Dynamics in Games: Algorithms and Learning

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Abstract

Game theory studies interactions between agents with specific aims, be they rational actors, genes, or computers. This course is intended to provide the main mathematical concepts and tools used in game theory with a particular focus on their connections to learning and convex optimization. The first part of the course deals with the basic notions: value, (Nash and Wardrop) equilibria, correlated equilibria. We will give several dynamic proofs of the minmax theorem and describe the link with Blackwell's approachability. We will also study the connection with variational inequalities.

The second part will introduce no-regret properties in on-line learning and exhibit a family of unilateral procedures satisfying this property. When applied in a game framework we will study the consequences in terms of convergence (value, correlated equilibria). We will also compare discrete and continuous time approaches and their analog in convex optimization (projected gradient, mirror descent, dual averaging). Finally we will present the main tools of stochastic approximation that allow to deal with random trajectories generated by the players.

Part A

BASIC TOOLS AND RESULTS

A.2 Extensions and related topics

This section deeply relies on the books:

Mertens J.- F., S. Sorin and S. Zamir, (2015) *Repeated Games*, Cambridge University Press.

Laraki R., J. Renault and S. Sorin (2019) *Mathematical Foundations of Game Theory*, Springer.

Correlated equilibria

This section deals with "correlated equilibrium", Aumann, 1974 [3], which is an extension of Nash equilibrium that has good properties from strategic, analytic and dynamic viewpoints.

Examples

Example 1 :

	l	r
T	$0,0$	$3,1$
B	$1,3$	$0,0$

This game has two 2 pure equilibria that are not symmetrical and a mixed, symmetric one $(3/4,1/4)$ which is dominated in terms of payoff.
The use of a public fair coin and of the plan: (T,r) if Head and (B,l) if Tail, induces the following distribution on profiles:

0	$1/2$
$1/2$	0

thus an efficient symmetric outcome, immune to unilateral deviations.

Example 2 :

	l	r
T	$2,7$	$6,6$
B	$0,0$	$7,2$

Introduce a signal space $\Omega = (X,Y,Z)$, with uniform probability $(1/3, 1/3, 1/3)$.
Assume that the players get private messages:
1 knows $a = \{X,Y\}$ or $b = \{Z\}$,
2 knows $\alpha = \{X\}$ or $\beta = \{Y,Z\}$.
Consider the strategies:
T if a, B if b for Player 1;
l if α, r if β for player 2.
They induce on the set of profiles S the correlation matrix:

$1/3$	$1/3$
0	$1/3$

and no deviation is profitable.
The formal model is as follows.

Information structure

An information structure $\mathscr{I}$ is given by:
- a probability space $(\Omega, \mathscr{A}, P)$
- a measurable map θ^i from $(\Omega, \mathscr{A})$ to A^i (signals of i), for each $i \in I$.

Let G be a finite game defined by $g : S = \Pi_i S^i \to \mathbb{R}^I$.

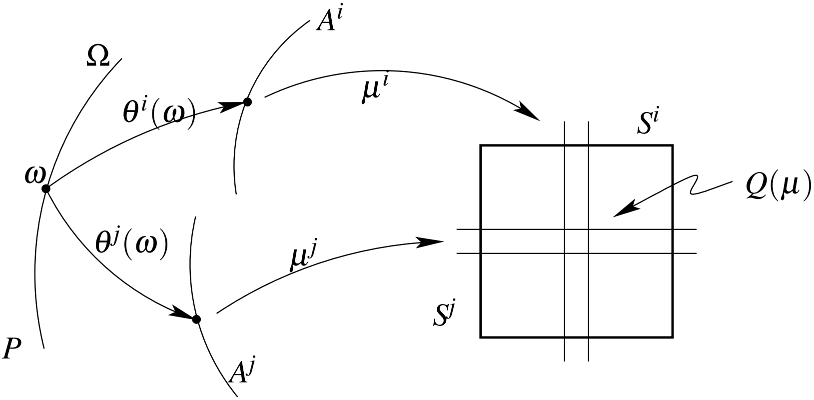
The game G extended by $\mathscr{I}$, denoted $[G, \mathscr{I}]$, is the game played in 2 stages:
- stage 0 : the random variable ω is selected according to P and the signal $\theta^i(\omega)$ is sent to player i.
- stage 1 : the players play the game G.

A strategy μ^i of player i in the game $[G, \mathscr{I}]$ is a map from A^i to S^i.
A profile μ of such elements is called a correlated strategy.

Definition 1.1

A correlated equilibrium of G is a Nash equilibrium of some extended game $[G, \mathscr{I}]$.

A profile μ of strategies in $[G, \mathscr{I}]$ maps the probability P on Ω to a probability $Q(\mu)$ on S:
random variable $\rightarrow$ signals $\rightarrow$ profile of moves.



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Explicitly, for each ω, $Q(\omega, \mu)$ is the probability on S given by $\prod_i \mu^i(\theta^i(\omega))$ and $Q(\mu)$ is the expectation w.r.t. P.

$CED(G)$ is the set of correlated equilibria distributions in G:

$$CED(G) = \cup_{\mathscr{I}} \{Q(\mu); \mu \text{ equilibrium in } [G, \mathscr{I}]\}$$

$CED(G)$ is a convex set: simply consider the convex combination of information structures.

A canonical information structure for G is given by:
$\Omega = S$; $\theta^i : S \to S^i, \theta^i(s) = s^i$.
Thus P is a probability on S and each player is informed upon his component.
A canonical correlated equilibrium is an equilibrium of $[G, \mathscr{I}]$ where:
i) $\mathscr{I}$ is a canonical information structure and
ii) equilibrium strategies are given by:

$$\mu^i(\omega) = \mu^i(s) = \mu^i(s^i) = s^i.$$

“Each player follows his signal (recommendation)".
The induced canonical correlated equilibrium distribution ($CCED$) is obviously P.

Properties

The main result is the following.

Theorem 1.1 (Aumann, 1974 [3])

$$CCED(G) = CED(G)$$

Proof :
Let μ be an equilibrium profile in an extension $[G, \mathscr{I}]$ and $Q = Q(\mu)$ the induced distribution.
Then Q belongs to $CCED(G)$.
In fact, each player i get less information: her move s^i rather than the signal a^i such that $\mu^i(a^i) = s^i$.
But s^i is a best reply to the correlated strategy of $-i$, conditional to a^i.
It is then enough to use the convexity of BR^i on $\Delta(S^{-i})$. ∎

The characterization in the finite case is given by:

Theorem 1.2

$CED = \cap_i CED^i$, with $Q \in CED^i$ iff:

$$\sum_{s^{-i} \in S^{-i}} [g^i(s^i, s^{-i}) - g^i(t^i, s^{-i})] Q(s^i, s^{-i}) \geq 0, \qquad \forall s^i, t^i \in S^i.$$

Proof :
Let $Q \in CCED(G)$.
Assume s^i is announced (i.e. its marginal $Q^i(s^i) = \sum_{s^{-i}} Q(s^i, s^{-i}) > 0$) and consider the conditional distribution on S^{-i}, $Q(.|s^i)$.
Then the equilibrium condition writes:

$$s^i \in BR^i(Q(.|s^i).$$

s^i is a best reply of player i to the distribution of the moves of $-i$, conditionally to s^i. ■

The approach in terms of equilibrium of an extended game is "ex-ante".

The previous characterization corresponds to an "ex-post" criteria.

Corollary 1.1

The set of CED is the convex hull of finitely many points.

Proof :
It is a subset of $\Delta(S)$ defined by a finite set of linear inequalities. ■

Complements

1) There exist correlated equilibria distributions outside the convex hull of (Nash) equilibria distributions. In the game:

$0,0$	$5,4$	$4,5$
$4,5$	$0,0$	$5,4$
$5,4$	$4,5$	$0,0$

the only equilibrium is symmetrical : $(1/3, 1/3, 1/3)$ with payoff 3.
The following is a CED:

0	$1/6$	$1/6$
$1/6$	0	$1/6$
$1/6$	$1/6$	0

inducing the payoff 9/2.

