Optimization and Games in Congested Networks

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Models to predict flows under congestion

Outline

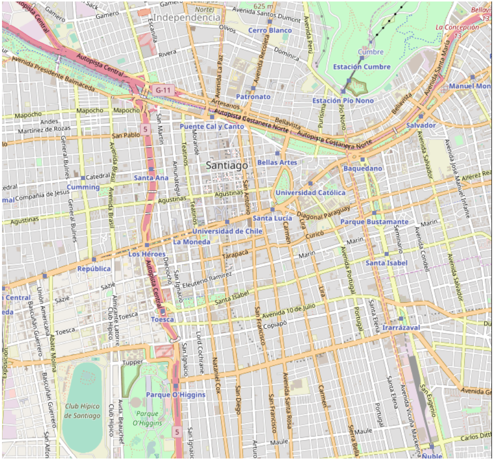
1. Traffic equilibrium
2. Transit equilibrium
3. Research opportunities
    - Adaptive dynamics
    - Dynamic equilibrium
    - TCP/IP multipath routing

Based on joint work with:

Jean-Bernard Baillon, Pablo Beltrán, Manuel Cepeda, Riccardo Colini-Baldeschi, José Correa, Miguel Dumett, Michael Florian, Cristóbal Guzmán, Emerson Melo, Panayotis Mertikopoulos, Marco Scarsini, Sylvain Sorin, Alfredo Torrico

1. **Traffic equilibrium**

Traffic flows under congestion



2266 nodes / 7636 arcs / 409 centroids

SANTIAGO
6.000.000 people
11.000.000 daily trips
1.750.000 car trips

Morning peak
500.000 car trips
29.000 OD pairs

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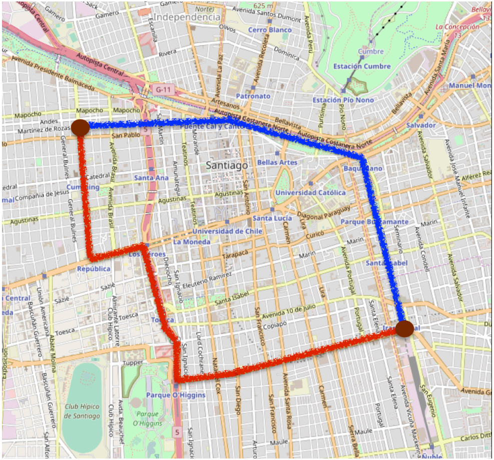

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Wardrop Equilibrium (Wardrop'52)

$$\text{Given} \begin{cases} \text{traffic network} & (V, A) \\ \text{travel demands} & g_i^d \geq 0 \\ \text{arc travel times} & t_a = s_a(v_a) \end{cases}$$

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Split $g_i^d = \sum_{r \in \mathcal{R}_i^d} x_r$ with $x_r \geq 0$ so that only shortest routes are used

$$\boxed{x_r > 0 \;\Rightarrow\; T_r = \tau_i^d}$$

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where

$$T_r = \textstyle\sum_{a \in r} s_a(v_a) \quad \text{(route times)}$$

$$v_a = \textstyle\sum_{r \ni a} x_r \quad \text{(total arc flows)}$$

$$\tau_i^d = \min_{r \in \mathcal{R}_i^d} T_r \quad \text{(minimal times)}$$

Variational characterization (Beckman-McGuire-Winsten'56)

$$(P) \quad \begin{cases} \text{Min} \quad \sum_a \int_0^{v_a} s_a(z)\,dz \\ \text{s.t. flow conservation} \end{cases}$$

$\Rightarrow$ There exists a unique equilibrium v^*

Dual characterization (Fukushima'84)

Change of variables: $v_a \leftrightarrow t_a$

$$(D) \qquad \mathop{\mathrm{Min}}_{t} \quad \underbrace{\sum_a \int_0^{t_a} s_a^{-1}(z)\, dz - \sum_{i,d} g_i^d \tau_i^d(t)}_{\substack{\phi(t) \\ \text{strictly convex}}}$$

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$$t \mapsto \tau_i^d(t) \quad = \quad \text{minimum travel time}$$

concave, non-smooth, polyhedral

$$\boxed{\tau_i^d = \min_{a \in A_i^+} [t_a + \tau_{j_a}^d]}$$

Bellman's equations

Quantifying inefficiency: Price-of-Anarchy

$$\text{Social cost} = \text{Total travel time} = \sum_{a\in A} x_a\, s_a(x_a)$$

$$\text{PoA} = \frac{\text{Cost of Equilibrium}}{\text{Minimal Cost}} \geq 1$$

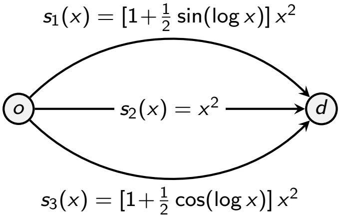
Theorem (Roughgarden-Tardos'2002, Roughgarden'2003)

