Computational Complexity in Games and Auctions

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EECS, MIT

Computational Complexity in Games and Auctions

- Complexity of Equilibria (day 1)
- Mechanism Design (day 2)

game theory

“reverse game theory”

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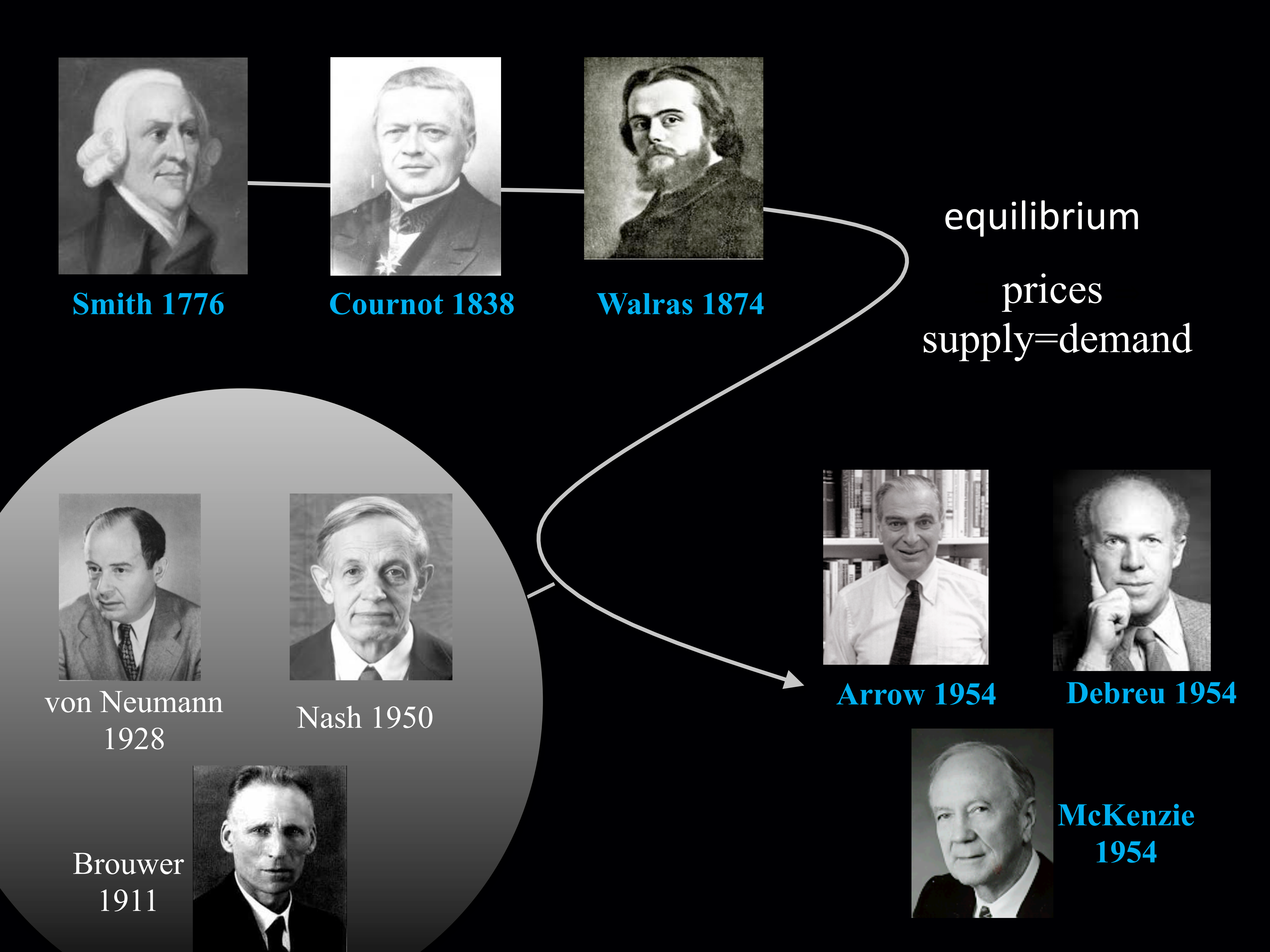
game theory