2) The same property holds in two-person zero-sum games: there exist correlated equilibria distributions outside the convex hull of optimal distributions, Forges, 1990 [12].
In the game:

0	0	1
0	0	-1
-1	1	0

the following is a CED:

$1/3$	$1/3$	0
$1/3$	0	0
0	0	0

while for optimal strategies $p(2,2) = 0$ implies $p(1,1) \geq 1/2$ and similarly for convex combinations.

3) Let us give an elementary proof of existence of correlated equilibrium via the minmax theorem: CED corresponds to the set of optimal strategies of a player in a finite 0-sum game, Hart and Schmeidler, 1989 [17].

Let G be a finite I-player game with payoff $g : S = \prod_i S^i \longrightarrow \mathbb{R}^I$.
Consider now the two player zero-sum game Γ with strategy sets S and $L = \cup_i (S^i \times S^i)$ and payoff $\gamma : S \times L \to \mathbb{R}$ defined by:

$$\gamma(s;(t^i,u^i)) = [g^i(t^i,s^{-i}) - g^i(u^i,s^{-i})]\mathbf{I}_{\{t^i=s^i\}}.$$

Player 1 proposes s and player 2 tests t^i against u^i if $\{t^i = s^i\}$.
Γ has a value v and optimal strategies.
Clearly one has:

Proposition 1.1

If $v \geq 0$ and $Q \in \Delta(S)$ is an optimal strategy of Player 1, then Q is a correlated equilibrium distribution in G.

Let us prove that $v \geq 0$.
Let $\pi \in \Delta(L)$, a mixed strategy of player 2. Define ρ^i, a transition probability on S^i, by:

$$\rho^i(t^i;u^i) = \pi(t^i,u^i), \quad \text{if } t^i \neq u^i$$

$$\rho^i(t^i;t^i) = 1 - \sum_{u^i \neq t^i} \pi(t^i,u^i).$$

Let now μ^i be a probability on S^i invariant by ρ^i:

$$\mu^i(t^i) = \sum_{u^i} \mu^i(u^i)\rho(u^i;t^i).$$

Let $\mu = \otimes_i \mu^i \in \Delta(S)$.
Note that the payoff $\gamma(\mu;\pi)$ can be decomposed as follows:

$$\gamma(\mu,\pi) = \sum_{s\in S} \times_j \mu^j(s^j) \sum_{i\in I} \sum_{(t^i,u^i)\in L^i} \pi(t^i,u^i)\gamma(s;(t^i,u^i)).$$

Let A_i be the term corresponding to i.

$$
\begin{aligned}
A_1 &= \sum_{s\in S} \otimes_j \mu^j(s^j) \sum_{(t^i,u^i)} \pi(t^i,u^i)\gamma(s;(t^i,u^i)) \\
&= \sum_{s^i} \mu^i(s^i) \sum_{u^i} \pi(s^i,u^i) \sum_{s^{-i}} \otimes_{j\neq i}\mu^j(s^j)\gamma(s;(s^i,u^i)) \\
&= \sum_{s^i} \mu^i(s^i) \sum_{u^i\neq s^i} \rho^i(s^i,u^i) \sum_{s^{-i}} \otimes_{j\neq i}\mu^j(s^j)[g^i(s^i,s^{-i}) - g^i(u^i,s^{-i})] \\
&= \sum_{s^i} \mu^i(s^i) \sum_{u^i} \rho^i(s^i,u^i)[g^i(s^i,\mu^{-i}) - g^i(u^i,\mu^{-i})] \\
&= \sum_{s^i,u^i} \mu^i(s^i)\rho^i(s^i,u^i)g^i(s^i,\mu^{-i}) - \sum_{s^i,u^i} \mu^i(s^i)\rho^i(s^i,u^i)g^i(u^i,\mu^{-i}) \\
&= \sum_{s^i} \mu^i(s^i)g^i(s^i,\mu^{-i}) - \sum_{u_i} \mu^i(u^i)g^i(u^i,\mu^{-i}) \\
&= 0
\end{aligned}
$$

Hence $\gamma(\mu,\pi) = 0$.
Thus $\forall \pi \in \Delta(L), \exists \mu \in \Delta(S)$ such that $\gamma(\mu,\pi) \geq 0$, so that the value of Γ is non negative, hence Q is a CED. ■

4) A superset of the set of correlated equilibria distributions which is important in applications is the Hannan set H, Hannan, 1957 [15]. Like the set of CED, it is a subset of $\Delta(S)$ obtained by intersection.

Definition 1.2

$H = \cap_i H^i$ *with:*

$$H^i = \{Q \in \Delta(S); g^i(s^i, Q^{-i}) \leq g^i(Q), \quad \forall s^i \in S^i\}.$$

This corresponds to distributions "immune to deviation before getting the signal".

In the case of a 0-sum game one has, for $z \in H$ with marginals z^1, z^2:

$$f(z) \geq f(s^1, z^2), \quad \forall s^1 \in S^1,$$

and the opposite inequality for player 2 hence the marginals z^1, z^2 are optimal strategies and $f(z)$ is equal to the value of the game.

Example :

For the 2-person 0-sum game:

0	1	-1
-1	0	1
1	-1	0

the distribution:

$1/3$	0	0
0	$1/3$	0
0	0	$1/3$

is in the Hannan set, but not a CED.

5) General formulation

Consider the framework of a game G on compact convex sets : $H^i : X = \prod_j X^j \to \mathbb{R}, i \in I$.

Definition 1.3

A correlated equilibrium distribution $\mu \in \Delta(X)$ satisfies :

$$\int_X H^i(x)\, d\mu(x) \geq \int_X H^i(\alpha^i(x^i), x^{-i})\, d\mu(x) \quad \forall i \in I \tag{1}$$

for every measurable function $\alpha^i : X^i \to X^i$.

Proposition 1.2 (Neyman, 1997 [28])

If a $\mathscr{C}^1$ game G has a concave potential W, then for every CED μ :

$$supp(\mu) \subset argmax_X W.$$

In particular if W is strictly concave there is a unique CED.

Proof :
Let us prove that $\mu \in CED$ iff :

$$\int_X W(x)\, d\mu(x) \geq W(y), \quad \forall y \in X. \tag{2}$$

1) Assume (2), then for any $\alpha : X \to X$ measurable :

$$\int_X [W(x) - W(\alpha(x))]\, d\mu(x) \geq 0$$

and in particular :

$$\int_X [H^i(x) - H^i(\alpha^i(x^i), x^{-i})]\, d\mu(x) = \int_X [W(x) - W(\alpha^i(x^i), x^{-i})]\, d\mu(x) \geq 0$$

so that $\mu \in CED$.

2) Assume by contradiction: $\int_X W(x)\,d\mu(x) < W(y)$ for some $y \in X$.
By concavity:

$$\sum_{i \in I} \langle \nabla_i W(x), y^i - x^i \rangle \geq W(y) - W(x),$$

hence integrating w.r. t. μ, there exists $i \in I$ with :

$$\int_X \langle \nabla_i W(x), y^i - x^i \rangle d\mu(x) > 0.$$

thus :

$$\int_X \langle \nabla_i H^i(x), y^i - x^i \rangle d\mu(x) > 0,$$

and for $\varepsilon > 0$ small enough :

$$\int_X H^i((x^i + \varepsilon(y^i - x^i), x^{-i}) d\mu(x) > \int_X H^i((x) d\mu(x)$$

so that μ is not a correlated equilibrium distribution. ■

6) For more notions of extensions of games, corresponding equilibria and representation properties, see Forges, 1986 [10], 1990 [11].

Approachability theory

We follow Blackwell, 1956 [7], see also MSZ, II.4 and Sorin, 2002 [31], Appendix B.

