Robust Solution Approaches for Challenging Network Optimization and Air Traffic Management Problems

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TRR 154

mathematics × models × computations = more energy

A COST PROJECT

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WIAS

ZIB

Gas networks

- Networks are inherently discrete

  valve a with switching variable $s_a \in \{0, 1\}$

$$\begin{aligned} s_a = 0 &\Rightarrow q_a = 0 \\ s_a = 1 &\Rightarrow p_i = p_j \end{aligned}$$

- Physics are inherently continuous

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho v^2 + p)}{\partial x} + g\rho\frac{\partial h}{\partial x} + \frac{\lambda}{2D}\rho\,|v|\,v = 0$$

$$\frac{\partial E}{\partial t} + \frac{\partial(Ev + pv)}{\partial x} + A\rho v g\frac{\partial h}{\partial x} + \pi\,D\,c_{\mathrm{HT}}\,(T - T_{\mathrm{soil}}) = 0.$$

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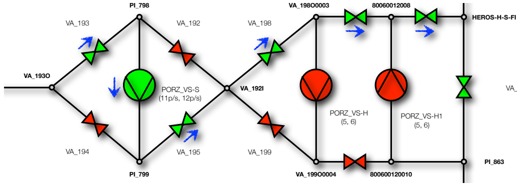

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TRR 154 on Gas Networks

Robust Optimization

Some References (not at all exhaustive)

- Soyster (1973). Convex Programming with Set-Inclusive Constraints and Applications to Inexact Linear Programming.
- Kouvelis, Yu 1997. Robust discrete optimization and its applications.
- Bertsimas, Sim several works on robust combinatorial optimization
- Ben-Tal, El Ghaoui, Nemirovski (2009). Robust optimization
- Gorissen, Yanikoglua, den Hertog (2015). A practical guide to robust optimization
- and many more...

Tractable Robust Counterparts for LP

Linear Inequality $\bar{a}^\top x \geq b$ with Polyhedral Uncertainty Set $\{\zeta \mid D\zeta \geq d\}$

$$(\bar{a} + P\zeta)^T x \geq b \qquad \forall \zeta : D\zeta \geq d \Leftrightarrow \bar{a}^T x + \min_{\{\zeta : D\zeta \geq d\}} (P^T x)^T \zeta \geq b$$

strong duality: $\min_\zeta \left\{ (P^T x)^T \zeta \mid D\zeta \geq d \right\} = \max_y \left\{ d^T y \mid D^T y = P^T x, y \geq 0 \right\} \Rightarrow$

$$\bar{a}^T x + \max_y \left\{ d^T y \mid D^T y = P^T x, y \geq 0 \right\} \geq b. \qquad (\star)$$

duality trick from robust optimization: If $(\star)$ is satisfied by some feasible y, then it is also for the maximum $\Rightarrow$ skip max.
x satisfies uncertain inequality iff $\exists\ y$ such that (x, y) satisfies

$$\boxed{\bar{a}^T x + d^T y \geq b, \quad D^T y = P^T x, \ y \geq 0}$$

Tractable Robust Counterparts for LP (and MIP)

Discussion

- Can easily be extended to conic uncertainty sets if strong duality holds
- Tractable robust counterparts are systems of linear inequalities over dual cones
- Integrality in problem variables x does not change such reformulations; they can still be applied.
- $\rightarrow$ Robust (mixed integer) linear optimization with 'standard' convex uncertainty sets is not (much) more difficult than the nominal problem.

Two-Stage Robust Optimization

Adjustable Uncertain Optimization Problem

$$\min c^T x$$
$$A_1(u)x_1 + A_2(u)x_2(u) \leq b \qquad \forall\, u \in \mathcal{U}, \mathcal{U} \text{ convex.}$$

x_1: variables on first stage, independent of u.
x_2: variables on second stage, adjustable, depending on u. $x_2(\cdot)$ is arbitrary function in u.
adjustable/two-stage robust optimization problem:

$$\min c^T x$$
$$s.t.\ \ x_1 \in \{x_1 \mid \forall\, u \in \mathcal{U}\ \exists\, x_2 \text{ such that } A_1(u)x_1 + A_2(u)x_2 \leq b\}$$

is already NP-hard for easy cases.

Two-Stage Robust Optimization

Affine Adjustability

$$\text{strict robustness: } \exists\,(x_1, x_2)\ \forall\, u \in \mathcal{U} \text{ with } A_1(u)x_1 + A_2(u)x_2 \le b,$$

$$\text{adjustable robustness: } \exists\, x_1\ \ \forall\, u \in \mathcal{U}\ \ \exists\, x_2 \text{ with } A_1(u)x_1 + A_2(u)x_2 \le b$$

restrictions on $x_2(u)$: often, affine adjustability is assumed:

$$x_2(u) = Qu + q, \text{ with appropriate } Q.$$

$$\text{(AJ)} \qquad \min c^T x_1$$
$$A_1(u)x_1 + A_2(u)[Qu + q] \le b \quad \forall\, u \in \mathcal{U}$$

- (AJ) is non-linear, but can be reformulated as positive semidefinite constraint
- result might be conservative, due to restricted adjustability.

Overview

- Robust Approaches to Gas Networks: strict + adjustable robustness, full adjustability
- Robust Optimization in Air Traffic Management

Robust Approaches in Energy (Networks)

Some References on Electricity (not at all exhaustive)

- Bacaud, Lemarechal, Renaud, Sagastizábal 2001. Bundle methods in stochastic optimal power management
- Zhao and Zeng 2012. Robust UC problem with demand response and wind energy
- Bertsimas et al. 2013 Adaptive robust optimization for security constrained UC
- Wang et al. 2013. 2-stage robust optimization for N-k contingency-constrained UC
- Ruiwei et al. 2014 2-stage network constrained robust UC problem
- Tahanan, van Ackooij, Frangioni, Lacalandra 2015. Large-scale UC under uncertainty
- Ruiz and Conejo 2015. Robust transmission expansion planning
- D'Ambrosio, Liberti, 2016. Power edge set problem

Optimization in Gas Networks

References (not at all exhaustive)

- Collins et al, 1978. (proof for uniqueness of flows in passive nets)
- De Wolf and Smeers 2000. (sequential linear programming approach for piecewise linearized problem)
- Borraz-Sanchez and Rios-Mercado 2004. (dynamic programming)
- Geißler et al., 2011. (linear relaxation approach gas and water networks (with active elements))
- Koch et al. Evaluating gas network capacities. SIAM, Philadelphia, PA, 2015. (book on modeling and optimization of gas networks, MIP/MINLP/NLP methods)
- Ríos-Mercado and Borraz-Sánchez 2015. (survey article)
- Gotzes et al 2016. (reduced description of passive network problem)

...not much out there on optimization approaches under uncertainty.

Nomination Validation Under Uncertainty

Aßmann, L, Stingl

- gas transport model: stationary, isothermal
- the pipes' roughness values is not known precisely, has a large effect on the flow value, and can only be measured with great effort.

Problem Statement

Is there a configuration of the active elements within the network such that there exists a feasible gas flow for each possible realization of the uncertain data?
solution approaches for the nominal setting: Koch et al (2015), Pfetsch et al. (2015)

Pressure Loss in a Pipe

Horizontal pipe with pressures p_{in}, p_{out} and flow q.
Roughness $k \in [k_{\min}, k_{\max}]$ is uncertain with a nominal value of $\hat{k}$.

Nominal Pressure Loss Equation

$$p_{\text{in}}^2 - p_{\text{out}}^2 = \phi(\hat{k})|q|q$$

Pressure Loss Under Uncertainty

$$\left\{p_{\text{in}}^2 - p_{\text{out}}^2 = \phi(k)|q|q\right\}_{k \in [k_{\min}, k_{\max}]}$$

Since $\phi(k)$ is monotonically increasing in k:

$$\left\{\, p_{\text{in}}^2 - p_{\text{out}}^2 = \phi|q|q \,\right\}_{\phi \in [\phi(k_{\min}), \phi(k_{\max})]}$$

Robust Nomination Validation

$$
\begin{aligned}
& \min_{p^2,q} \text{ operating costs} \\
& \sum_{a\in\delta^+(v)} q_a - \sum_{a\in\delta^-(v)} q_a = d_v && \forall\, v \in V && \text{flow conservation} \\
& p_i^2 - p_j^2 = {\color{red}\phi_a} q_a |q_a| && \forall\, a = (i,j) \in A, {\color{red}\forall\, \phi_A \in U_a} && \text{pressure loss} \\
& p_v^2 \in [\underline{p}_v^2, \overline{p}_v^2] && \forall\, v \in V && \text{pressure bounds} \\
& q_a \in [\underline{q}_a, \overline{q}_a] && \forall\, a \in A && \text{flow bounds} \\
& \text{active elements}
\end{aligned}
$$

...this is a mixed-integer two-stage robust optimization problem with a non-convex quadratic lower level.

