

Optimal Control Techniques for Sampled-Data Control Systems with Medical Applications

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Control system of the form

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0, \quad x(T) = x_T$$

and an optimal control of the Mayer type

$$\min_{u(\cdot)} \varphi(x(T))$$

with $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$.

- *Permanent Control* : $u : [0, T] \mapsto U$, $u \in \mathcal{U}$ where \mathcal{U} are the absolutely continuous maps valued in U .
- *Sampled Data Case* $u \in \mathcal{U}_{sampled}$: fix $n \in \mathbb{N}$,
 - n sampling times

$$0 < t_1 < \dots < t_n < T.$$

- $(n+1)$ -amplitudes

$$\eta = (\eta_0, \dots, \eta_n) \in [0, 1]^{n+1}$$

The control is constant over $[t_i, t_{i+1}]$.

Numerical schemes

In the **permanent case**, the optimal control can be computed using

- *Direct scheme* : the problem is transformed into a finite dimensional optimization problem using
 - discretization scheme for the dynamics
 - discretization scheme for the control
- *Indirect scheme* : the problem can be analyzed using Pontryagin Maximum Principle which leads to Hamiltonian equations

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}$$

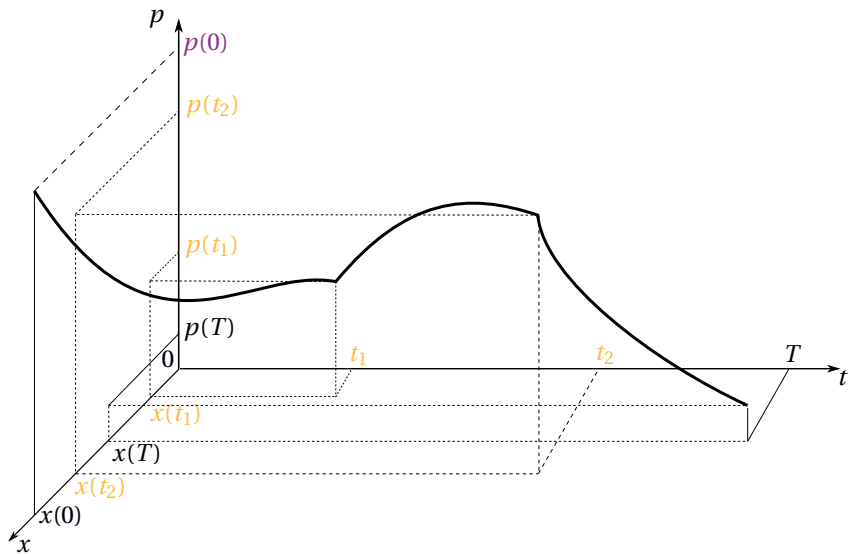
$$H(x, p, u) = \max_{v \in U} H(x, p, v)$$

with $H(x, p, v) = p \cdot f(x, v)$, p being the adjoint vector satisfying

$$p(T) = p_0 \frac{\partial \varphi}{\partial x}(x(T)) \quad (\text{transversality condition}).$$

This necessary optimality condition can be handled using a *shooting method*

Shooting method



Force-Fatigue muscular model²

FES input i . Dirac impulses δ at times $t = 0, t_1, t_2, \dots, t_N$.

$$i(t) = \sum_{i=0}^N R_i \eta_i \delta(t - t_i), \quad \eta_i \in [0, 1]$$

where

$$R_i := \begin{cases} 1, & \text{for } i = 0, \\ 1 + (\bar{R} - 1) \exp\left(-\frac{t_i - t_{i-1}}{\tau_c}\right), & \text{for } i = 1, \dots, N, \end{cases}$$

takes into account the *tetanic* contraction.

FES signal E_s .

$$E_s(t) = \frac{1}{\tau_c} \sum_{i=0}^N R_i \eta_i \mathbf{H}(t - t_i) \exp\left(-\frac{t - t_i}{\tau_c}\right)$$

\mathbf{H} : Heaviside

². based on the Ding et al. / Hill-Huxley model

The FES signal drives the evolution of the dynamics :

$$\begin{aligned}\dot{C}_N(t) &= -\frac{C_N(t)}{\tau_c} + E_s(t; t_i, \eta_i), \\ \dot{F}(t) &= -F(t) \gamma(t) + A \beta(t).\end{aligned}$$

where the Hill functions are given by

$$\beta(t) := \frac{C_N(t)}{K_m + C_N(t)}, \text{ and } \gamma(t) := \frac{1}{\tau_1 + \tau_2 \beta(t)}.$$

(A, K_m, τ_1, τ_2) are the fatigue parameters.