Presentation

Consider a two person game defined by A, a $I \times J$-matrix with entries in $\mathbb{R}^K$: $A_{ij} \in \mathbb{R}^K$ is the vector payoff, or outcome, if Player 1 plays $i \in I$ and player 2 plays $j \in J$.
The game is played in discrete time for infinitely many stages: at each stage $n = 1, 2, ...$, after having observed the past history h_{n-1} of actions used from stage 1 to stage $n-1$, i.e. $h_{n-1} = (i_1, j_1, ..., i_{n-1}, j_{n-1}) \in \mathscr{H}_{n-1} = (I \times J)^{n-1}$, Player 1 chooses $x_n \in X = \Delta(I)$ and player 2 chooses $y_n \in Y = \Delta(J)$.
Then a couple $(i_n, j_n) \in I \times J$ is selected according to the product probability $x_n \otimes y_n$, and the game goes to stage $n+1$ with the history $h_n = (i_1, j_1, ..., i_n, j_n) \in \mathscr{H}_n$.

A strategy σ of Player 1 in the repeated game is a sequence $\sigma = (s_1, ..., s_n, ...)$ with $s_n : \mathscr{H}_{n-1} \to \Delta(I)$ for each n.
(Similarly for a strategy τ of player 2).

A couple (σ, τ) naturally defines a probability distribution $\mathbf{P}_{\sigma,\tau}$ over the set of plays $\mathscr{H}_\infty = (I \times J)^\infty$, endowed with the product σ-algebra, and $\mathbf{E}_{\sigma,\tau}$ is the associated expectation.

Every play $h = (i_1, j_1, ..., i_n, j_n, ...)$ of the game induces a sequence of vector outcomes $z(h) = (z_1 = A_{i_1 j_1}, ..., z_n = A_{i_n j_n}, ...)$ with values in $\mathbb{R}^k$.

Denote by $\bar{z}_n$ the Cesaro-average outcome up to stage n:

$$\bar{z}_n(h) = \frac{1}{n} \sum_{k=1}^{n} A_{i_k j_k} = \frac{1}{n} \sum_{k=1}^{n} z_k.$$

Definition 2.1

A set $C \subset \mathbb{R}^K$ is approachable by Player 1 if, for any $\varepsilon > 0$, there exists a strategy σ and N such that, for any strategy τ of Player 2 and any $n \geq N$:

$$E_{\sigma,\tau}(d_n) \leq \varepsilon,$$

where d_n is the euclidean distance $d(\bar{z}_n, C)$.

A set $C \subset \mathbb{R}^K$ is excludable by Player 1 if for some $\delta > 0$, the set $C^c_\delta = \{z; d(z,C) \geq \delta\}$ is approachable by her.

A dual definition holds for Player 2.

From the definitions it is enough to consider closed sets C and even their intersection with the closed ball of radius $\|A\|$.
Given x in $X(=\Delta(I))$, define $[xA] = \text{co}\,\{\sum_i x_i A_{ij}; j \in J\}$, and similarly $[Ay]$, for y in $Y(=\Delta(J))$. If Player 1 uses x, his expected payoff will be in $[xA]$, whatever being the move of player 2.

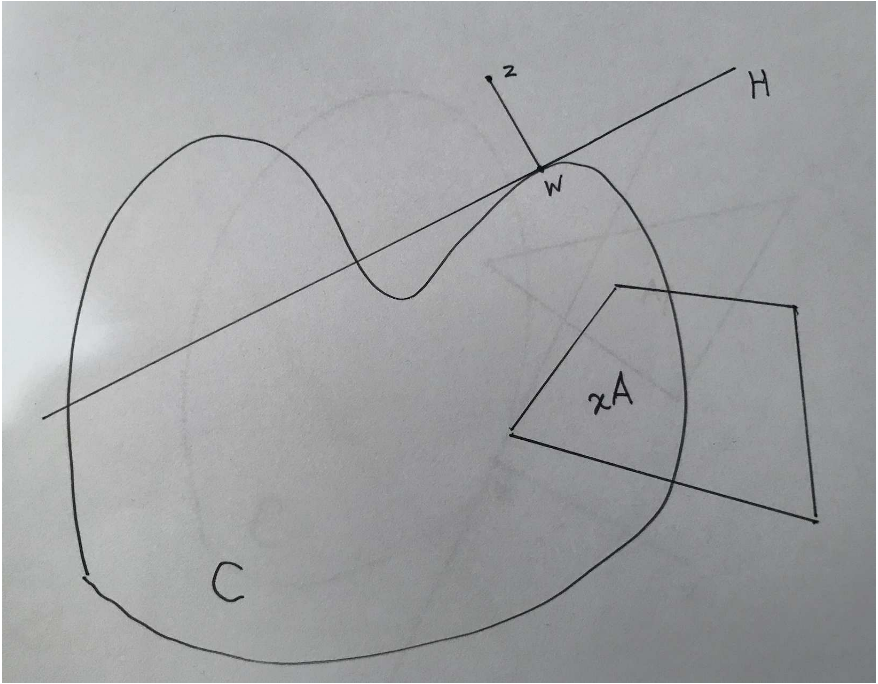
B-sets and sufficient condition

The first result is a sufficient condition for approachability based on the following notion:

Definition 2.2

A closed set C in $\mathbb{R}^K$ is a **B***-set for Player 1 if:*
for any $z \notin C$, there exists a closest point $w = \mathscr{P}_C(z)$ in C to z and a mixed move $x = x(z)$ in X, such that the hyperplane trough w orthogonal to the segment $[wz]$ separates z from $[xA]$. Explicitly:

$$\langle z - w, u - w \rangle \leq 0, \quad \forall u \in [xA].$$


z
H
w
xA
C

Theorem 2.1

*Let C be a **B**-set for Player 1.*
Then C is approachable by that player.
Explicitly, a strategy satisfying $\sigma(h_{n+1}) = x(\bar{z}_n)$, whenever $\bar{z}_n \notin C$, gives:

$$E_{\sigma\tau}(d_n) \leq \frac{2\|A\|}{\sqrt{n}}, \quad \forall\tau \text{ strategy of player 2,}$$

and d_n converges $P_{\sigma\tau}$ a.s. to 0, more precisely:

$$P(\exists n \geq N; d_n^2 \geq \varepsilon) \leq \frac{8\|A\|^2}{\varepsilon N}.$$

Proof :
Let Player 1 use a strategy σ as above. Denote $w_n = w(\bar{z}_n)$.

The property of $x(\bar{z}_n)$ implies that:

$$\langle E(z_{n+1}|h_n) - w_n, \bar{z}_n - w_n \rangle) \leq 0$$

since $E(z_{n+1}|h_n)$ belongs to $[x(\bar{z}_n)A]$.
Hence the previous equation in the deterministic case (see Part A1):

$$d_{n+1}^2 \leq (1 - \frac{2}{n+1})\, d_n^2 + (\frac{1}{n+1})^2 \|z_{n+1} - \bar{z}_n\|^2,$$

gives here by taking conditional expectation with respect to the history h_n:

$$\mathsf{E}(d_{n+1}^2|h_n) \leq (1 - \frac{2}{n+1})\, d_n^2 + (\frac{1}{n+1})^2 \mathsf{E}(\|z_{n+1} - \bar{z}_n\|^2|h_n). \qquad (3)$$

So that we obtain:

$$\mathsf{E}(d_{n+1}^2) \leq (\frac{n-1}{n+1})\ \mathsf{E}(d_n^2) + (\frac{1}{n+1})^2\ 4\|A\|^2,$$

and by induction:

$$\mathsf{E}(d_n^2) \leq \frac{4\|A\|^2}{n}.$$

This gives in particular the convergence in probability of d_n to 0.
Now introduce the random variable:
$W_n = d_n^2 + \|A\|^2 \sum_{m=n+1}^{\infty} (\frac{1}{m^2} E(\|z_m - \bar{z}_m\|^2 | h_n)$. We have from (3):

$$E(W_{n+1}|h_n) \leq W_n,$$

thus W_n is a positive supermartingale hence converges $P\ a.s.$ to 0.
More precisely Doob's maximal inequality, see e.g. Neveu, 1972 [27], gives :

$$P(\exists n \geq N; d_n^2 \geq \varepsilon) \leq \frac{E(W_N)}{\varepsilon} \leq \frac{8\|A\|^2}{\varepsilon N}.$$

■

In particular one obtains:

Corollary 2.1

For any x in X, $[xA]$ is approachable by Player 1, with the constant strategy x.