Some References

- tractable robust counterpart in sense of Ben-Tal, Nemirovski, et al. (e.g. MPS-SIAM, 2002) or Bertsimas et al. (e.g. SIAM Review, 2011): (how) can it be applied due to
  - non-convex character of our problem?
  - full adjustability (which decision rules?)
- robust problems in optimal control: direct treatment of bilevel structure; the inner (here: passive) problem is approximated (see, e.g., Diehl et al., 2006, 2008, Laas/ Ulbrich, 2017, Diehl, 2013)
  - if the inner problem is non-convex (as in our problem), these methods may fail to detect the worst-case scenario!
- further interesting related work based on partitioning of uncertainty set in order to cope with full adjustability as well as binary variables (Bertsimas, Dunning 2014, Postek, den Hertog, 2014, ...)

Difficulty of the Problem

We face a robust

mixed-integer	
two-level optimization problem with	
non-convex quadratic lower level	

Difficulty of the Problem

We face a robust

	potential simplification
mixed-integer	continuous (passive networks)
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non-convex quadratic lower level	piecewise-linear relaxed lower level

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Linearized Pressure Loss Equation

Approximating the pressure loss equation through a piecewise linear function $\tau(q)$ with given error $\varepsilon > 0$, e.g.,
Geißler et al. (2012):

$$\tau(q) - \varepsilon \leq \phi(\hat{k})q|q| \leq \tau(q) + \varepsilon$$

Mixed-integer linear constraints for linearized pressure loss equation:

$$\tau(q) - \varepsilon \leq p_{\text{in}}^2 - p_{\text{out}}^2 \leq \tau(q) + \varepsilon$$

Mixed-Integer Linear Pressure Loss Under Uncertainty

With $[c_{\min}, c_{\max}] = [\phi(\hat{k})^{-1}\phi(k_{\min}),\ \phi(\hat{k})^{-1}\phi(k_{\max})]$:

$$\left\{ c(\tau(q) - \varepsilon) \leq p_{\text{in}}^2 - p_{\text{out}}^2 \leq c(\tau(q) + \varepsilon) \right\}_{c \in [c_{\min}, c_{\max}]}$$

Pressure Loss Equation with Uncertain Scaling

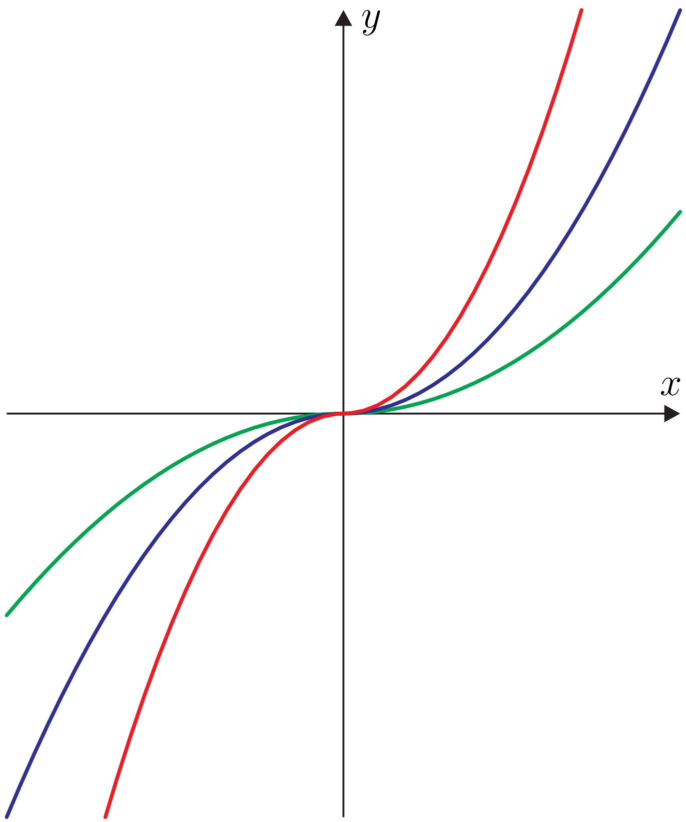


Figure : Nonlinear Function with Uncertain Scaling.

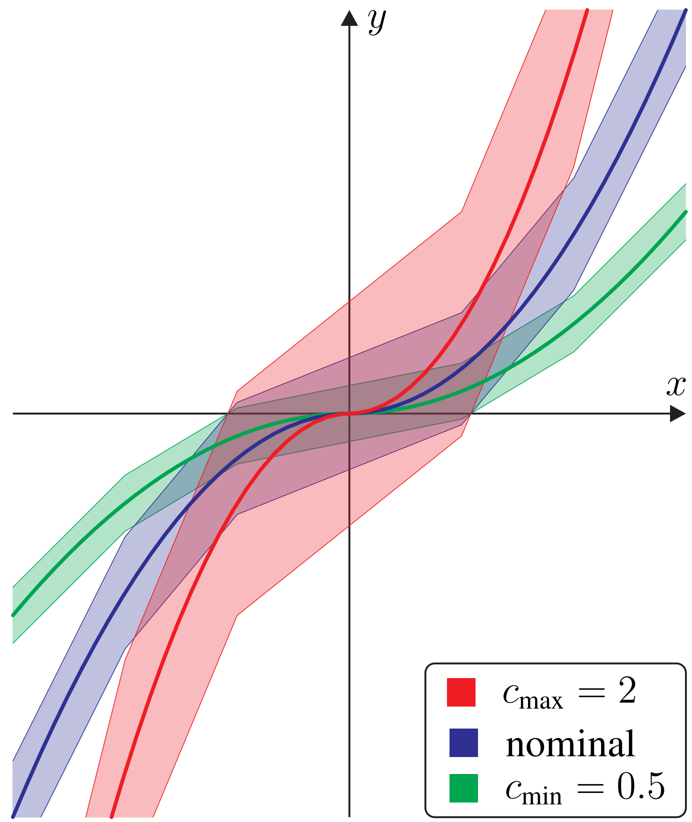


Figure : Linearization of Nonlinear Function with Uncertain Scaling.

Strict Robust Counterpart for Robust Nomination Validation

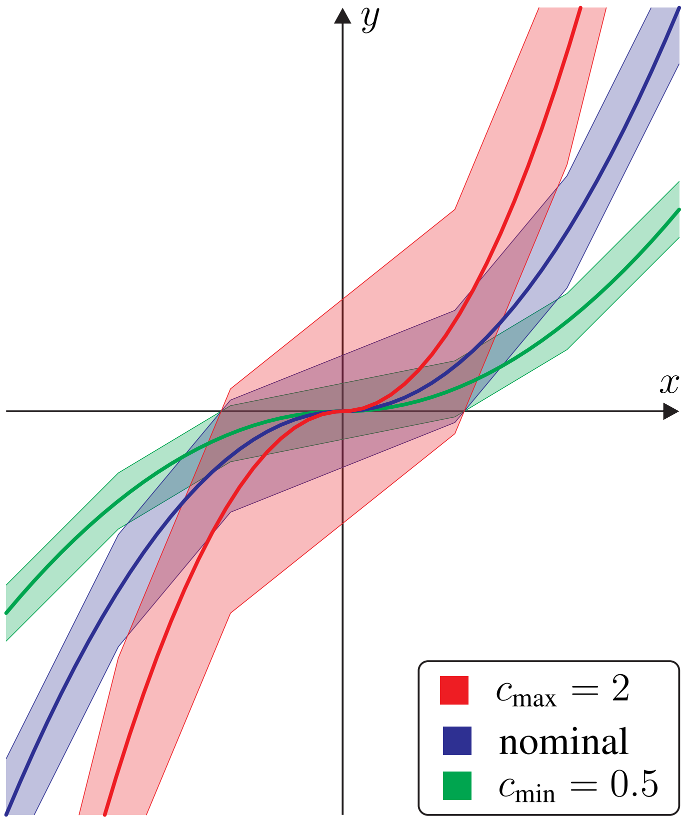


Figure : Uncertain Linearization.

