Chennai, India

In-vehicle congestion

Sao Paulo Metro Station

Platform congestion

London Tube

Bus stop congestion

Problem

$$\text{Given} \begin{cases} \text{transit network} & (V, A) \\ \text{travel demands} & g_i^d \geq 0 \\ \text{arc travel times} & t_a = s_a(v_a) \\ & \end{cases}$$

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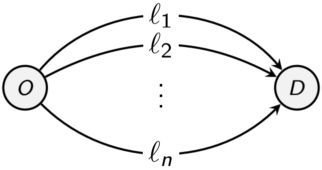
Determine:

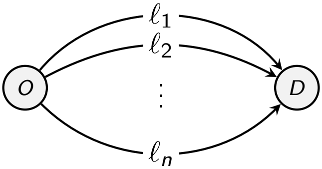
- passenger flows on line segments
- travel times for each OD pair
- waiting times at bus stops

Common lines – uncongested (Chriqui-Robillard'75)



Travel times	$t_1 \leq \ldots \leq t_n$
Frequencies	$\mu_1, \ldots, \mu_n$
Strategies	$s \subseteq L$

Common lines – uncongested (Chriqui-Robillard'75)



Travel times $t_1 \leq \ldots \leq t_n$
Frequencies $\mu_1, \ldots, \mu_n$
Strategies $s \subseteq L$

$$T_s = W_s + \sum_{a \in s} t_a \, \pi_s^a = \frac{1 + \sum_{a \in s} t_a \, \mu_a}{\sum_{a \in s} \mu_a}$$

$$\tau = \min_{s \subseteq L} T_s$$

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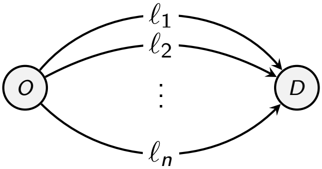

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Theorem (Chriqui-Robillard'75)

Optimal strategy: $s^* = \{1, 2, \ldots, k^*\}$ *take the* k^* *fastest lines.*

$\Rightarrow$ linear time algorithm: $s^* = \{a : t_a \leq T_{s^*}\}$

Extension to networks

Spiess'1984, Gendreau'1984, Nguyen-Pallottino'1988, Spiess-Florian'1989, Fernández-De Cea'1989, Wu-Florian-Marcotte'1994, Bouzaiene-Gendreau-Nguyen'1995

Extension to networks

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These models overlook capacity effects.

- ✓ On-board: increased discomfort
- ✓ Bus stops: increased waiting times
- ✓ Boardings & alightings: increased travel times
- ✗ Boarding probabilities: changes in flow distribution

Key point: estimate correctly the flows !

Denied boardings deferred to other services

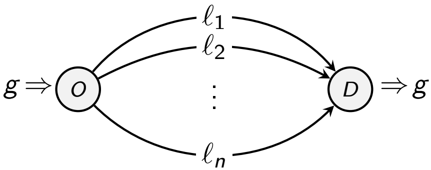
Sydney

"Sydney bus and train commuters say overcrowding is still their main public transport concern."
http://www.abc.net.au/news/stories/2010/12/31/3104302.htm?site=sydney

"Buses in Sydney on the busiest routes are often overcrowded and do not stop for passengers, with an extraordinary 22% of people missing their service."
http://www.2ue.com.au/blogs/2ue-blog/crowded-buses-just-not-stopping/20111130-1o6f1.html

Emma Freijinger, November 2013

Common lines with congestion



Split the demand $g = \sum_{s \subseteq L} x_s$ so that only optimal strategies are used

$$x_s > 0 \quad \Rightarrow \quad T_s \triangleq W_s + \sum_{a \in s} t_a \, \pi_s^a \ \text{ is minimal}$$

$$\begin{cases} W_s = W_s(x) & ? \\ \pi_s^a = \pi_s^a(x) & ? \end{cases}$$

... queueing theory

Bulk queues $M|M|1$

$$v \longrightarrow \bigcirc \longrightarrow \mu, c \qquad (\lambda < \mu C)$$

Little's formula yields the waiting time

$$W(v) = \frac{1}{v}\mathbb{E}(L) = \frac{1}{v}\frac{\rho}{1-\rho}$$

where $\rho = \rho(v) \in [0,1]$ is the solution of

$$\mu(\rho + \rho^2 + \cdots + \rho^c) = v.$$

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Definition

Effective frequency is defined as $f(v) \triangleq \frac{1}{W(v)}$

REMARK: for light demand $v \sim 0$ we have $f(v) \sim \mu$.