It follows that a necessary condition for a set C to be approachable by Player 1 is that for any y in Y, $[Ay] \cap C \neq \emptyset$, otherwise C would be excludable by Player 2.

In fact this condition is also sufficient for convex sets.

Convex case

Theorem 2.2

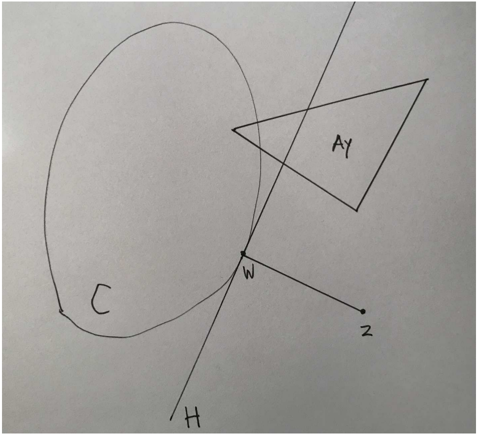
Assume C closed and convex in $\mathbb{R}^K$.
C is either approachable or excludable.
*More precisely C is a **B**-set for Player 1 iff :*

$$[Ay] \cap C \neq \emptyset, \qquad \forall y \in Y. \tag{4}$$

*In particular a set is approachable iff it is a **B**-set.*

Proof :
By the previous Corollary, it is enough to prove that (4) implies that C is a **B**-set.
The idea is to reduce by projection the property to the one-dimentional case and to use the minmax theorem.

In fact, let $z \notin C$, $w = \Pi_C(z)$ its projection on C , and consider the game with real payoff matrix $B = \langle w - z, A \rangle$. Since $[Ay] \cap C \neq \emptyset$ for all $y \in Y$, this implies that its value is at least $\min_{c \in C} \langle w - z, c \rangle = \langle w - z, w \rangle$. Hence there exists an optimal strategy $x \in X$ of Player 1 such that $\langle w - z, \sum_i x_i A_{ij} \rangle \;\geq\; \langle w - z, w \rangle$ for any $j \in J$, which shows that xA is on the opposite side of the hyperplane H to z, and the result follows. ∎



The previous proof gives also the following practical criteria:

Corollary 2.2

A closed convex set C is a **B**- *set for Player 1 iff, for any α in $\mathbb{R}^K$:*

$$\texttt{val}\langle\alpha, A\rangle \geq \min_{c\in C}\langle\alpha, c\rangle,$$

where `val` *is the* $\max_X \min_Y = \min_Y \max_X$ *operator.*

Extensions

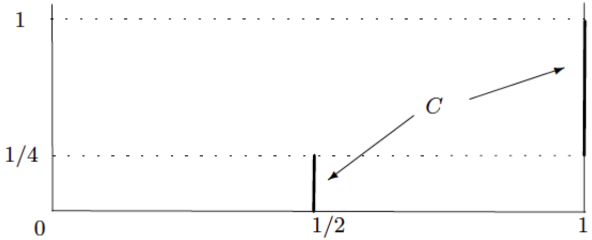
1. In dimension 1, any set is either approachable or excludable. More precisely let v (resp. v') be the value of A when Player 1 maximizes (resp. minimizes).
If $v \geq v'$, C is approachable by Player 1 iff $C \cap [v', v] \neq \emptyset$ and excludable by Player 2 otherwise.
If $v \leq v'$, C is approachable iff $[v, v'] \subset C$ and excludable by Player 2 otherwise.

2. Extension to random payoffs, uniformly bounded in L^2.

3. There exist sets that are neither approachable nor excludable. Consider the game:

	l	r
T	$(0,0)$	$(0,0)$
B	$(1,0)$	$1,1)$

and the set $C = \{(1/2, y);\ 0 \leq y \leq 1/4\} \cup \{(1, y);\ 1/4 \leq y \leq 1\}$.



If Player 1 plays n times Bottom and then Bottom or Top for the next n stages depending whether $\bar{z}_n^2$ is greater than 1/2 or not, the average $\bar{z}_{2n}$ will be in C.
However Player 2 can prevent the average payoff to remain near C by either playing left (and forcing the horizontal axis) or right (and forcing the diagonal).

This leads to the following definition of weak approachability.

4. Weak approchability

Definition 2.3

A set C in $\mathbb{R}^K$ is weakly approachable by Player 1 if for any $\varepsilon > 0$ there exists N such that for any $n \geq N$ there is a strategy $\sigma = \sigma(n)$ satisfying, for any strategy τ of Player 2:

$$E_{\sigma,\tau}(d_n) \leq \varepsilon.$$

C is weakly excludable by Player 1 if for some $\delta > 0$, the set $C^c_\delta = \{z;\ d(z,C) \geq \delta\}$ is weakly approachable by her.

The main difference is that the strategy may now depend of the lenght of the game.

In the example above the described strategy of Player 1 shows that C is weakly approachable.
In fact, the sequence of un-normalized cumulative outcomes, $(\frac{1}{n}\sum_{\ell \leq m} z_\ell)$, $m = 1, \ldots, n$ defines a piecewise linear curve and Player 1's objective is to reach C at stage$n$.
It is thus natural to consider the game played in continuous time between times 0 and 1 with position $\int_0^t z_u du$ at time t.
Again in the previous example, Player 1 can generate a curve with slope between 0 and 1 (controlled by Player 2) and horizontal speed 1, but he can also stop the game. Clearly he is thus able to reach C at time 1.

This result is general since one has:

Theorem 2.3 (Vieille, 1992 [34])

Any set is either weakly approachable or weakly excludable.

Sketch of the proof:
The proof uses the theory of differential games of fixed duration.
A) The idea is to consider v_n as the value of the discretisation of a differential game Γ played between time 0 and 1.
Formally the deterministic dynamic in $\mathbb{R}^K$ is given by:

$$\dot{z}_t = \alpha_t A \beta_t$$

where α and β are the controls of the players with $\alpha_t \in \Delta(I)$ and $\beta_t \in \Delta(J)$ for $t \in [0,1]$, and the terminal payoff is $r(z_1)$ (there is no payoff on the trajectory), where r is a smooth function like $r(z) = 1 - d(z, C)$.

For any initial point $z \in \mathbb{R}^K$ and time $\xi \in [0,1]$, one defines two discretisations $\Gamma_n^-(\xi,z)$ (resp. $\Gamma_n^+(\xi,z)$) of Γ played on the time interval $[\xi,1]$, where the state starts from z at time ξ, the controls are constant on $(\frac{m}{n}(1-\xi), \frac{m+1}{n}(1-\xi))$ and Player 1 (resp. Player 2) is playing first. Their values $W_n^-(\xi,z)$ (resp. $W_n^+(\xi,z)$) satisfy:

$$W_n^-(\xi,z) = \max_{\alpha_1 \in \Delta(I)} \min_{\beta_1 \in \Delta(J)} \dots \max_{\alpha_n \in \Delta(I)} \min_{\beta_n \in \Delta(J)} r\left(z + (1-\xi)\frac{1}{n}\sum_{m=1}^{n} \alpha_m A \beta_m\right)$$

$$W_n^+(\xi,z) = \min_{\beta_1 \in \Delta(J)} \max_{\alpha_1 \in \Delta(I)} \dots \min_{\beta_n \in \Delta(J)} \max_{\alpha_n \in \Delta(I)} r\left(z + (1-\xi)\frac{1}{n}\sum_{m=1}^{n} \alpha_m A \beta_m\right)$$

Moreover the following recursive equations (dynamic programming) hold:

$$W_n^-(\xi, z) = \max_{\alpha \in \Delta(I)} \min_{\beta \in \Delta(J)} W_{n-1}^-(\xi + \frac{1}{n}(1-\xi), z + \frac{1}{n}(1-\xi)\alpha A \beta)$$

$$W_n^+(\xi, z) = \min_{\beta \in \Delta(J)} \max_{\alpha \in \Delta(I)} W_{n-1}^+(\xi + \frac{1}{n}(1-\xi), z + \frac{1}{n}(1-\xi)\alpha A \beta)$$

The main results from the theory of differential games with fixed duration that we use are, see e.g. Souganidis, 1999 [32]:

1) W_n^- and W_n^+ converge to some functions W^- and W^+ as $n \to \infty$.