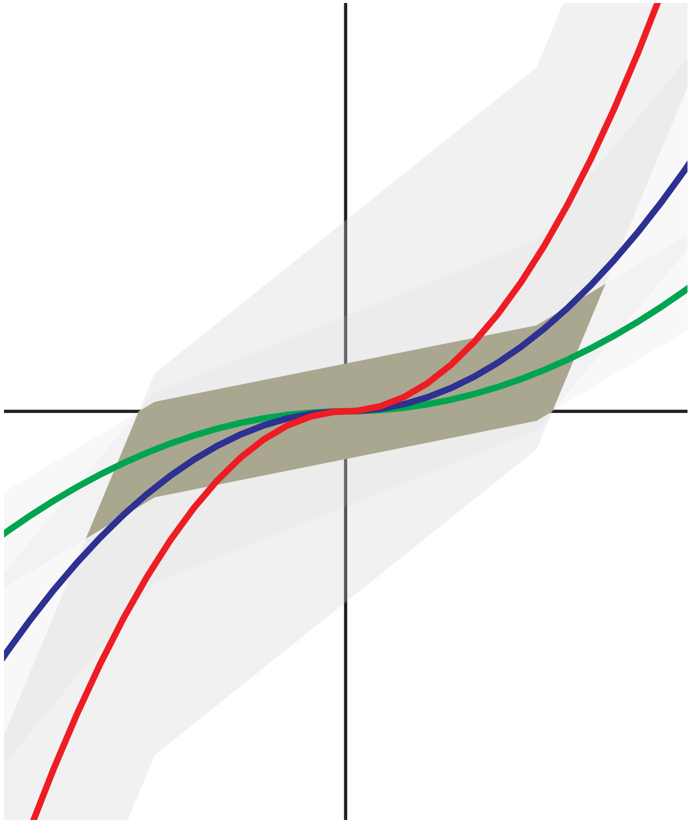
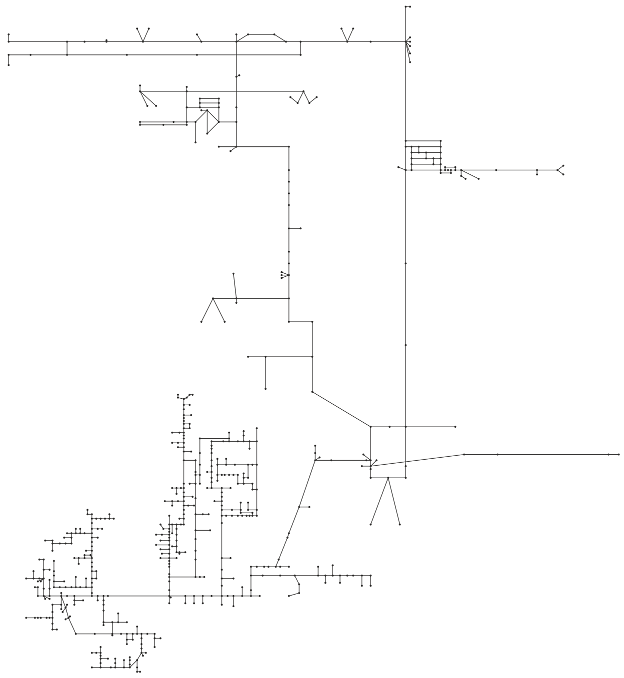


Figure : Robustification.

Numerical Results for the Strict Robust Counterpart



For each pipe, the following uncertain roughness is assumed:

$$k_a \in [\hat{k}_a, (1 + r_{\max})\hat{k}_a].$$

$r_{\max}$	MIP obj.	MIP runtime	NLP obj.
nominal	214.22	44.25 s	444.20
0.001	214.22	236.12 s	infeas.
0.01	214.22	165.12 s	444.20
0.05	infeas.		
0.1	infeas.		
0.5	infeas.		

Table : Objective values of nominal and robustified instances.

...can be computed fast (via a MIP), but yields (too) conservative solutions.

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two-level optimization problem with	single-level, affine adjustability, etc.
non-convex quadratic lower level	piecewise-linear relaxed lower level

Adjustable Robustness for Nomination Validation

Relaxing one of the assumptions for strict robustness leads to adjustable robustness:

- solutions must be feasible for all possible realizations of the uncertain data
- ~~variables must be fixed before uncertainty becomes known~~

$\rightarrow$ solution can depend on the uncertain data:

"Here-and-Now" or First Stage Variables

must be fixed before the uncertainty becomes known

"Wait-and-See" or Second Stage Variables

may adjust themselves to the realized uncertainty

Affine Adjustable Robustness for Nomination Validation

- random recourse
- assume affine linear second-stage variables $\Rightarrow$ reformulation:
- one SDP constraint for each original constraint
- each SDP constraint has size $(\dim(U) + 1 \times \dim(U) + 1)$
- exact reformulation for $\dim(U) = 1$, approximative otherwise
- restricted adjustability of pressure and flow since the binary auxilliary variables for the piecewise linear model cannot be cast on the second stage

Numerical Results for Affinely Adjustable Robustness

Software used: Python, MISDP plugin for SCIP (Mars, Schewe (2012))

- passive network
- linear pressure loss equation
- pressure loss equation is approximated through 8 sampling points

		runtime [s]		
topology	#bin.	mean	min	max
1 pipe	7	1.02	0.86	1.33
2 pipes	14	1.10	0.93	1.41
3 pipes	21	1.41	1.04	2.38
triangle	21	2.14	1.33	3.33
square	28	5.50	2.63	10.21
chorded square	35	15.26	5.70	36.08

Comparison of Strict and Adjustable Models for Nomination Validation

- 5 sampling points per pressure loss equation
- uncertainty set on each arc: $c_a \in [0.96, 2], \quad a \in A$
- 6 different approximation errors: $\varepsilon \in \{0.5, 1, 2, 3, 4, 5\}$

	robustification		
topology	nominal	strict	adjustable
1 pipe	6	2	6
2 pipes	6	2	6
3 pipes	6	2	6
triangle	6	4	6
square	6	4	4
chorded square	6	3	4

...(somewhat) less conservative, but costly.

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New Approach: Robustness Equals Set Containment

D. Aßmann, L, M. Stingl (FAU), J. Vera (Tilburg) https://opus4.kobv.de/opus4-trr154/frontdoor/index/index/docId/136

uncertain parameter $u \in \mathcal{U} \subseteq \mathbb{R}^n$
problem variables $x \in \mathbb{R}^m$
constraints $F(u, x) \geq 0$
feasible combinations $\mathcal{B} := \{u \in \mathcal{U}, x \in \mathbb{R}^m \mid F(u, x) \geq 0\}$

Robust Feasibility as Set Containment

Deciding robust feasibility

$$\forall\, u \in \mathcal{U} \quad \exists\, x \quad \text{such that} \quad F(u, x) \geq 0$$

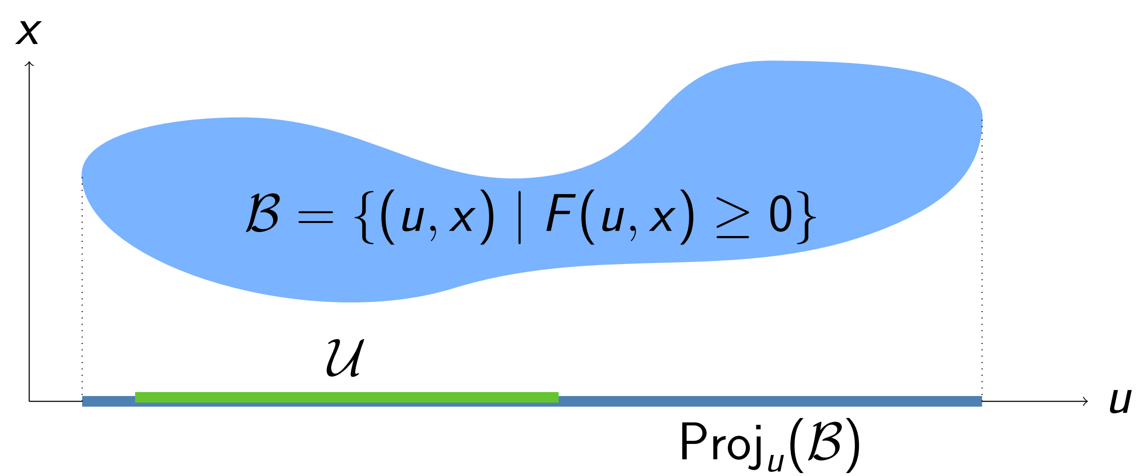
is equivalent to set containment problem

$$\mathcal{U} \subseteq \text{Proj}_u(\mathcal{B}).$$

robust optimization: two-stage adjustable nonlinear problem w. empty first stage

+ nonlinear/nonconvex constraints
+ no decision rules → full adjustability

Solution Approach: Robustness Equals Set Containment



$$\mathcal{U} \subseteq \text{Proj}_u(\mathcal{B})?$$

Polynomial SDP Relaxation Hierarchy

let $K := \{x \mid g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$ $\quad (f, g_i$ poly.):

$$\begin{aligned}\min_{x \in K} f(x) &= \sup_{\lambda \in \mathbb{R}} \lambda \text{ s.t. } f(x) - \lambda \geq 0 \quad \forall x \in K \\ &= \sup_{\lambda \in \mathbb{R}} \lambda \text{ s.t. } f(x) - \lambda \in \mathcal{P}(K) \\ &\geq \sup_{\lambda \in \mathbb{R}} \lambda \text{ s.t. } f(x) - \lambda \in \Sigma_d(K)\end{aligned}$$