Strategies with congestion

Let $s \subseteq L$ and denote $f_a(v_a)$ the effective frequency of each line $a \in L$.

Lemma (C-Correa'2001)

Assume Poisson arrivals with rate x_s. Then the expected flows v_a on the lines $a \in s$ are the unique solution of the system

$$(E) \qquad v_a = x_s \frac{f_a(v_a)}{\sum_{b \in s} f_b(v_b)}$$

Moreover, setting $f_a = f_a(v_a)$ we have

$$\begin{aligned} W_s &= 1/\textstyle\sum_{b \in s} f_b \\ \pi_s^a &= f_a/\textstyle\sum_{b \in s} f_b \\ T_s &= \frac{1+\sum_{a \in s} t_a f_a}{\sum_{a \in s} f_a} \end{aligned}$$

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When there are several x_s's we have $v_a = v_a(x)$, $f_a = f_a(x)$,...
No analytic expressions available... yet !

Common lines – Equilibrium model

We postulate $f_a = f_a(v_a)$ as a decreasing function of the line's load, and we denote $v = v(x)$ the unique solution of the system

$$(E) \qquad v_a = \textstyle\sum_s x_s\, \pi_s^a = \displaystyle\sum_{s \ni a} x_s\, \frac{f_a(v_a)}{\sum_{b\in s} f_b(v_b)}$$

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Equilibrium: Split the demand $g = \sum_s x_s$ with $x_s \geq 0$ so that

$$(W) \qquad x_s > 0 \;\Rightarrow\; T_s(v) = \tau(v)$$

where

$$\begin{aligned} T_s(v) &= \frac{1 + \sum_{a \in s} t_a \, f_a(v_a)}{\sum_{a \in s} f_a(v_a)} \\ \tau(v) &= \min_{s \subseteq L} T_s(v) \end{aligned}$$

Characterization & Existence

Theorem (C-Correa'2001)

Let $\bar{v}_a(\alpha)$ be the inverse function of $v_a \mapsto \frac{v_a}{f_a(v_a)}$ and denote $\tau_\alpha \triangleq \tau(\bar{v}(\alpha))$. Then the equilibrium flows v_a are the optimal solutions of

$$\min_{(\alpha,v)} \sum_{a\in L} \left[t_a v_a + \int_0^\alpha [\tau_\xi - t_a]_+ \bar{v}_a'(\xi)\, d\xi \right]$$

$$\textstyle\sum_{a\in L} v_a = g$$

$$0 \leq v_a \leq \bar{v}_a(\alpha).$$

In particular, there exists an equilibrium.

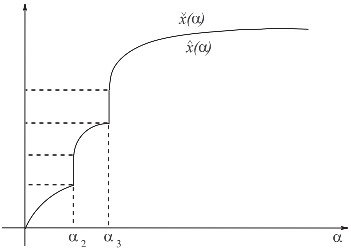
REMARK: Once α is known, the flows v are easily obtained.

Characterization & Existence

Theorem (C-Correa'2001)

Let α *be the unique solution of* $\sum_{t_a<\tau_\alpha} \bar{v}_a(\alpha) \leq g \leq \sum_{t_a\leq\tau_\alpha} \bar{v}_a(\alpha)$.
Then v *is an equilibrium iff* $\sum_{a\in L} v_a = g$ *with*

$$\frac{v_a}{f_a(v_a)} \begin{cases} = \alpha & \text{if } t_a < \tau_\alpha \\ \leq \alpha & \text{if } t_a = \tau_\alpha \\ = 0 & \text{if } t_a > \tau_\alpha \end{cases}$$

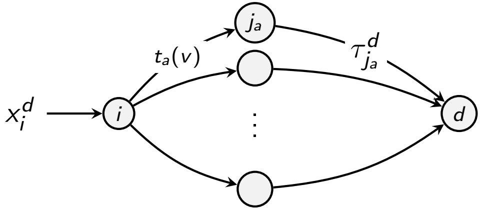


Comments

- Existence + characterization + conditions for uniqueness
- Constant time for some ranges of demand
- Co-existence of multiple equilibrium strategies
- Inefficient equilibria... Braess-type paradox
- Model consistent with simulations

Network Transit Equilibrium

Family of common line problems coupled by flow conservation.