2) W^- is a viscosity solution, see e.g. Crandall, Ishii and Lions (1992) [9], on $[0,1]$ of the equation:

$$\frac{\partial U}{\partial t} + \max_{\alpha \in \Delta(I)} \min_{\beta \in \Delta(J)} \langle \nabla U, \alpha A \beta \rangle = 0$$

$$U(1,z) = r(z)$$

3) this solution is unique.

A similar result for W^+ and the property (Isaacs):

$$\max_{\alpha\in\Delta(I)}\min_{\beta\in\Delta(J)}\langle \nabla U, \alpha A \beta\rangle = \min_{\beta\in\Delta(J)}\max_{\alpha\in\Delta(I)}\langle \nabla U, \alpha A \beta\rangle$$

finally imply:

$$W^- = W^+.$$

Denote this value by W.

Hence if $W(0,1) = 1$, for any $\varepsilon > 0$ there exists N such that if $n \geq N$ Player 1 can force an outcome within ε of C in Γ_n^-.

B) The second part of the proof consists in showing that Player 1 can generate almost the same trajectory in the initial discrete time repeated game.

Consider first G_{rn}.
Inductively Player 1 will play by blocks of length r: on the m-th block he computes the optimal control α_m of stage m in Γ_n^- (given a past generated in the differential game by β_ℓ, $\ell < m$, which is the average behavior of Player 2 on block ℓ) and play the mixed move α_m i.i.d. for r stages.
Use the approachability theorem to prove that if d_m denotes the distance between $\alpha_m A \beta_m$ and the average outcome on block m in G_{rn}, one has: $E(d_m) \leq \frac{2\|A\|}{\sqrt{r}}$.
It follows that for r large enough the average payoff in G_{rn} is within ε of one trajectory compatible with an optimal strategy in Γ_n^- hence within 2ε of C.
Finally if $m = rn + q$ the error in approximating G_m by G_{rn} is at most of the order $1/r$.
C) If $W(0,1) < 1$, Player 2 can force the outcome to belong to the complement of a δ-neighborhood of C for δ small enough and a similar construction shows that C is weakly excludable. ∎

5. The **B**-sets are the minmal approachable sets.

Proposition 2.1 (Spinat, 2002 [33])

Any approachable set contains a **B***-set.*

In particular this implies that the “light" definition of approachability (convergence in expectation) is in fact equivalent to the “heavy " one (cv a.s.).
This also shows that there are always robust approachability strategies (for example: independent of ε and of the previous own moves).
Note however that the a.s. convergence does not extend to a stochastic framework.

6. Extension to infinite dimension, Lehrer, 2002 [19].

7. General active states, Lehrer, 2003 [20].

8. A dual representation of **B**-sets.

Definition 2.4

The set of proximal normals to $C \subset \mathbb{R}^K$ at $w \in C$ is:

$$P_C(w) = \{p \in \mathbb{R}^K ; w = \mathscr{P}_C(w+p)\}. \tag{5}$$

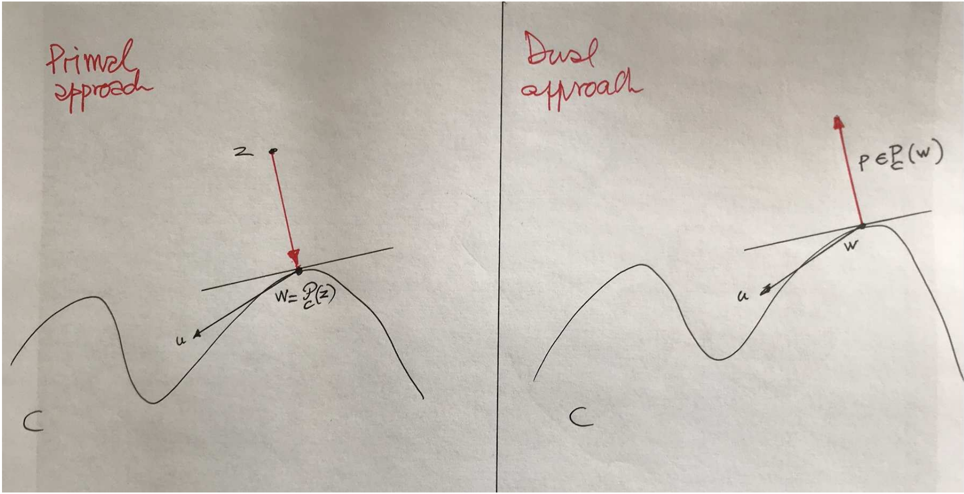
Recall:

A closed set C in $\mathbb{R}^K$ is a **B**-set for Player 1 if:
for any $z \notin C$, there exists a closest point $w = \mathscr{P}_C(z)$ in C to z and a mixed move $x = x(z)$ in X, such that the hyperplane trough w orthogonal to the segment $[wz]$ separates z from $[xA]$.

$$\langle z - w, u - w \rangle \leq 0, \quad \forall u \in [xA].$$

Lemma 2.1 (As Soulaimani, Quincampoix and Sorin, 2009 [1])
*C is a **B**-set iff :*

$$\min_{x \in X} \max_{y \in Y} \langle p, xAy - w \rangle \leq 0, \qquad \forall w \in C, \forall p \in P_C(w). \tag{6}$$



Primal approach
z
W = P_C(z)
u
C
Dual approach
p ∈ P_C(w)
w
u
C

8. Approachability and viability [second link with differential games]: As Soulaimani, Quincampoix and Sorin, 2009 [1].

Recall: [first link with differential games] Weak approachability and differential games with fixed duration

The analysis using these two points of view - asymptotic approach versus uniform approach - is fundamental in the study of all dynamical models.

Consider a differential game with dynamics $w_t \in \mathbb{R}^K$ given by :

$$\dot{w}_t = f(w_t, \alpha_t, \beta_t), \qquad w_0 = w(0).$$

where $(\alpha_t, \beta_t) \in U \times V$ with U, V convex, compact.
Assume in addition i) $f : \mathbb{R}^K \times U \times V \to \mathbb{R}^K$ continuous, ii) $f(., u, v)$ L-Lipschitz for all $(u, v) \in U \times V$, (iii) f affine w.r.t. u.

Given strategies (α, β), let $\{W_t(w(0), \alpha, \beta); t \geq 0\}$ be the associated trajectory.