- replace $\min_{x \in K} f(x)$ with hierarchy of SDP relaxations;
  Sherali-Adams ('90), Lováxz-Schrijver ('91), Parrilo ('00), Lasserre (2001)
- relaxation hierarchy main ideas:
  1. formulate optimization problem in terms of nonnegative polynomials
  2. sum of squares condition ($p = \Sigma_i q_i^2$) can be checked easily with SDP
  3. replace nonnegativity by weaker degree bounded SOS formulation
     $\rightarrow$ max degree $2d \equiv$ level of hierarchy d

Deciding Robust Feasibility

Assume $\mathcal{B} = \mathcal{G} \cap \mathcal{H}$:

$$\mathcal{G} := \{u, x \mid u \in \mathcal{U},\ h(u, x) = 0\}, \qquad \text{(gas: flow equation)}$$
$$\mathcal{H} := \{u, x \mid f_i(u, x) \geq 0,\ i \in I\}. \qquad \text{(gas: pressure constrs.)}$$

Furthermore, assume extension x for $u \in \mathcal{U}$ is unique.

Lemma (Elimination of Projection)

$$\mathcal{U} \subseteq \text{Proj}_u(\mathcal{B}) \iff \mathcal{G} \subseteq \mathcal{H}$$

- proof idea: exploit uniqueness of x: $\mathcal{G} = \{u, x \mid u \in \mathcal{U},\ x = x(u)\}$

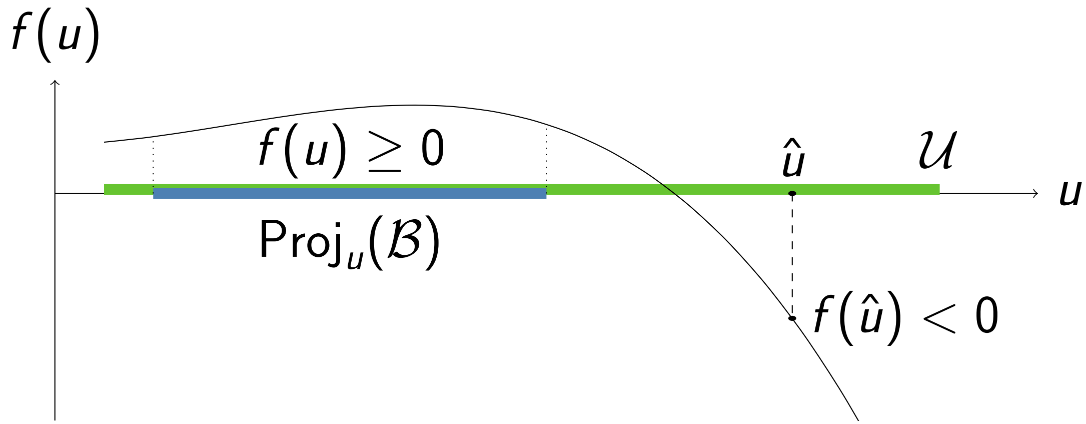
Minimize Constraints to Check Set Containment

$$\begin{aligned} \mathcal{U} \subseteq \text{Proj}_u(\mathcal{B}) &\iff \mathcal{G} \subseteq \mathcal{H} \\ &\iff \inf_{(u,x)\in\mathcal{G}} f_i(u, x) \geq 0 \quad (\forall\, i \in I) \end{aligned}$$

$\implies$ global optimal solutions needed, use polynomial optimization

Deciding Robust Infeasibility

- infeasible iff. $\mathcal{U} \not\subseteq \text{Proj}_u(\mathcal{B}) \iff \mathcal{U} \setminus \text{Proj}_u(\mathcal{B}) \neq \emptyset$
- idea: find function f that is non-negative on $\text{Proj}_u(\mathcal{B})$ and negative on $\mathcal{U} \setminus \text{Proj}_u(\mathcal{B})$



- as optimization problem, objective $-\infty$ iff. robust infeasible:

$$\inf_{f,u} \left\{ f(u) \;\middle|\; \begin{array}{l} u \in \mathcal{U} \\ f \in \{f \mid f(u) \geq 0 \quad \forall\, u \in \text{Proj}_u(\mathcal{B})\} \end{array} \right\}$$

- any feasible f with $f(u) < 0$ certifies violation of set containment

Polynomial Approximation of Robust Infeasibility Problem

- abstract problem: $\inf\{f(u) \mid u \in \mathcal{U},\ f \geq 0 \text{ on } \mathrm{Proj}_u(\mathcal{B})\}$
- restrict functions to nonnegative polynomials p over $\mathrm{Proj}_u(\mathcal{B})$, approximate with sum of square polynomials
- replace $p(u)$ with weaker $\int_{\mathcal{U}} p \,\mathrm{d}\mu = \int_{\mathcal{U}} \sum_{\alpha} p_\alpha u^\alpha \mathrm{d}\mu = \sum_{\alpha} p_\alpha \int_{\mathcal{U}} u^\alpha \mathrm{d}\mu$
  $\rightarrow$ requires moments $\int_{\mathcal{U}} u^\alpha \mathrm{d}\mu$

Lemma (Elimination of Projection)

The following problems have same objective value:

$$(1) \quad \inf_p \sum_\alpha p_\alpha \int_{\mathcal{U}} u^\alpha \mathrm{d}\mu, \quad p \in \mathcal{P}[\mathrm{Proj}_u(\mathcal{B})]$$

$$(2) \quad \inf_{\tilde{p}} \sum_{\alpha,\beta} \tilde{p}_{\alpha,\beta} \int_{\mathcal{U}} u^\alpha x^\beta \mathrm{d}\mu, \quad \tilde{p}_{\alpha,\beta} = 0 \quad \forall\, \beta \neq 0, \quad \tilde{p} \in \mathcal{P}[\mathcal{B}]$$

Deciding Robustness for Nomination Validation in Gas Networks

- lots of absolute values due to pressure drop equation:

$$\pi_i - \pi_j = \phi_a q_a |q_a|$$

  (π denote squared pressures)

-

trees: no absolute values ✓

one cycle: partitions of $\mathcal{U}$ allow elimination of $|\cdot|$ ✓

$\geq$ one cycle(s): binaries + bigM,
case distinction

The Passive Nomination Validation Problem

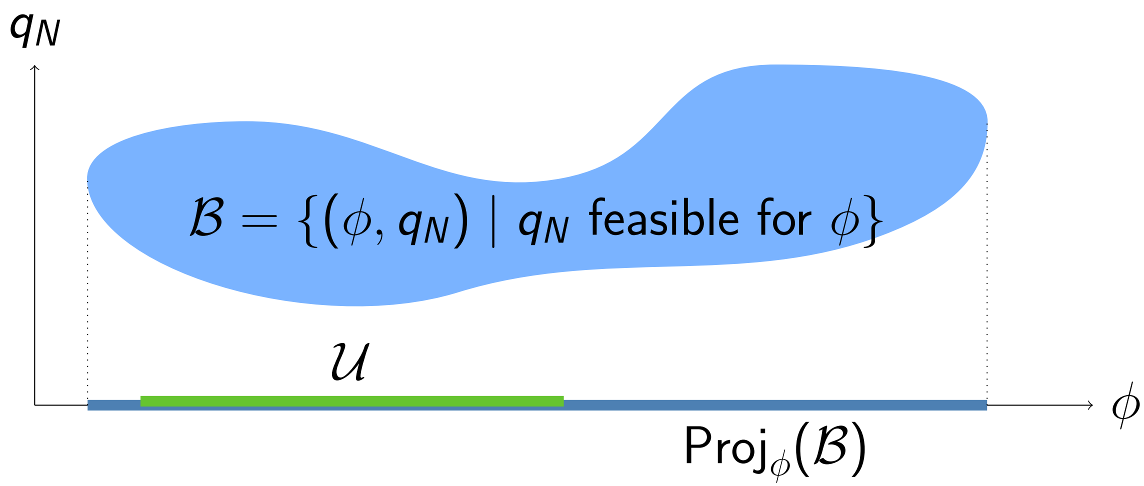
- eliminate variables (Gotzes et al. (2016)) $\rightarrow$ only cycle flows q_N remaining
- set of feasible (roughness, flow)-tuples:

$$\mathcal{B} := \left\{ \phi \in \mathcal{U}, q_N \in \mathbb{R}^{|N|} \;\middle|\; \begin{array}{c} \mathcal{A}_N^T g(\phi, q_N) = \operatorname{diag}(\phi_N) q_N \, |q_N| \\ \underline{\pi}_i + g_i(\phi, q_N) \leq \overline{\pi}_j + g_j(\phi, q_N) \quad \forall\, i, j \in V \end{array} \right\}$$