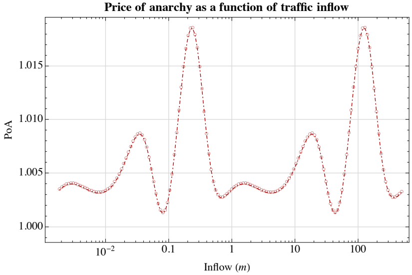
- PoA $\leq \frac{4}{3}$ *for affine costs.*
- PoA $\sim O(d/\log d)$ *for polynomials of degree d.*

Worst-case bounds overly pessimistic on real instances.
In practice PoA ≈ 1 under low or high congestion.
True for a large class of costs but false in general !

Quantifying inefficiency: Price-of-Anarchy

PoA may oscillate and stay away from 1 even for simple networks with smooth strongly convex costs:





Theorem (Colini-C-Mertikopoulos-Scarsini, 2016, 2017)

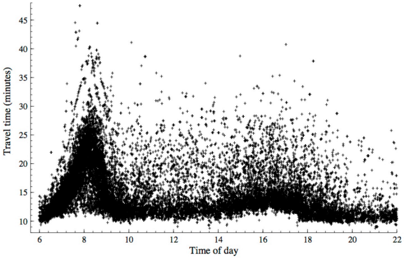
- *Regularly varying costs:* PoA $\to 1$ *in the high congestion regime.*
- *Polynomial costs:* PoA $\to 1$ *plus sharp convergence rates.*

What if travel times are uncertain?

Copenhagen – *Source: DTU Transport (www.transport.dtu.dk)*

Figure 2: Example of real time illustration of congestion (Source: Vejdirektoratet, www.trafikken.dk)

Figure 7: Observations of travel time by time of day. Frederikssundsvej, inward direction



Stochastic User Equilibrium (Dial'71, Fisk'80)

Drivers have different assessments for route travel times

$$\left.\begin{array}{l}\tilde{t}_a = t_a + \epsilon_a \\ \tilde{T}_r = \sum_{a\in r} \tilde{t}_a \\ \tilde{\tau}_i^d = \min_{r\in\mathcal{R}_i^d} \tilde{T}_r\end{array}\right\} \text{ random variables}$$

with $t_a = s_a(v_a)$ and $v_a = \sum_{r\ni a} x_r$ as before, and $\mathbb{E}(\epsilon_a) = 0$.

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Demand splits according to the pbb of each route being optimal

$$\boxed{x_r = g_i^d\, \mathbb{P}(\tilde{T}_r = \tilde{\tau}_i^d)}$$

LOGIT MODEL (Dial'71, Fisk'80)

ϵ_r i.i.d. Gumbel noise (supported by Gnedenko's theorem)

$$x_r = g_i^d \frac{\exp(-\beta T_r)}{\sum_{s\in\mathcal{R}_i^d}\exp(-\beta T_s)}$$

Drawback: independence is unlikely

PROBIT MODEL (Daganzo'82)

ϵ_r correlated Normal noise

No closed form equations $\Rightarrow$ Montecarlo

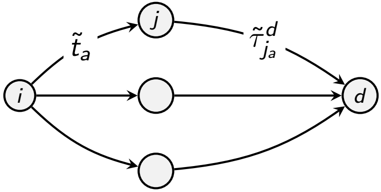
Drawback: tractable only for very small networks

Markovian Traffic Equilibrium (Akamatsu'00, Baillon-C'06)

Routing as a stochastic dynamic programming process

$$\left.\begin{aligned} \tilde{t}_a &= t_a + \epsilon_a \\ \tilde{T}_r &= \textstyle\sum_{a\in r} \tilde{t}_a \\ \tilde{\tau}_i^d &= \min_{r\in\mathcal{R}_i^d} \tilde{T}_r \end{aligned}\right\} \text{ random variables}$$

At every intermediate node i, users select a *random optimal arc*



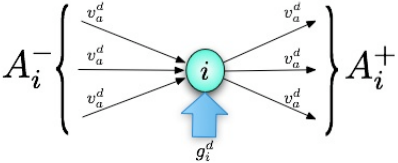
$$\operatorname*{argmin}_{a\in A_i^+} \tilde{t}_a + \tilde{\tau}^d_{j_a}$$

$\Rightarrow$ Markov chain for each destination d

MTE equations

Expected in-flow

$$x_i^d = g_i^d + \sum_{a\in A_i^-} v_a^d$$



leaves node i according to

$$v_a^d = x_i^d\,\mathbb{P}(\tilde{t}_a + \tilde{\tau}_{j_a}^d \leq \tilde{t}_b + \tilde{\tau}_{j_b}^d\ \forall\, b \in A_i^+)$$

where $t_a = s_a(v_a)$ and $v_a = \sum_d v_a^d$

Variational formulation

$$\tilde{\tau}_i^d = \min_{a \in A_i^+} \{\tilde{t}_a + \tilde{\tau}_{j_a}^d\}$$

Theorem (Baillon-C'06)