Theorem 2.4 (Cardaliaguet, 1996 [8])

Let C be closed in $\mathbb{R}^K$ and C^ε be a closed $\varepsilon-$neighborhood.
C is a discriminating domain for Player 1, i.e. :

$$\forall w(0) \in C, \quad \forall \varepsilon > 0, \quad \exists \alpha, \quad \forall \beta : \quad W_t(w(0), \alpha, \beta) \in C^\varepsilon, \quad \forall t \geq 0, \quad (7)$$

iff

$$\max_{v \in V} \min_{u \in U} \langle p, f(w, u, v) \rangle \leq 0, \qquad \forall w \in C, \forall p \in P_C(w). \qquad (8)$$

Consider the average expected outcome in the continuous version of the approachability game:

$$w_t = \frac{1}{t}\int_0^t x_s A y_s$$

so that :

$$w_t + t\,\dot{w}_t = x_t A y_t$$

hence :

$$\dot{w}_t = \frac{1}{t}[x_t A y_t - w_t].$$

With a time change we obtain the differential game with dynamics :

$$f(w,u,v) = uAv - w.$$

Thus we deduce, using:

$$\max_{v\in V}\min_{u\in U}\langle p,\ uAv - w\rangle = \min_{u\in U}\max_{v\in V}\langle p,\ uAv - w\rangle$$

the following :

Lemma 2.2

A discriminating domain for the differential game defined by f is a **B***-set for the repeated game defined by A.*

Corollary 2.3

*A **B**-set is approachable.*

Proof :
Let C be a **B**-set hence a discriminating domain.
Let $w(0) \in C$ and any $\omega(0) \in \mathbb{R}^K$. Let α be a discriminating strategy for Player 1, given $w(0)$ and β any strategy of Player 2.
One has:

$$\dot{W}_t(w(0), \alpha, \beta) = \alpha_t A \beta_t - W_t(w(0), \alpha, \beta)$$

$$\dot{W}_t(\omega(0), \alpha, \beta) = \alpha_t A \beta_t - W_t(\omega(0), \alpha, \beta),$$

thus :

$$\dot{W}_t(w(0), \alpha, \beta) - \dot{W}_t(\omega(0), \alpha, \beta) = W_t(w(0), \alpha, \beta) - W_t(\omega(0), \alpha, \beta)$$

and $\|W_t(w(0), \alpha, \beta) - W_t(\omega(0), \alpha, \beta)\| \leq e^{-t}\|w(0) - \omega(0)\|$.
But $W_t(w(0), \alpha, \beta) \in C$ hence $d(W_t(\omega(0), \alpha, \beta), C) \to 0$ uniformy in β.
Finally the same property holds in the discrete time game with random trajectories. ∎

Corollary 2.4

If C is approachable in the expected deterministic repeated game, C contains a **B***-set.*

Proof :
Consider a strategy σ that approach C and show that the closure of the set of limit points of all trajectories compatible with σ, is a discriminating domain. ∎

9. Approachability with signals, Perchet, 2011 [29] (see Part B.1).

10. Approachability plays a crucial role in games with incomplete information, Aumann and Maschler,1995 [4], Kohlberg, 1975 [18], MSZ.

Fictitious play

Discrete fictitious play and continuous best reply

Consider the repetition of a game $g : S = \prod S^i \to \mathbb{R}^I$.

The discrete fictitious play procedure (DFP) satisfies the following: given an n-(stage) history $h_n = (x_1 = \{x_1^i\}_{i=1,\dots,I}, x_2, \dots, x_n) \in S^n$, the move x_{n+1}^i of each player i at stage $n+1$ is a best reply to the "average moves" of her opponents.

For $n \geq 3$ there are two variants:

- **independent FP**:

for each i, let $\overline{x}_n^i = \frac{1}{n}\sum_{m=1}^{n} x_m^i$ and $\overline{x}_n^{-i} = \{\overline{x}_n^j\}_{j \neq i}$.

Player i computes, for each of her opponents $j \in I$, the empirical distribution on her moves and considers the product distribution. Then, her next move satisfies:

$$x_{n+1}^i \in BR^i(\overline{x}_n^{-i})$$

so that:

$$\overline{x}_{n+1}^i - \overline{x}_n^i = \frac{x_{n+1}^i - \overline{x}_n^i}{n+1}$$

which can also be written as :

$$\overline{x}_{n+1}^i - \overline{x}_n^i \in \frac{1}{(n+1)}[BR^i(\overline{x}_n^{-i}) - \overline{x}_n^i]. \tag{9}$$

- **correlated FP**:

one defines a point $\tilde{x}_n^{-i}$ in $\Delta(S^{-i})$ by :

$$\tilde{x}_n^{-i} = \frac{1}{n}\sum_{m=1}^{n} x_m^{-i}$$

which is the empirical distribution of joint moves of the opponents $-i$. Here the dynamics is given by:

$$x_{n+1}^i \in BR^i(\tilde{x}_n^{-i})$$

We will consider here the first approach (unless otherwise specified).

The continuous counterpart of this difference inclusion, is the differential inclusion called continuous fictitious play (CFP):

$$\dot{X}_t^i \in \frac{1}{t}[BR^i(X_t^{-i}) - X_t^i]. \tag{10}$$

The change of time $Z_s = X_{e^s}$ leads to :

$$\dot{Z}_s^i \in [BR^i(Z_s^{-i}) - Z_s^i] \tag{11}$$

which is the (continuous time) best response (or best reply) dynamic (BR) introduced by Gilboa and Matsui, 1991 [14].

General properties

Convergence

Definition 3.1

A process z_n (discrete) or z_t (continuous) converges in a metric space M to $Z \subset M$ if $d(z_n, Z)$ or $d(z_t, Z)$ goes to 0 as n or $t \to \infty$.

Proposition 3.1

If (DFP) or (CFP) converges to a point x, x is a Nash equilibrium.

Proof :

If x is not a Nash then $d(x, BR(x)) = \delta > 0$.
Hence by uppersemicontinuity $d(y, BR(z)) \geq \delta/2 > 0$ for each y and z in a neighborhood of x, which prevents convergence. ■

The dual property is clear:

Proposition 3.2

If x is a Nash equilibrium, it is a rest point of (DFP) and (CFP).

Remark

(DFP) is “previsible": in the game with payoffs

$\sqrt{2}$	0
0	1

if Player 1 follows (DFP) her move is always pure, since the past frequency of Left, say y, is rational; so that unilateral (DFP) guarantees only 0.

Potential games

Recall that x_n converges to $NE(G)$ if $d(x_n, NE(G))$ goes to 0.
Since g is continuous and X is compact, an equivalent property is that for any $\varepsilon > 0$ and n large enough, x_n is an ε-equilibrium.

Proposition 3.3 (Monderer and Shapley, 1996 [23])

In finite potential games (DFP) converges to $NE(g)$.

Proof :
Let F be a potential for g. Since F is multilinear, one has:

$$\begin{aligned} F(\bar{x}_{n+1}) - F(\bar{x}_n) &= F(\bar{x}_n + \frac{1}{n+1}(x_{n+1} - \bar{x}_n)) - F(\bar{x}_n) \\ &\geq \sum_i \frac{1}{n+1}[F(x^i_{n+1}, \bar{x}^{-i}_n) - F(\bar{x}_n)] - \frac{K_1}{(n+1)^2} \qquad (12) \end{aligned}$$

for some constant K_1 independent of n.

Let $a_{n+1} = \sum_i [F(x^i_{n+1}, \bar{x}^{-i}_n) - F(\bar{x}_n)] = \sum_i [g^i(x^i_{n+1}, \bar{x}^{-i}_n) - g^i(\bar{x}_n)]$ (≥ 0 by definition of (DFP)).
Adding the previous inequality implies

$$F(\bar{x}_{n+1}) \geq \sum_{m=1}^{n+1} \frac{a_m}{m} - K_2$$

for some constant K_2. Since $a_m \geq 0$ and F is bounded, $\sum_{m=1}^{n+1} \frac{a_m}{m}$ converges. This implies

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N} a_n = 0, \tag{13}$$

(In fact for M large enough, $\sum_{m\geq M} \frac{a_m}{m} \leq \varepsilon$ hence a fortiori $\frac{1}{N}\sum_{m\geq M}^{N} a_m \leq \varepsilon$ and the remaining term is bounded.)