- $g_i(\phi, q_N)$ is aggregated pressure drop between root node and node $i \in V$

Robust feasible if and only if:

$$\mathcal{U} \subseteq \operatorname{Proj}_\phi(\mathcal{B}) = \{\phi \,|\, \exists\, q_N \colon (\phi, q_N) \in \mathcal{B}\}\,.$$



Deciding Robust Feasibility on Tree Networks is Easy!

- tree G, root r, arcs directed away from r; no cycle $\to$ no q_N

$$\mathcal{B} := \left\{ \phi \in \mathcal{U} \,\middle|\, \underline{\pi}_i + g_i(\phi) \le \overline{\pi}_j + g_j(\phi) \quad \forall\, i, j \in V \right\}$$

  with

$$g_i(\phi) = \sum_{a \in \mathrm{Path}(r,i)} \phi_a \underbrace{q_a^2}_{\text{const!}}$$

- robust feasible iff. $\mathcal{U} \subseteq \mathrm{Proj}_\phi(\mathcal{B}) = \mathcal{B}$
- set containment (polyhedral $\mathcal{U}$) can be checked with LPs (Mangasarian (2002))

Lemma

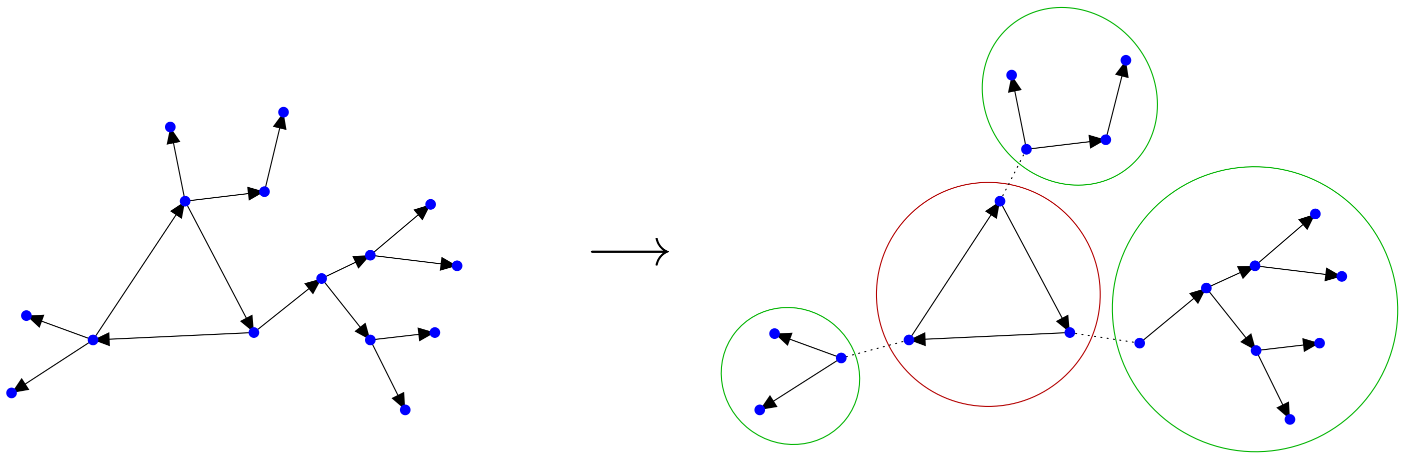
Given a tree G, the gas network problem is robust feasible if and only if

$$\pi_r \in [\underline{\pi}_r^*, \overline{\pi}_r^*]$$

where $\underline{\pi}_r^*, \overline{\pi}_r^*$ are optimal values of two LPs.

Extension to Tree + Edge Networks

- add edge to tree topology $\longrightarrow$ cycle is created
- decompose graph into cycle and subtrees
- determine condition for robust feasibility of all subtrees
- remove trees and update pressure bounds of cycle nodes



$\longrightarrow$ computational complexity mostly depends on size of cycle

Numerical Experiments: Test Instances

- focus on cycle network, preprocess attached trees
- instance: cycle with $n = 2, \ldots, 7$ nodes
- variable uncertainty set:

$$\mathcal{U}(c) := \times_{a \in A}[1, c]$$

- two special sets:

$$\begin{aligned} \mathcal{U}_{\text{feas}} &:= \mathcal{U}(2) \\ \mathcal{U}_{\text{infeas}} &:= \mathcal{U}(4) \end{aligned}$$

Deciding Robust Feasibility

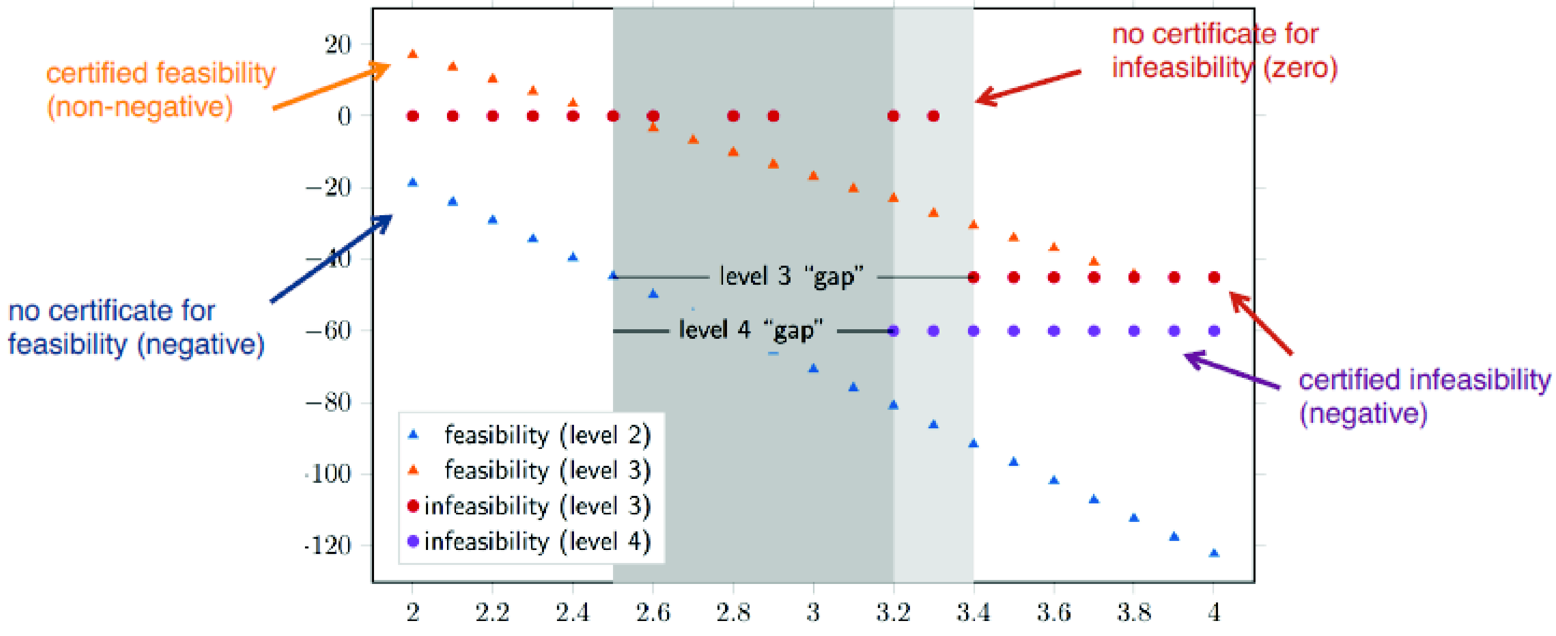
- application of feasibility method (minimization) to $\mathcal{U}_{\text{feas}}$

nodes	$\lvert I \rvert$	# success at level 2	3	4	mean runtime 2	3	4
2	2	1	1	0	0.03 s	0.04 s	0.11 s
3	6	5	1	0	0.04 s	0.11 s	0.61 s
4	12	11	1	0	0.05 s	0.32 s	3.90 s
5	20	19	1	0	0.08 s	1.23 s	26.68 s
6	30	29	1	0	0.15 s	4.53 s	148.40 s
7	42	42	0	0	0.24 s	15.72 s	809.94 s

$\rightarrow$ almost all subproblems confirm set containment at level 2

Strength of Robust Feasibility and Infeasibility Methods

- vary uncertainty set $U(c) = \times_{a \in A}[1, c]$ for $c \in [2, 4]$ on problem 4



- remark: 'gap' may be further tightened by driving level d up ...