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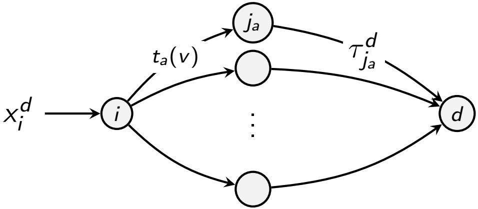


Generalized Bellman Equations:

$$\tau^d_d = 0 \; ; \; \tau^d_i = \min_{s \subseteq A_i^+} \frac{1+\sum_{a\in s}[t_a(v)+\tau^d_{j(a)}]f_a(v)}{\sum_{a\in s} f_a(v)}$$

Flow Conservation:

$$x^d_i \triangleq g^d_i + \sum_{A_i^-} v^d_a = \sum_{A_i^+} v^d_a$$

Existence of equilibria: Kakutani's Fixed Point Theorem

Characterization

Theorem (Cepeda-C-Florian'2006)

(v_a^d) are equilibrium flows iff there exist (α_i^d) such that

$$\frac{v_a^d}{f_a(v)} \begin{cases} = \alpha_i^d & \text{if } t_a(v) + \tau_{j(a)}^d(v) < \tau_i^d(v) \\ \leq \alpha_i^d & \text{if } t_a(v) + \tau_{j(a)}^d(v) = \tau_i^d(v) \\ = 0 & \text{if } t_a(v) + \tau_{j(a)}^d(v) > \tau_i^d(v) \end{cases}$$

These are precisely the optimal solutions of

$$\min_{(v_a^d)} \sum_d \left[\sum_{a \in A} t_a(v) v_a^d + \sum_{i \in V} \max_{a \in A_i^+} \frac{v_a^d}{f_a(v)} - \sum_{i \in V} g_i^d \tau_i^d(v) \right]$$

$$s.t. \; g_i^d + \sum_{A_i^-} v_a^d = \sum_{A_i^+} v_a^d$$

Applications

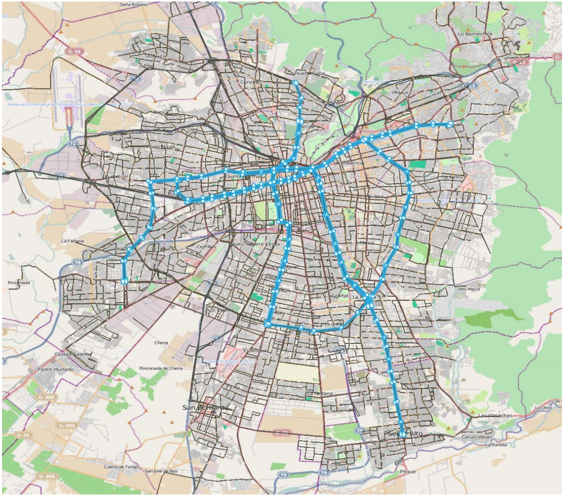
Model implemented as a macro within EMME software (INRO).

Has been used in

- 2005: Sao Paulo
- 2010: San Francisco, Bangkok
- 2011: Sidney, Mexico City, Los Angeles,
- 2012: Santiago
- 2013: Brisbane, Rio de Janeiro
- 2014: Stockholm
- 2016: Toronto

Source: Michael Florian (INRO)

Application: Santiago (developed by SECTRA, 2012)



Emme Model – Model Construction

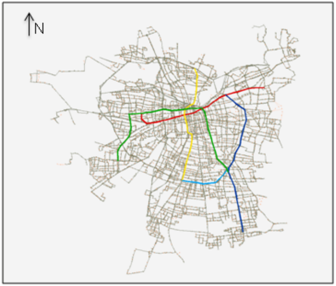
Trunk Routes (2011 coverage)

Feeder Routes (2011 coverage)

Regular nodes:	20,093
Stops and stations:	12,500
Centroids:	2,916
Regular links:	60,528
Connectors:	24,714

(software size and license)

Metro Lines (2011 coverage)