Now a consequence of (13) is that, for any $\varepsilon > 0$,

$$\frac{\#\{n \leq N; \bar{x}_n \notin NE^{\varepsilon}(F)\}}{N} \to 0, \qquad \text{as } N \to \infty. \tag{14}$$

In fact there exists $\delta > 0$ such that $\bar{x}_n \notin NE^{\varepsilon}(F)$ implies $a_{n+1} \geq \delta$.
Inequality (14) in turns implies that $\bar{x}_n$ belongs to $NE^{2\varepsilon}(F)$ for n large enough.
Otherwise $\bar{x}_m \notin NE^{\varepsilon}(F)$ for all m in a neighborhood of n of relative size $0(\varepsilon)$ non negligible.
(This is a general property of Cesaro mean of Cesaro means). ■

The next proof is due to Harris, 1998 [16], in the finite case and Benaim, Hofbauer and Sorin, 2005 [5], in the compact case. Assume (H): H^i is defined on a product X of compact convex subsets X^i of an euclidean space, $\mathscr{C}^1$ and concave in x^i.

Proposition 3.4

Under (H), (BR) converges for potential games.

Proof

Let F be a potential for the game $\{H^i; i \in I\}$ and
$W(x) = \sum_i [G^i(x) - F(x)]$ where $G^i(x) = \max_{s \in X^i} F(s, x^{-i})$.
Thus x is a Nash equilibrium iff $W(x) = 0$.
Let x_t be a solution of (BR) and consider $f_t = F(x_t)$. Then

$$\dot{f}_t = \sum_i D_i F(x_t)\, \dot{x}_t^i.$$

By concavity one obtains:

$$F(x_t^i, x_t^{-i}) + D_i F(x_t^i, x_t^{-i})\, \dot{x}_t^i \geq F(x_t^i + \dot{x}_t^i, x_t^{-i})$$

which implies :

$$\dot{f}_t \geq \sum_i [F(x_t^i + \dot{x}_t^i, x_t^{-i}) - F(x_t)] = W(x_t) \geq 0,$$

hence f is increasing but bounded.
f is thus constant on the limit set $L(\mathbf{x})$.

By the previous majoration, for any accumulation point x^* one has $W(x^*) = 0$ and x^* is a Nash equilibrium. ∎

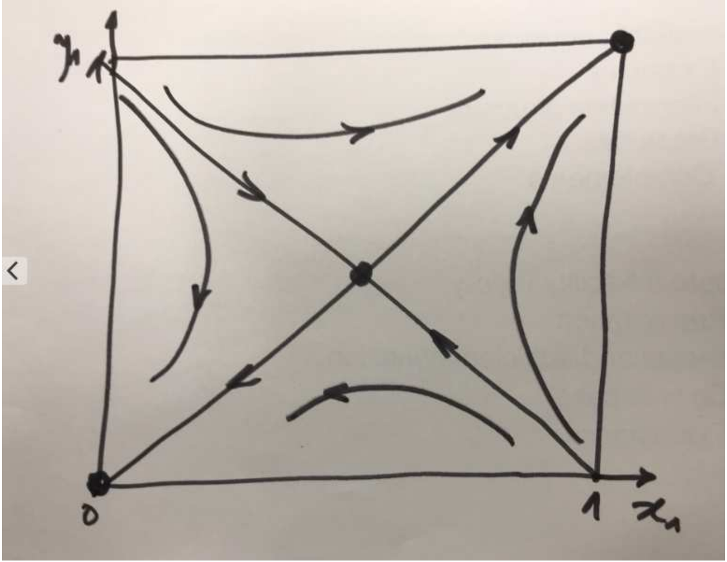
Note that one cannot expect uniform convergence: see the standard symmetric coordination game below.

The only attractor that contains $NE(F)$ is the diagonal.

In particular convergence of (BR) or (CFP) does not imply directly convergence of (DFP)

$(1,1)$	$(0,0))$
$(0,0)$	$(1,1)$

Note that the equilibrium $(1/2, 1/2)$ is unstable but the time to go from $(1/2^+, 1/2^-)$ to $(1,0)$ is not bounded.



y_1
1
0
1
x_1

Further results

Monderer, Samet and Sela, 1997 [21], introduce a comparison between the anticipated payoff at stage n: $E^i_n = F^i(x^i_n, \bar{x}^{-i}_{n-1})$, and the average payoff up to stage n (excluded): $A^i_n = \frac{1}{n-1}\sum_{p=1}^{n-1} F^i(x_p)$.

Proposition 3.5

Assume (DFP) for player i (with 2 players or correlated (DFP)), then

$$E^i_n \geq A^i_n. \tag{15}$$

Proof :
By definition of (DFP), one has:

$$\sum_{m=1}^{n-1} F^i(x^i_n, x^{-i}_m) \geq \sum_{m=1}^{n-1} F^i(s, x^{-i}_m), \quad \forall s \in X^i. \tag{16}$$

Write $(n-1)E^i_n = b_n = \sum_{m=1}^{n-1} a(n,m)$ for the left hand side.

By choosing $s = x_{n-1}^i$ one obtains

$$b_n \geq a(n-1, n-1) + b_{n-1}$$

hence by induction

$$E_n^i \geq A_n^i = \sum_{m=1}^{n-1} a(m,m)/(n-1).$$

∎

Corollary 3.1

The average payoffs converge to the value for (DFP) in the zero-sum case.

Proof :

Recall that in this case E_n^1 (resp. E_n^2) converges to v (resp. $-v$), since $\bar{x}_n^{-i}$ converges to the set of optimal strategies of $-i$. ■

A last property is the following (the argument is already in Shapley, 1964 [30]). Note that it is not stated in the usual state variable $(\bar{x}_n)$ but is related to Myopic Adjustment Dynamics.

Proposition 3.6 (Monderer and Sela, 1996 [22])

Assume (DFP) for player i with 2 players.Then :

$$F^i(x^i_n, x^{-i}_{n-1}) \geq F^i(x_{n-1}). \tag{17}$$

Proof :

In fact the (DFP) property implies :

$$F^i(x^i_{n-1}, \overline{x}^{-i}_{n-2}) \geq F^i(x^i_n, \overline{x}^{-i}_{n-2}) \tag{18}$$

and:

$$F^i(x^i_n, \overline{x}^{-i}_{n-1}) \geq F^i(x^i_{n-1}, \overline{x}^{-i}_{n-1}). \tag{19}$$

Hence if equation (17) is not satisfied, adding to (18) and using the linearity would contradict (19). ■

Remark

The two results above correspond to unilateral properties.

Shapley's example

Shapley, 1964 [30], Monderer and Sela, 1996 [22].

$(0,0)$	(a,b)	(b,a)
(b,a)	$(0,0)$	(a,b)
(a,b)	(b,a)	$(0,0)$

Let $a > b > 0$.
Note that the only equilibrium is $(1/3, 1/3, 1/3)$.

Proposition 3.7

(DFP) does not converge.

Proof 1:
Starting from a Pareto entry, the improvement principle (17) implies that one will stay on Pareto entries. Hence the sum of the stage payoffs will always be $(a+b)$. If (DFP) converges then it converges to $(1/3, 1/3, 1/3)$ so that the anticipated payoff converges to the Nash payoff $\frac{a+b}{3}$ which contradicts inequality (15).

Proof 2:
Add a line to the Shapley matrix G defining a new matrix:

G' =

$(0,0)$	(a,b)	(b,a)
(b,a)	$(0,0)$	(a,b)
(a,b)	(b,a)	$(0,0)$
$(c,0)$	$(c,0)$	$(c,0)$

with $2b > a > b > c > \frac{a+b}{3}$.
By the improvement principle (17), starting from a Pareto entry one will stay on the Pareto set, hence line 4 will not be played so that (DFP) in G' is also (DFP) in G.
If there were convergence it would be to a Nash equilibrium hence to $(1/3,1/3,1/3)$ in G, thus to $((1/3,1/3,1/3,0);(1/3,1/3,1/3))$ in G'.
But a best reply for Player 1 to $(1/3,1/3,1/3)$ in G' is the fourth line, contradiction.

Proof 3:
Following Shapley, 1964 [30], let us study explicitely the (DFP) path.
Starting from (12), there is a cycle : 12, 13, 23, 21, 31, 32, 12,..
Let $r(ij)$ be the duration of the corresponding entry and a_1, a_2, a_3 the different cumulative payoffs of Player 1 at the beginning of the cycle i.e. if it occurs at stage $n+1$, given by:

$$a_i = \sum_{m=1}^{n} A_{ij_m}$$

(proportional to the payoff of i against $\bar{y}_n$).