Conclusions

- (affinely) adjustable robust optimization is too conservative

full adjustability:

- use polynomial optimization to decide robustness
- separate methods for feasibility / infeasibility
- gas: tree, tree+edge topologies: easy (LP)
- gas: complexity in size of cycle

Difficulty of the Problem

We face a robust

	potential simplification
mixed-integer	continuous (passive networks)
two-level optimization problem with	single-level, affine adjustability, etc.
non-convex quadratic lower level	piecewise-linear relaxed lower level

Two-Stage Robust Gas Network Operation with Compressors

Assmann, L, Stingl

Assumptions: no compressor in a cycle, compressor changes squared pressures

$$\pi_v - \pi_u = \Delta_a, \Delta_a \in [\underline{\Delta}_a, \overline{\Delta}_a] \subseteq \mathbb{R}_{\geq 0}, \quad (u, v) = a \in A_{\mathrm{cs}}$$

cost-minimum operation of compressors yields:

$$\begin{aligned} \min\ & w^T \Delta \\ \mathcal{A}^+ q &= q^{\mathrm{nom}+}, \\ {\mathcal{A}^+}^T \pi &= F(\phi, q, \Delta), \\ \Delta &\in [\underline{\Delta}, \overline{\Delta}] \subseteq \mathbb{R}_{\geq 0}^{|A_{\mathrm{cs}}|} \\ \pi &\in [\underline{\pi}, \overline{\pi}] \subseteq \mathbb{R}_{\geq 0}^{|V|}, \\ q &\in \mathbb{R}^{|A|}. \end{aligned}$$

$$F_a(\phi, q, \Delta) = \begin{cases} -\phi_a q_a |q_a| & \text{if } a \in A_{\mathrm{pi}}, \\ \Delta_a & \text{if } a \in A_{\mathrm{cs}}. \end{cases}$$

Two-Stage Robust Gas Network Operation with Compressors

explicit algebraic description of feasible set (Gotzes et al. (2016)) can be generalized by compressors $\Rightarrow$

$$\min_{\Delta} \left\{ w^\top(\Delta) \,\middle|\, \exists\, \Delta \in \mathbb{R}^{n_1}\ \forall\, u \in U\ \exists\, y \in \mathbb{R}^{n_2} \text{ with } g(y,u) = 0,\ h(\Delta, y, u) \le 0 \right\},$$

1. $g(y,u) = 0$ admit a unique flow-pressure solution $y^*(u)$ for all $u \in \mathcal{U}$.
2. $h(\Delta, y, u) \le 0$ are separable: $h(\Delta, y, u) = s(\Delta) + t(y,u)$.

Lemma

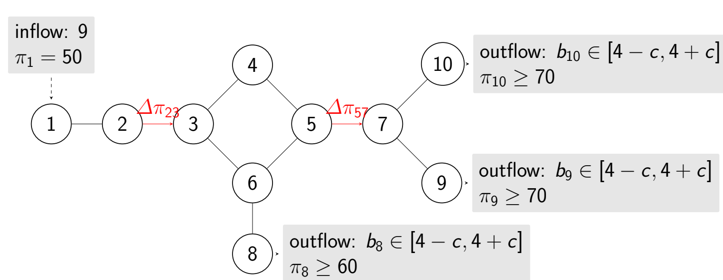
The set of feasible first stage decisions Δ is given by

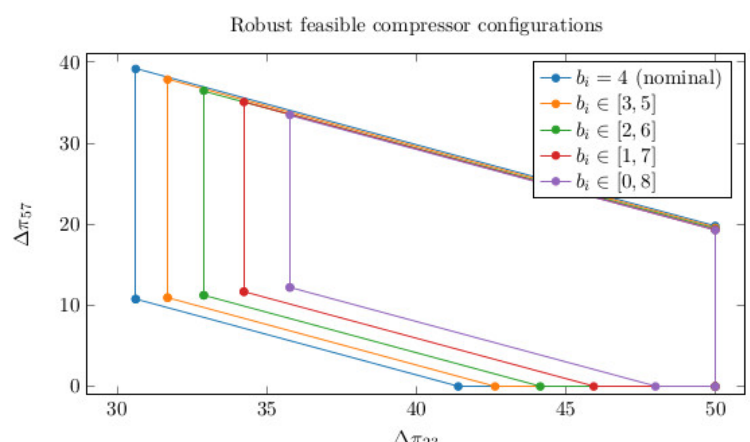
$$\left\{ \Delta \in \mathbb{R}^{n_1} \,\middle|\, s(\Delta) \le b \right\}, \text{ with}$$

$$b_i = -\max_{u \in \mathcal{U}} \left\{ t_i(y,u) \,\middle|\, g(y,u) = 0,\ y \in \mathbb{R}^{n_2} \right\}.$$

Algorithmically Tractable Version of the Problem

- replace in the computation of b_i all non-convex functions by piecewise relaxations (overestimators)
- leads to somewhat more conservative solutions, while preserving robust feasibility
- two-stage robust problem then can be solved simply as a mixed-integer linear problem!
- running times $< 0.01s$ for net10





Summary up to now

- (affinely) adjustable robust optimization is too conservative

full adjustability:

- use polynomial optimization to decide robustness
- separate methods for feasibility / infeasibility
- gas: tree, tree+edge topologies: easy (LP)
- gas: complexity in size of cycle

Overview

- Robust Approaches to Gas Networks: strict + adjustable robustness, full adjustability
- Robust Optimization in Air Traffic Management

Robust Pre-Tactical Planning Air Traffic Management

Hupp, Kapolke, L, Martin, Weismantel

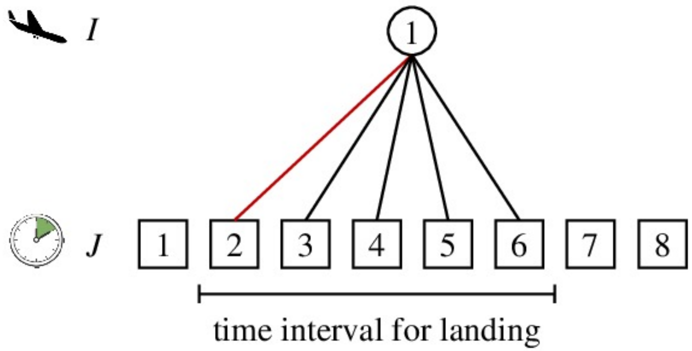
source: tagaytayhighlands.net

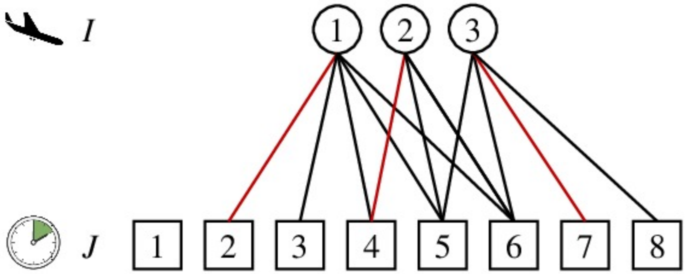
⇒ optimization of runway utilization is one of the main challenges in ATM

goal: runway schedules that are robust against uncertainties

Pre-Tactical Planning Phase

- assign aircraft to time windows
- for each aircraft: possible time interval for landing