Trunk routes: 390 service-direction / 3,700 vehicles / 7,800 km

Feeder routes: 430 service-direction / 2,200 vehicles / 5,100 km

Source: SECTRA

Metro: 150 vehicles (trains) / 5 lines / 200 km

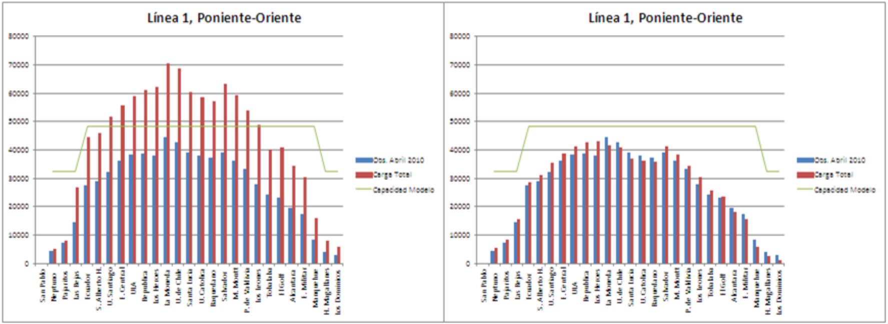
Gobierno de Chile | Ministerio de Transportes y Telecomunicaciones

13

Flows on all transit modes

Emme Model – Results with Capacity Considerations

Etapas Escenario Sin Rest. Capacidad	Etapas Escenario Con Rest. Capacidad



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Emme Model – Results with Capacity Considerations

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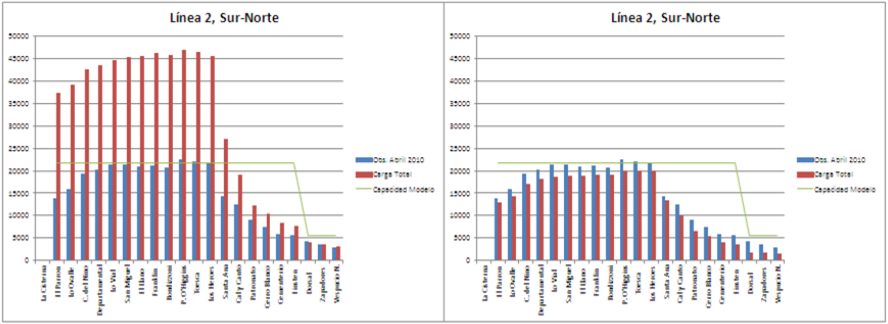

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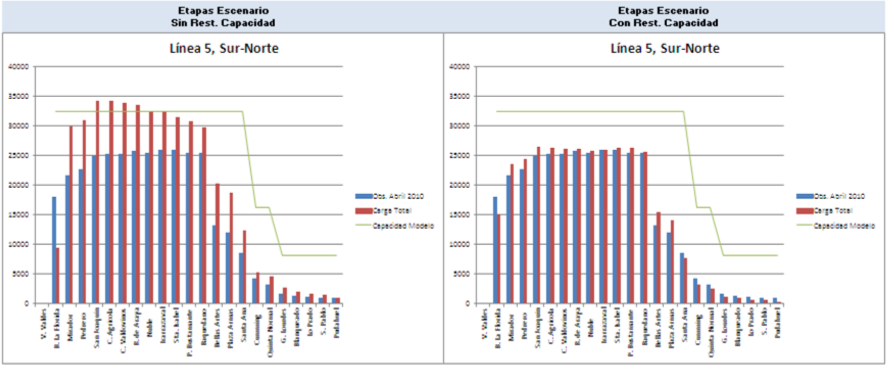