Thus, after $r(12)$ stages of (12) and $r(13)$ stages of (13)

$$a'_1 = a_1 + r(12)a + r(13)b$$
$$a'_2 = a_2 + r(12)0 + r(13)a$$

and then Player 1 switches to move 2, hence one has:

$$a'_2 \geq a'_1$$

but also:

$$a_1 \geq a_2$$

(because 1 was played) so that :

$$a'_2 - a_2 \geq a'_1 - a_1$$

which gives:

$$r(13)(a-b) \geq r(12)a$$

and by induction at the next round:

$$r'(11) \geq [\frac{a}{(a-b)}]^6 r(11)$$

so that exponential growth occurs and the empirical distribution does not converge. ■

RSP and Shapley triangle

We follow the analysis of (BR) in RSP (rock-scissors-paper game) done by Gaunersdorfer and Hofbauer, 1995 [13].

Consider the following symmetric game with $a > 0, b > 0$.

$(0,0)$	$(a,-b)$	$(-b,a)$
$(-b,a)$	$(0,0)$	$(a,-b)$
$(a,-b)$	$(-b,a)$	$(0,0)$

Note that the only Nash is $(1/3,1/3,1/3)$ with payoff $(a-b)/3$.
Let $V(x) = \max_y yAx$ and define $v_t = V(x_t)$ on a (BR) path.
Then, if $e^i = BR(x_t)$ on some intervall:

$$\dot{v}_t = e^i A\dot{x}_t = e^i Ae^i - e^i Ax_t$$

hence since $e^i Ae^i = 0$:

$$\dot{v}_t = -v_t.$$

If $a \geq b$ then $V(x) \geq (a-b)/3 \geq 0$ for all x, hence v_t decreases to $(a-b)/3$ and x_t converges to the equilibrium which is reached in finite time if $a-b>0$.

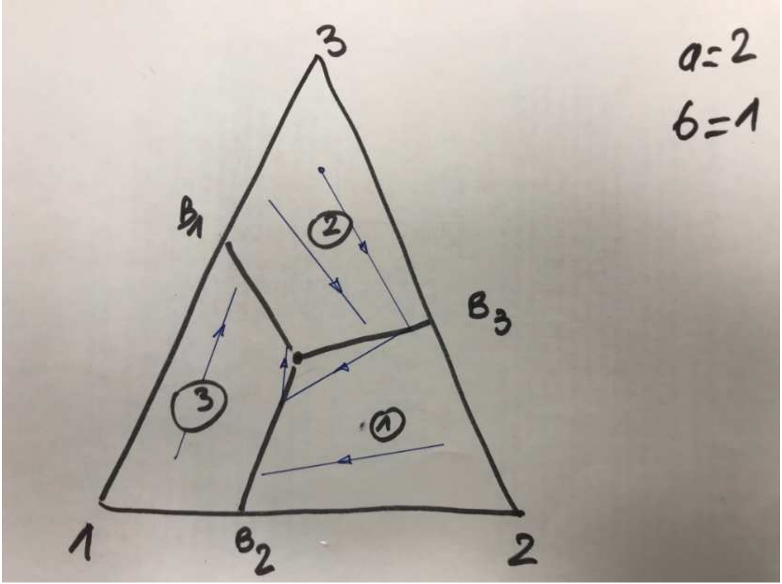
In the other case, v_t converges to 0 that is larger than $(a-b)/3$. Then x_t follows a cycle which is called the “Shapley triangle".
The coordinates are given by:

$$B_3 = (0, \frac{a+b}{2a+b}, \frac{a}{2a+b})$$

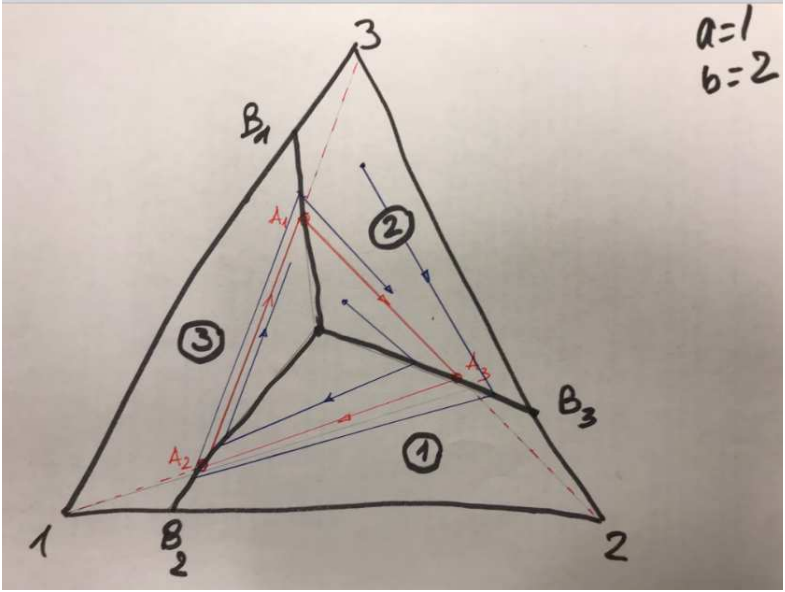
$$A_3 = (\frac{a^2}{a^2+b^2+ab}, \frac{b^2}{a^2+b^2+ab}, \frac{ab}{a^2+b^2+ab})$$

A_1, A_3 and e_2 are on a line and the segment $[A_1, A_3]$ is characterized by

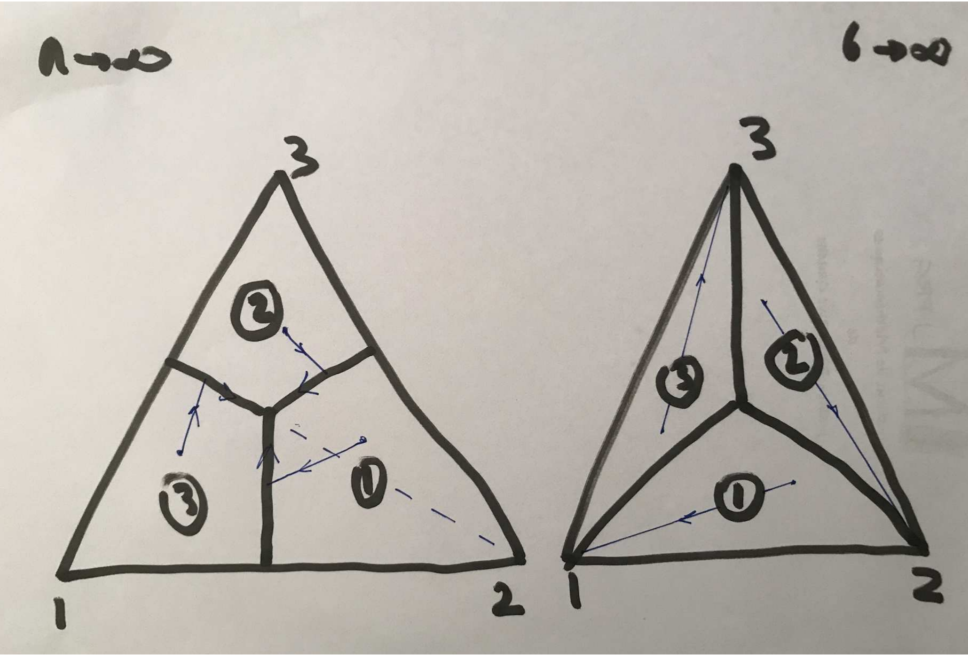
$$F(2,x) = 0 \geq F(i,x), \quad i = 1,3.$$



$a > b$:E is stable



$a < b$: Shapley triangle (in red)


a → ∞
b → ∞
3
2
3
1
1
2
3
2
3
1
1
2

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