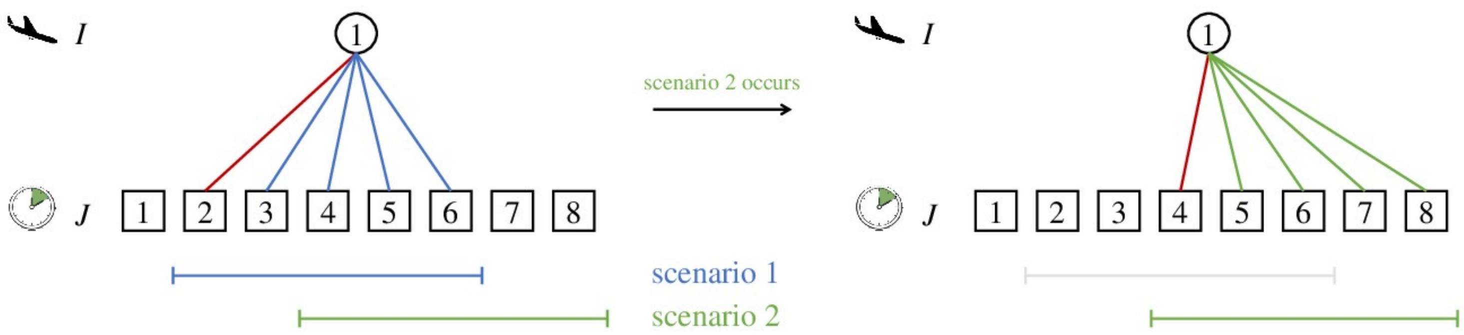


objective:

maximize punctuality i.e. minimize deviation from the (published) flight plan

Pre-Tactical Planning Phase

uncertainty: time interval for landing



⇒ if scenario 2 occurs, plan of scenario 1 might become infeasible

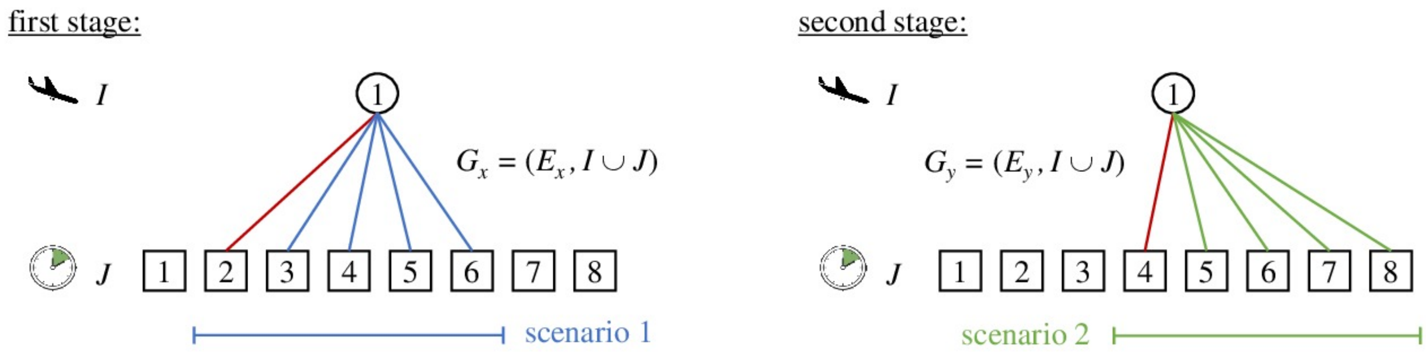
⇒ the aircraft may need to be replanned

Modelling Issues for Robust Pre-Tactical Planning

- fairness
- (recoverable) robust optimization for the plans made the evening before, say
- fast algorithms for recovery actions during operation in case of large disturbances

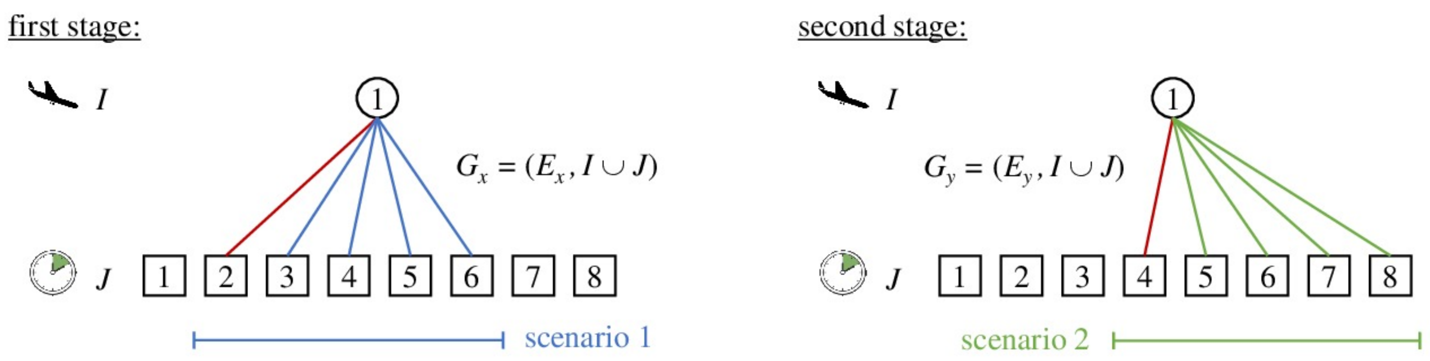
(Not covered here in further detail.)

Pre-Tactical Planning Phase



two-stage optimization task:

Pre-Tactical Planning Phase



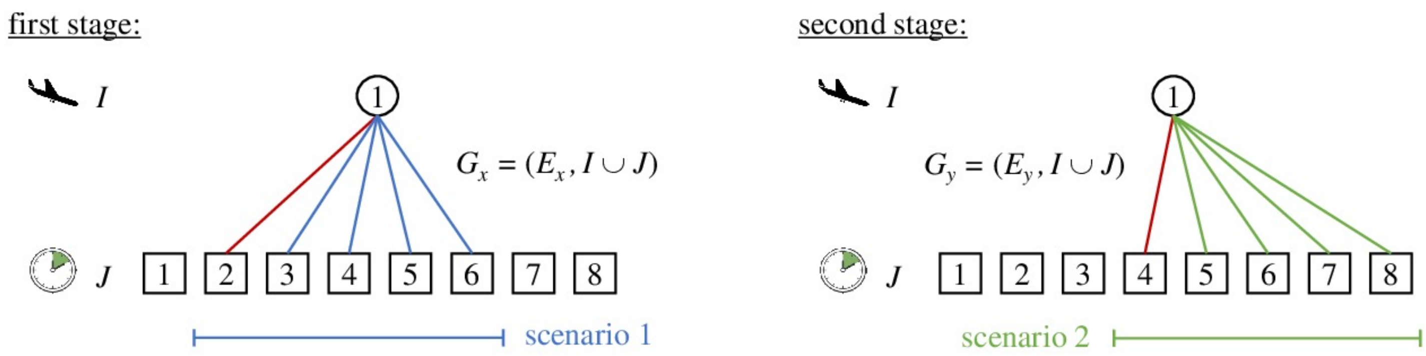
two-stage optimization task:

$$x_{ij} = \begin{cases} 1, & \text{if aircraft } i \text{ is assigned to time window } j \text{ on the } \textbf{first stage} \\ 0, & \text{otherwise} \end{cases}$$

$$y_{ij} = \begin{cases} 1, & \text{if aircraft } i \text{ is assigned to time window } j \text{ on the } \textbf{second stage} \\ 0, & \text{otherwise} \end{cases}$$

Pre-Tactical Planning Phase

For reasons of fairness: restrict replanning for each aircraft by at most r



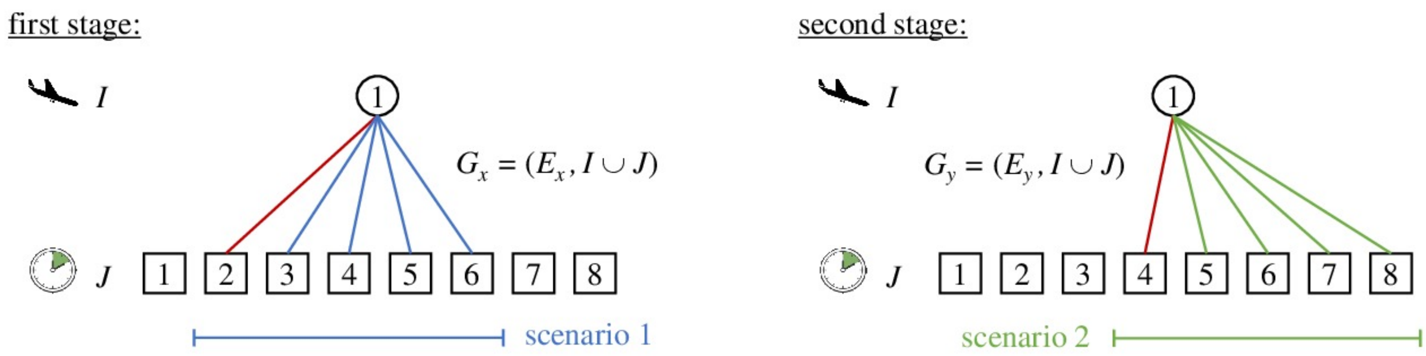
Special knapsack constraint for **each** aircraft:

$$|2x_{12} + 3x_{13} + 4x_{14} + 5x_{15} + 6x_{16} - 4y_{14} - 5y_{15} - 6y_{16} - 7y_{17} - 8y_{18}| \leq r$$

$$x_{ij} \in \{0, 1\} \text{ first stage variables,} \quad y_{ij} \in \{0, 1\} \text{ second stage variables}$$

