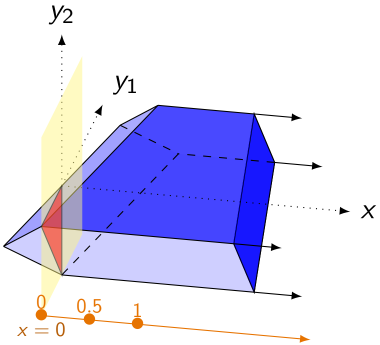
$$x = 0$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

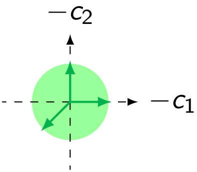


P and P_x

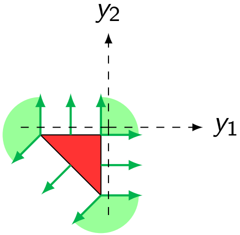
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

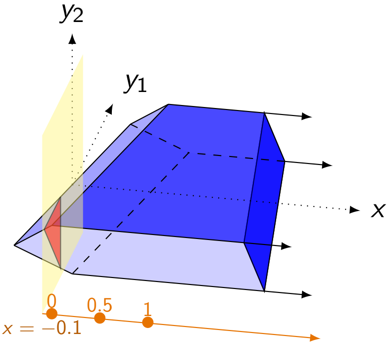
$$x = -0.1$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

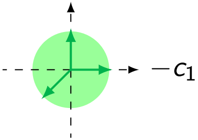


P and P_x

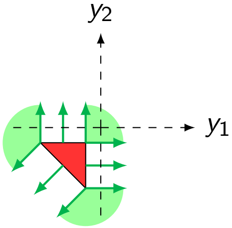
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

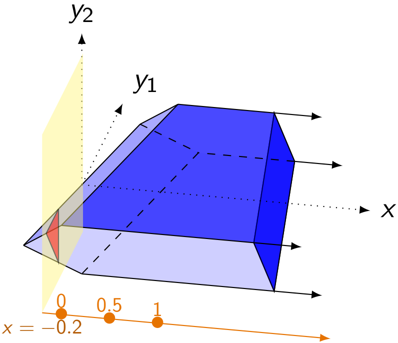
$x = -0.2$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

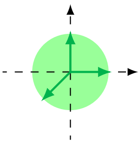


P and P_x

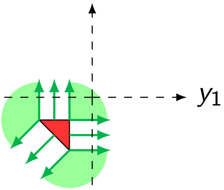
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x, y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

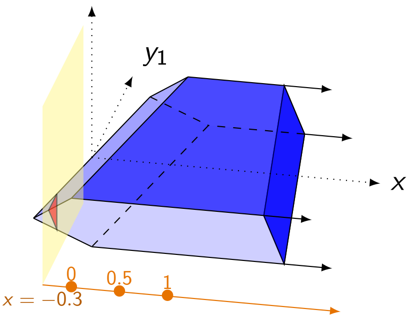
$$x = -0.3$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$

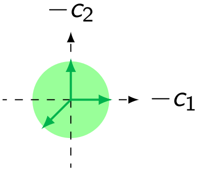


P and P_x

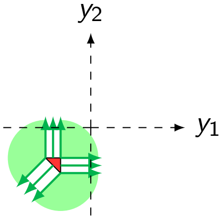
$\mathcal{N}(P_x)$ is piecewise constant with x.

$$P := \{(x,y) \mid Tx + Wy \leqslant h\} \quad \text{and} \quad P_x := \{y \mid Tx + Wy \leqslant h\}$$

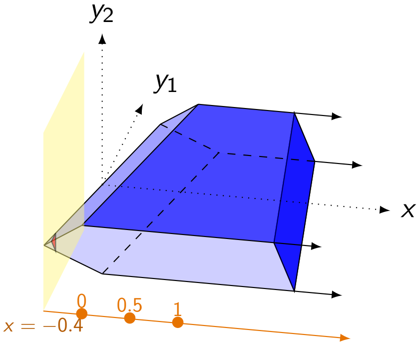
$$x = -0.4$$



$\mathcal{N}(P_x)$



P_x and $\mathcal{N}(P_x)$



P and P_x

General cost $\mathbf{c}$ is equivalent to discrete cost $\check{\mathbf{c}}$ for all x

Theorem (Quantization of the cost distribution)

Let $\mathcal{R} = \bigwedge_{\sigma \in \mathcal{C}(P,\pi)} -\mathcal{N}_\sigma$, *then* **for all** $x \in \mathbb{R}^n$

$$V(x) = \sum_{R \in \mathcal{R}} \check{p}_R \min_{y \in \mathbb{R}^m} \check{c}_R^\top y + \mathbb{I}_{Tx+Wy \leqslant h}$$

where $\check{p}_R := \mathbb{P}\big[\mathbf{c} \in \mathrm{ri}(R)\big]$ *and* $\check{c}_R := \mathbb{E}\big[\mathbf{c} \,|\, \mathbf{c} \in \mathrm{ri}(R)\big]$

Bonus: This quantization method works for *every distribution of* $\mathbf{c}$!

General cost $\mathbf{c}$ is equivalent to discrete cost $\check{\mathbf{c}}$ for all x

Theorem (Quantization of the cost distribution)

Let $\mathcal{R} = \bigwedge_{\sigma \in \mathcal{C}(P,\pi)} -\mathcal{N}_\sigma$, *then* **for all** $x \in \mathbb{R}^n$

$$V(x) = \sum_{R \in \mathcal{R}} \check{p}_R \min_{y \in \mathbb{R}^m} \check{c}_R^\top y + \mathbb{I}_{Tx+Wy \leqslant h}$$

where $\check{p}_R := \mathbb{P}\big[\mathbf{c} \in \mathrm{ri}(R)\big]$ *and* $\check{c}_R := \mathbb{E}\big[\mathbf{c} \,|\, \mathbf{c} \in \mathrm{ri}(R)\big]$

Bonus: This quantization method works for *every distribution of* $\mathbf{c}$!

Extension to multistage and stochastic constraints

Theorem

All results generalize to stochastic constraints with finite support and multistage

- $\rightsquigarrow$ *All V_t are polyhedral* *(easy)*
- $\rightsquigarrow$ *The regions where $(V_t)_t$ is affine do not depend on the $(\mathbf{c}_t)_t$* *(harder)*
- $\rightsquigarrow$ *We have an exact discretization method working for all $(\mathbf{c}_t)_t$* *(harder)*

Details and complexity results can be found in our preprint :
`https://hal.archives-ouvertes.fr/hal-02929361/`