The problem fits in the sampled data control frame with :

- $u_0 = \eta_0 e^{-t/\tau_c} / \tau_c$ on $[0, T]$
- $u_1 = u_0(t_1) + \eta_1 R_1 e^{-(t-t_1)/\tau_c} / \tau_c$ on $[t_1, T]$
- \vdots

Note that in this form each control splits into

- *Head* : restricting to $[t_i, t_{i+1}]$
- *Tail* : restricting to $[t_i, T]$

Optimal control problems considered.

- **(OCP1)** $\max_{t_i, \eta_i} F(T)$
- **(OCP2)** $\min_{t_i, \eta_i} \int_0^T |F(t) - F_{ref}|^2 dt$ (F_{ref} : reference force).

Main theoretical results



$$c_N(T) = \frac{1}{\tau_c} \sum_{i=0}^n e^{-(t-t_i)/\tau_c} (T - t_i) R_i$$

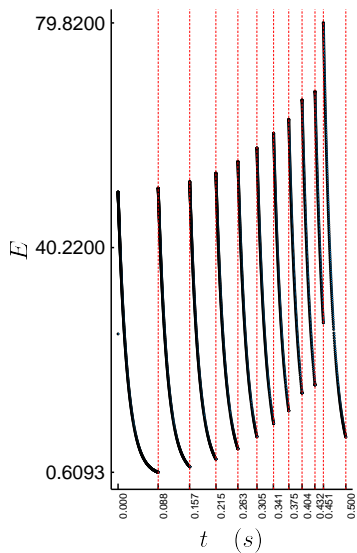
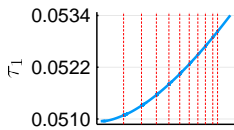
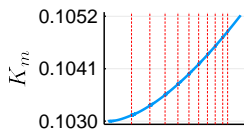
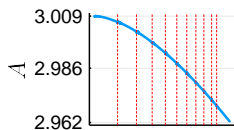
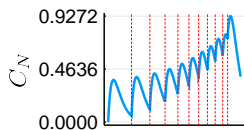
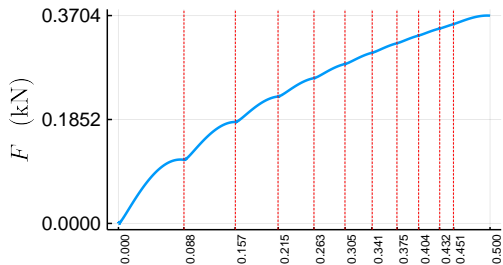
$F(T)$ is a piecewise C^∞ mapping with respect to η_i, t_i and the problem fits in a **finite dimensional optimization problem**.

- *The non-fatigue model* for a sequence of train, fatigue parameters P satisfy a dynamics of the form

$$\dot{P}(t) = \frac{P(t) - P_{rest}}{\tau_p} + \alpha_p F(t).$$

- First order necessary optimality conditions can be obtained adapting [5] and are described in [2]. They were numerically implemented and compared with a direct optimization scheme in [3].

Numerical results using direct method



Theorem (see [2])

If $(\eta_0^*, \eta_1^*, \dots, \eta_N^*, t_1^*, \dots, t_N^*)$ is optimal, then there exists p satisfying the co-state equation and the transversality condition.

Moreover, the necessary conditions are :

(i) the inequality

$$\left(\int_{t_i^*}^T p_1(s) b(s) ds \right) \tilde{\eta}_i \leq 0,$$

for all $i = 0, \dots, n$ and all admissible perturbation $\tilde{\eta}_i$ of η_i^* ;

(ii) and the inequality

$$\begin{aligned} NC_i := & \left(-p_1(t_i^*) b(t_i^*) G(t_{i-1}^*, t_i^*) \eta_i^* + b(-t_i^*) \eta_i^* \int_{t_i^*}^T p_1(s) b(s) ds \right. \\ & \left. + b(-t_i^*) (\bar{R} - 1) \eta_{i+1}^* \int_{t_{i+1}^*}^T p_1(s) b(s) ds \right) \tilde{t}_i \leq 0, \end{aligned}$$

for all $i = 1, \dots, n$ and all admissible perturbation \tilde{t}_i of t_i^* .

Work in progress

- We need efficient algorithms (real-time application) :

- Explicit expression of

$$(t_1, \dots, t_n, \eta_0, \dots, \eta_n) \rightarrow F(T)$$

to apply a direct optimization scheme

- *LQ* methods
 - MPC methods
- Online parameter estimation coupled with optimization methods [1]
 - Extension to the non-isometric case with **joint angle variable** to produce a motion [4]

References

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