$\tau_i^d = \mathbb{E}(\tilde{\tau}_i^d)$ is the unique solution of the stochastic Bellman equations

$$\tau_d^d = 0 \quad ; \quad \tau_i^d = \mathbb{E}(\min_{a \in A_i^+} \{t_a + \tau_{j_a}^d + \varepsilon_a^d\}).$$

Moreover $t \mapsto \tau_i^d(t)$ is concave & smooth, and MTE is characterized by

$$(D) \qquad \operatorname*{Min}_t \ \phi(t) \triangleq \sum_a \int_0^{t_a} s_a^{-1}(x)\, dx - \sum_{i,d} g_i^d \tau_i^d(t).$$

...same formulation as Wardrop equilibrium !

Method of Successive Averages

- Compute current arc travel times $t_a^k = s_a(v_a^k)$
- Solve stochastic Bellman equations
- Compute invariant measures of Markov chains $\tilde{v}_a^d$
- Aggregate flows $\tilde{v}_a = \sum \tilde{v}_a^d$
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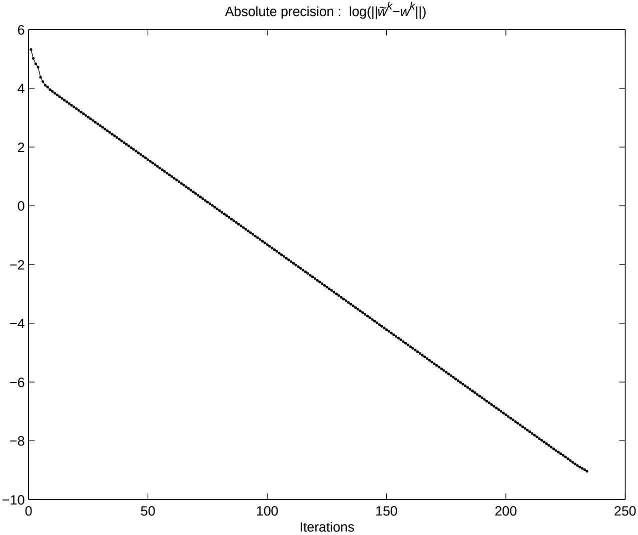
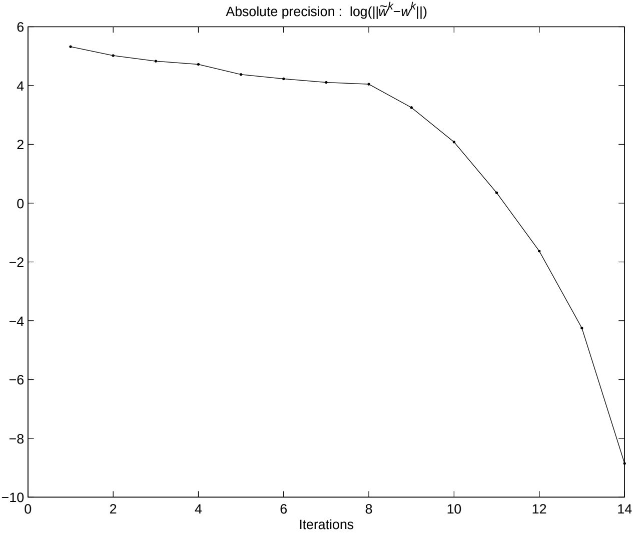
Theorem (Baillon-C'06)

$\sum \alpha_k = \infty$ *and* $\sum \alpha_k^2 < \infty \Rightarrow$ *convergence to MTE*

Stochastic MSA iterations



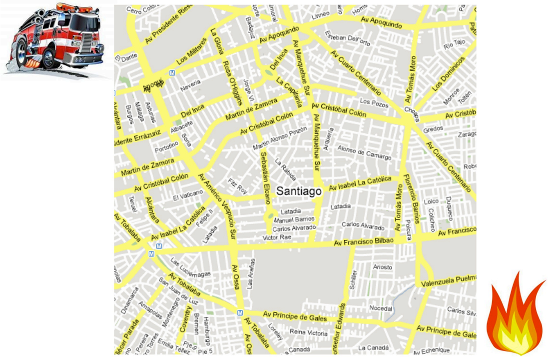
Stochastic MSA-Newton iterations



Risk-averse routing

So far we focused on expected travel times $\equiv$ risk-neutral approach.

What is the risk of a route with random travel time $\tilde{T}_r = \sum_{a \in r} \tilde{t}_a$?