Pre-Tactical Planning Phase

For reasons of fairness: restrict replanning for each aircraft by at most r



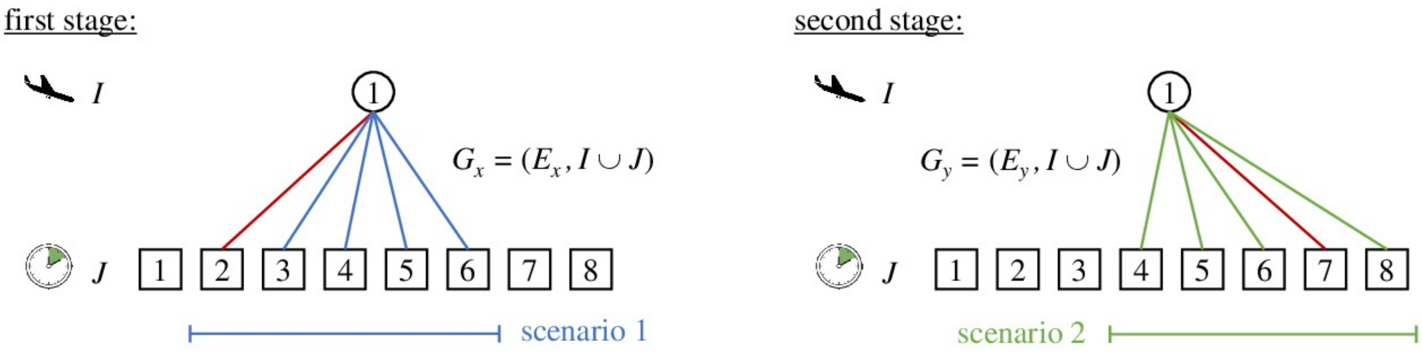
Special knapsack constraint for **each** aircraft:

$$|2x_{12} + 3x_{13} + 4x_{14} + 5x_{15} + 6x_{16} - 4y_{14} - 5y_{15} - 6y_{16} - 7y_{17} - 8y_{18}| \leq 2$$

$x_{ij} \in \{0, 1\}$ first stage variables, $y_{ij} \in \{0, 1\}$ second stage variables

Pre-Tactical Planning Phase

For reasons of fairness: restrict replanning for each aircraft by at most r



Special knapsack constraint for **each** aircraft:

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$$x_{ij} \in \{0, 1\} \text{ first stage variables,} \qquad y_{ij} \in \{0, 1\} \text{ second stage variables}$$

(Towards) robust bipartite b-Matching Problem (RMP)

- minimize deviation from scheduled times
- assign each aircraft to one time window on the first and second stage
- assign at most b aircraft to a time window
- restrict replanning action

Robust bipartite b-Matching Problem (RMP)

$$\min_{x,y} \sum_{(i,j)\in E_x} c^x_{ij} x_{ij} + \sum_{(i,j)\in E_y} c^y_{ij} y_{ij}$$

$$\sum_{\substack{j\in J:\\(i,j)\in E_x}} x_{ij} = 1, \qquad \sum_{\substack{j\in J:\\(i,j)\in E_y}} y_{ij} = 1 \quad \forall\, i \in I \qquad \text{(aircraft)}$$

$$\sum_{\substack{i\in I:\\(i,j)\in E_x}} x_{ij} \le b, \qquad \sum_{\substack{i\in I:\\(i,j)\in E_y}} y_{ij} \le b \quad \forall\, j \in J \qquad \text{(time windows)}$$

$$\left| \sum_{\substack{j\in J:\\(i,j)\in E_x}} j \cdot x_{ij} - \sum_{\substack{j\in J:\\(i,j)\in E_y}} j \cdot y_{ij} \right| \le r_i \quad \forall\, i \in I \qquad \text{(replanning constraints)}$$

$$x_e,\; y_f \in \{0,1\} \qquad \forall\, e \in E_x,\; f \in E_y$$

Mixed Integer Reformulations

- Approach [Bader,Hildebrand,Weismantel,Zenklusen (2016)]
- Solve:

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & x \in \mathbb{Z}^n \end{aligned}$$

Mixed Integer Reformulations

- Approach [Bader,Hildebrand,Weismantel,Zenklusen (2016)]
- Solve:

$$\begin{aligned} \max \quad & c^\top x \\ \text{s.t.} \quad & Ax \leq b \\ & \sout{x \in \mathbb{Z}^n} \quad x \in \mathbb{R}^n, Wx \in \mathbb{Z}^k \end{aligned}$$

Given: Polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, A, b integral
Goal: Find $W \in \mathbb{Z}^{k \times n}$

$$\operatorname{conv}(\{x \in P \mid x \in \mathbb{Z}^n\}) = \operatorname{conv}(\{x \in P \mid Wx \in \mathbb{Z}^k\})$$

- k instead of n integrality constraints

Affine TU Decomposition of Matrix A:

$A, \overline{A}, U, W$ integer matrices

$$A = \overline{A} + UW$$

such that

$$\begin{pmatrix} \overline{A} \\ W \end{pmatrix}$$

is totally unimodular (TU).

Theorem [Bader,Hildebrand,Weismantel,Zenklusen (2016)]:

Let

$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$,

$A = \overline{A} + UW$ be affine TU decomposition,

then

$$\operatorname{conv}(\{x \in P \mid x \in \mathbb{Z}^n\}) = \operatorname{conv}(\{x \in P \mid Wx \in \mathbb{Z}^k\}).$$

Robust bipartite b-Matching Problem with One Replanning Constraint for All Aircraft (RMP1)

$$\min_{x,y} \quad \sum_{ij \in E_x} c^x_{ij} x_{ij} + \sum_{ij \in E_y} c^y_{ij} y_{ij}$$

$$\sum_{\substack{j \in J: \\ (i,j) \in E_x}} x_{ij} = 1, \qquad \sum_{\substack{j \in J: \\ (i,j) \in E_y}} y_{ij} = 1 \qquad \forall\, i \in I \qquad \text{(aircraft} \rightarrow M\text{)}$$

$$\sum_{\substack{i \in I: \\ (i,j) \in E_x}} x_{ij} \leq b, \qquad \sum_{\substack{i \in I: \\ (i,j) \in E_y}} y_{ij} \leq b \qquad \forall\, j \in J \qquad \text{(time windows} \rightarrow M\text{)}$$

$$\left| \sum_{i \in I} \sum_{\substack{j \in J: \\ (i,j) \in E_x}} j \cdot x_{ij} - \sum_{i \in I} \sum_{\substack{j \in J: \\ (i,j) \in E_y}} j \cdot y_{ij} \right| \leq r \qquad \text{(sum of replannings} \rightarrow R\text{)}$$

$$x_e,\ y_f \quad \in \quad \{0, 1\} \qquad \forall\, e \in E_x,\ f \in E_y$$

Affine TU Decomposition for RMP1 with One Replanning Constraint

Goal: affine TU decomposition

$$\begin{pmatrix} M \\ R \end{pmatrix} = \overline{A} + UW \quad \text{and} \quad \begin{pmatrix} \overline{A} \\ W \end{pmatrix} \quad \text{is TU.}$$

(work in progress.)

Conclusions

- tradeoff between conservatism and algorithmic tractability can nicely be seen in gas network operation.
- full adjustability via polynomial optimization
- (or via piecewise linearization when compressors are involved, with somewhat increased conservatism)
- affine TU decompositions can successfully be applied for pretactical planning under uncertainty.

Thank you very much!