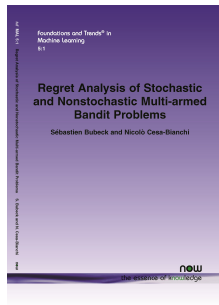


Lecture 1: Introduction to regret analysis

Sébastien Bubeck

Machine Learning and Optimization group, MSR AI

Microsoft®
Research



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Performance measure: The regret is the difference between the player's accumulated loss and the minimum loss she could have obtained had she known all the adversary's choices:

$$R_T := \mathbb{E} \sum_{t=1}^T \ell_t(i_t) - \min_{i \in [n]} \mathbb{E} \sum_{t=1}^T \ell_t(i) =: L_T - \min_{i \in [n]} L_{i,T}.$$

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What's it about? Full information game is about *hedging*, while bandit game also features the fundamental tension between *exploration* and *exploitation*.

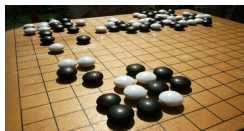
Applications

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AI for games



Brain computer interface



Medical trials



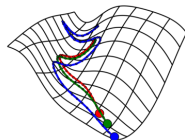
Packets routing



Ad placement



Hyperparameter opt



Hedging with multiplicative weights [Freund and Schapire 96, Littlestone and Warmuth 94, Vovk 90]

Assume for simplicity $\ell_t(i) \in \{0, 1\}$. MW keeps weights $w_{i,t}$ for each action, plays from normalized weights, and update as follows:

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Key insight: if i^* does not make a mistake on round t then we get “closer” to δ_{i^*} (i.e., we learn), and otherwise we might get confused but i^* had to pay for it.

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Theorem

For any $\eta \in [0, 1/2]$ and $i \in [n]$,

$$L_T \leq (1 + \eta)L_{i,T} + \frac{\log(n)}{\eta}.$$

By optimizing η one gets $R_T \leq 2\sqrt{T \log(n)}$.

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Note that $\Omega(\sqrt{T \log(n)})$ is the best one could hope for.

Potential based analysis

Define $\psi(t) = \sum_{i=1}^n w_{i,t}$. One has:

$$\psi(t+1) = \sum_{i=1}^n (1 - \eta \ell_t(i)) w_{i,t} = \psi(t)(1 - \eta \langle p_t, \ell_t \rangle),$$

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The mirror descent framework (Lec. 2) will give a principled approach to derive both the MW algorithm and its analysis.

A principled game-theoretic approach to regret analysis

[Abernethy, Warmuth, Yellin 2008; Rakhlin, Sridharan, Tewari 2010; B., Dekel, Koren, Peres 2015]

Let us focus on an oblivious adversary, that is he chooses $\ell_1, \dots, \ell_T \in \mathcal{L}$ at the beginning of the game.

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where the swap of min and max comes from Sion's minimax theorem.

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In other words we can study the minimax regret by designing a strategy for a *Bayesian* scenario where $\ell \sim \nu$ and ν is known.

A Doob strategy [B., Dekel, Koren, Peres 2015]

Since we know ν , we also know the *distribution* of i^* . In fact as we make observations, we can update our knowledge of i^* with the *posterior distribution*. Denote \mathbb{E}_t for this posterior distribution (e.g., in full information $\mathbb{E}_t := \mathbb{E}[\cdot | \ell_1, \dots, \ell_{t-1}]$).

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The regret of this strategy can be controlled via the *movement* of this Doob martingale (recall $\|\ell_t\|_\infty \leq 1$)

$$\mathbb{E} \sum_{t=1}^T \langle p_t - \delta_{i^*}, \ell_t \rangle = \mathbb{E} \sum_{t=1}^T \langle p_t - p_{t+1}, \ell_t \rangle \leq \mathbb{E} \sum_{t=1}^T \|p_t - p_{t+1}\|_1 .$$

How stable is a martingale?

Question: is a martingale in Δ_n “stable”? Following famous inequality is a possible answer (proof on the next slide):

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Thus we have recovered the regret bound of MW (in fact with an optimal constant) by a purely geometric argument!

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Now essentially by definition one has (recall that $I(X, Y) = H(X) - H(X|Y) = \mathbb{E}_Y \text{Ent}(p_{X|Y}; p_X)$):

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Proof concluded by telescopic sum and maximal entropy being $\log(n)$.

A more general story: M-cotype

Let us generalize the setting. In *online linear optimization*, the player plays $x_t \in K \subset \mathbb{R}^n$, and the adversary plays $\ell_t \in \mathcal{L} \subset \mathbb{R}^n$. We assume that there is a norm $\|\cdot\|$ such that $\|x_t\| \leq 1$ and $\|\ell_t\|^* \leq 1$.

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$$\mathbb{E} \sum_{t=1}^T \langle \ell_t, x_t - x^* \rangle = \mathbb{E} \sum_{t=1}^T \langle \ell_t, x_t - x_{t+1} \rangle \leq \mathbb{E} \sum_{t=1}^T \|x_t - x_{t+1}\|.$$

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The norm $\|\cdot\|$ has M -cotype (C, q) if for any martingale (x_t) one has:

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In particular this gives a regret in $C T^{1-1/q}$.

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Important: these are “dimension-free arguments”, if one brings the dimension in the bounds then the story changes.

What about the bandit game? [Russo, Van Roy 2014]

So far we only talked about the *hedging* aspect of the problem. In particular for the full information game the “learning” part happens automatically. This is captured by the fact that the **variation in the posterior is lower bounded by the instantaneous regret**:

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Importantly note that the cotype inequality for ℓ_1 is proved by relating the ℓ_1 variation squared to the mutual information between OPT and the feedback. Thus a weaker inequality that would suffice is:

$$\mathbb{E}_t \langle \mathbf{p}_t - \delta_{i^*}, \ell_t \rangle \leq C \sqrt{I_t(i^*, (i_t, \ell_t(i_t)))},$$

which would lead to a regret in $C\sqrt{T \log(n)}$.

The Russo-Van Roy analysis

Let $\bar{\ell}_t(i) = \mathbb{E}_t \ell_t(i)$ and $\bar{\ell}_t(i, j) = \mathbb{E}_t(\ell_t(i) | i^* = j)$. Then

$$\mathbb{E}_t \langle p_t - \delta_{i^*}, \ell_t \rangle = \sum_i p_t(i) (\bar{\ell}_t(i) - \bar{\ell}_t(i, i)),$$

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$$I_t((i_t, \ell_t(i_t)), i^*) = \sum_{i, j} p_t(i) p_t(j) \text{Ent}(\mathcal{L}_t(\ell_t(i) | i^* = j) \| \mathcal{L}_t(\ell_t(i)))$$

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$$\sqrt{n \sum_i p_t(i)^2 (\bar{\ell}_t(i) - \bar{\ell}_t(i, i))^2} \leq \sqrt{n \sum_{i, j} p_t(i) p_t(j) (\bar{\ell}_t(i) - \bar{\ell}_t(i, j))^2}.$$

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Thus one obtains

$$\mathbb{E}_t \langle \mathbf{p}_t - \delta_{i^*}, \ell_t \rangle \leq \sqrt{n I_t((i_t, \ell_t(i_t)), i^*)}.$$