Two natural axioms for a risk map $X \mapsto \phi(X)$

Standard risk maps ? Markowitz, VaR(α), CVaR(α)

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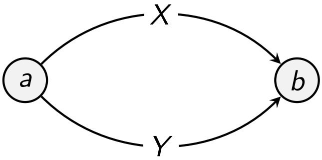

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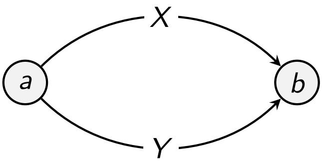

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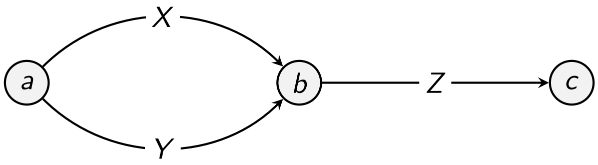

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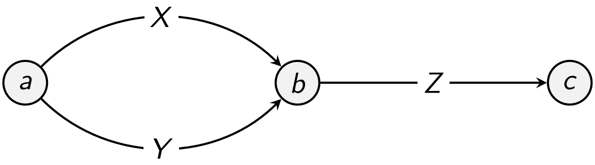

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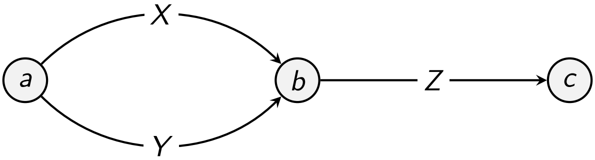

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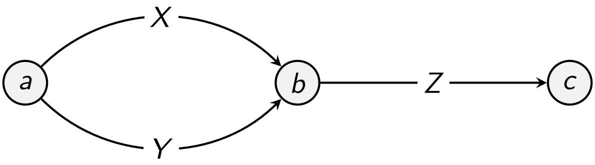


Remark: Independent link travel times $\Rightarrow \phi(\tilde{T}_r) = \sum_{a \in r} \phi(\tilde{t}_a)$

- Risk-optimal path $\equiv$ shortest path with lengths $\ell_a = \phi(\tilde{t}_a)$.
- Risk-equilibrium $\equiv$ Wardrop model for $s_a(v_a) = \phi(\tilde{t}_a)$ with $\tilde{t}_a \sim F(v_a)$.

Theories of choice & Risk aversion

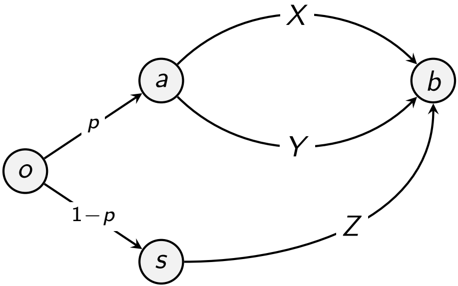
Expected utility: $\phi(X) = u^{-1}(\mathbb{E}[u(X)])$
Exagerate the **cost** *of bad outcomes through a convex increasing* $u : \mathbb{R} \to \mathbb{R}$
(VonNeuman-Morgenstern'1947)

Distorted risk measure: $\phi(X) = \mathbb{E}(X^h)$ *with* $\mathbb{P}(X^h \leq x) = h(\mathbb{P}(X \leq x))$
Exagerate the **probability** *of bad outcomes through a distortion map*
$h : [0,1] \to [0,1]$ *increasing from 0 to 1 with* $h(s) \leq s$ *(Yaari'1987)*

Rank-dependent utility: $\phi(X) = u^{-1}(\mathbb{E}[u(X^h)])$
(Schmeidler'1989; Quiggin'1993)

One additional axiom

Each of these theories of choice are characterized by monotonicity, weak continuity, and some form of independence axiom:



$$\phi(X) \leq \phi(Y) \Rightarrow \phi(\mathcal{L}(p, X, Z)) \leq \phi(\mathcal{L}(p, Y, Z)).$$

Entropic risks

Theorem (C-Torrico'2015)

Among expected utilities, distorted risk measures, and rank dependent utilities, the only add-consistent risk measures are

$$\phi_\beta(X) = \frac{1}{\beta}\ln(\mathbb{E}[e^{\beta X}]).$$

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- Mixtures of entropic risks $\int_{\mathbb{R}} \phi_\beta(X) dH(\beta)$ are also add-consistent.
- In atomic spaces there are other *exotic* add-consistent measures.

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- In atomic spaces there are other *exotic* add-consistent measures.
- Open questions:
  - Are mixtures of entropic risks the only add-consistent measures in non-atomic probability spaces ?
  - Risk-optimal paths without independence ?
  - Populations with heterogeneous risk-aversion ?
  - Price-of-anarchy for risk averse equilibrium ?

2. **Transit equilibrium**

Hong Kong MTR

Many large cities suffer from overcrowded transit systems