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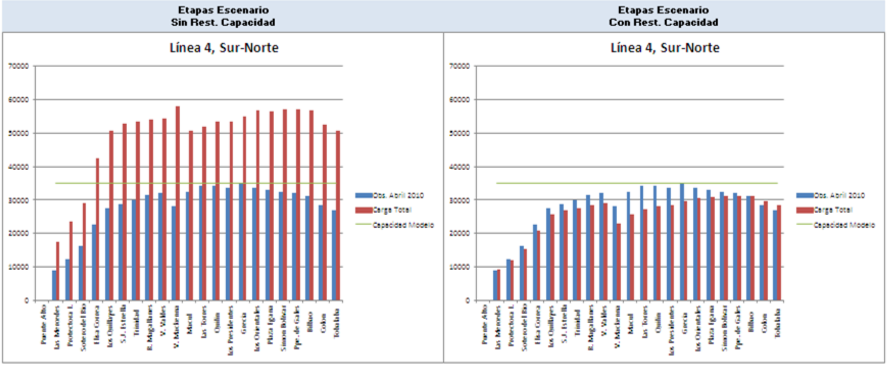
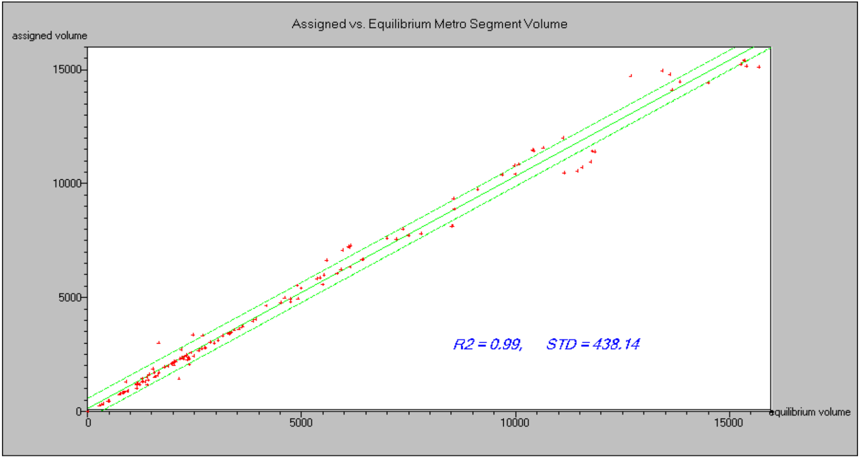


Source: SECTRA

Assigned vs. metro counts segment volume



Emma Freijinger, November 2013

3 Research opportunities

- Adaptive dynamics
- Dynamic equilibrium
- TCP/IP multipath routing

Adaptive dynamics

- Are drivers fully rational? Do they have full information?
- Do myopic adaptive dynamics support equilibrium?
- Recent results by C-Melo-Sorin'2010 and Bravo'2011 provide partial answers and lead to other notions of equilibria

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- Are drivers fully rational? Do they have full information?
- Do myopic adaptive dynamics support equilibrium?
- Recent results by C-Melo-Sorin'2010 and Bravo'2011 provide partial answers and lead to other notions of equilibria
- Many open questions remain !
  - almost sure convergence under small noise
  - speeds of convergence of stochastic adaptive dynamics
  - multiplicity and bifurcations of equilibria
  - large population asymptotics
  - more realistic adaptive dynamics
  - robustness under model specification

Dynamic equilibrium

- Traffic equilibrium under automated guidance software tools
- Dynamic equilibrium more appropriate than static equilibrium
- Recent progress for fluid queuing networks:
  "derivatives of equilibria $\equiv$ *normalized thin flows with resetting"*
  - characterization & computation of equilibria: Koch-Skutella'2010
  - characterization & existence of equilibria: C-Correa-Larré'2015
  - long-term behavior of equilibria: C-Correa-Olver'2017

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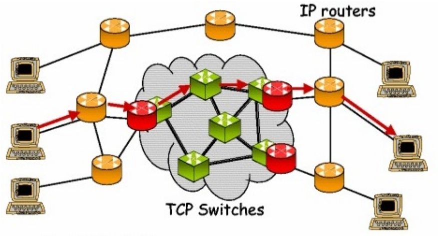
- Many open questions remain !
  - finite convergence of reconstruction algorithm
  - efficient computation of NTFR's and equilibria
  - multiple origin-destination networks
  - other link dynamics: LWR, spillbacks

TCP/IP multipath routing

- Each source $s \in S$ transmits packets from o_s to d_s
- TCP = At which rate? / IP = Along which route?
- Random delays $\tilde{t}_a$ = queuing + transmission + propagation
- Finite queuing buffers $\Rightarrow$ packet loss probabilities p_a



- Congestion control – TCP Reno/Tahoe/Vegas
  Sources adjust transmission rates in response to congestion.
  Feedback mechanism: *higher congestion* $\Leftrightarrow$ *smaller rates*.

Congestion control and multipath routing

- Kelly *et al.* 1998: Steady states of TCP protocols are
  - Equilibria for an associated potential game
  - TCP reverse-engineered as a decentralized asynchronous algorithm that solves a network optimization problem

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- Many open questions remain !
  - Convergence analysis accounting for stochastics & delays
  - Increase transmission rates: multi-path routing
  - Combine NUM with MTE (C-Guzmán'2014)
  - Design stable packet-level protocols

Thanks!

Reprints available at

https://sites.google.com/site/cominettiroberto/