“reverse game theory”

Econo
mics

equilibrium

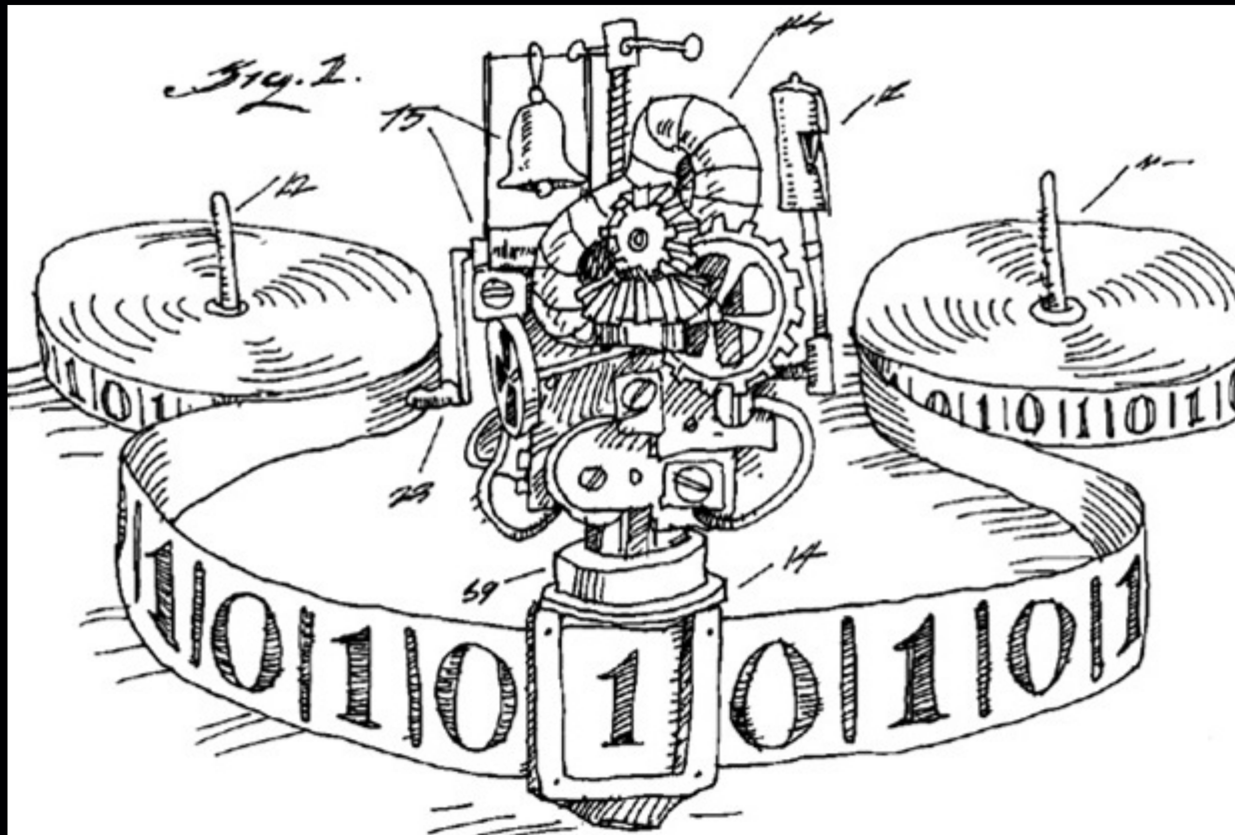

Smith 1776
Cournot 1838
Walras 1874
equilibrium
prices
supply=demand
von Neumann
1928
Nash 1950
Brouwer
1911
Arrow 1954
Debreu 1954
McKenzie
1954

[Myerson'99]: "Nash's theory of non-cooperative games should now be recognized as one of the outstanding intellectual advances of the twentieth century. The formulation of Nash equilibrium has had a fundamental and pervasive impact in Economics and the Social Sciences which is comparable to that of the discovery of the DNA double helix in the biological sciences."

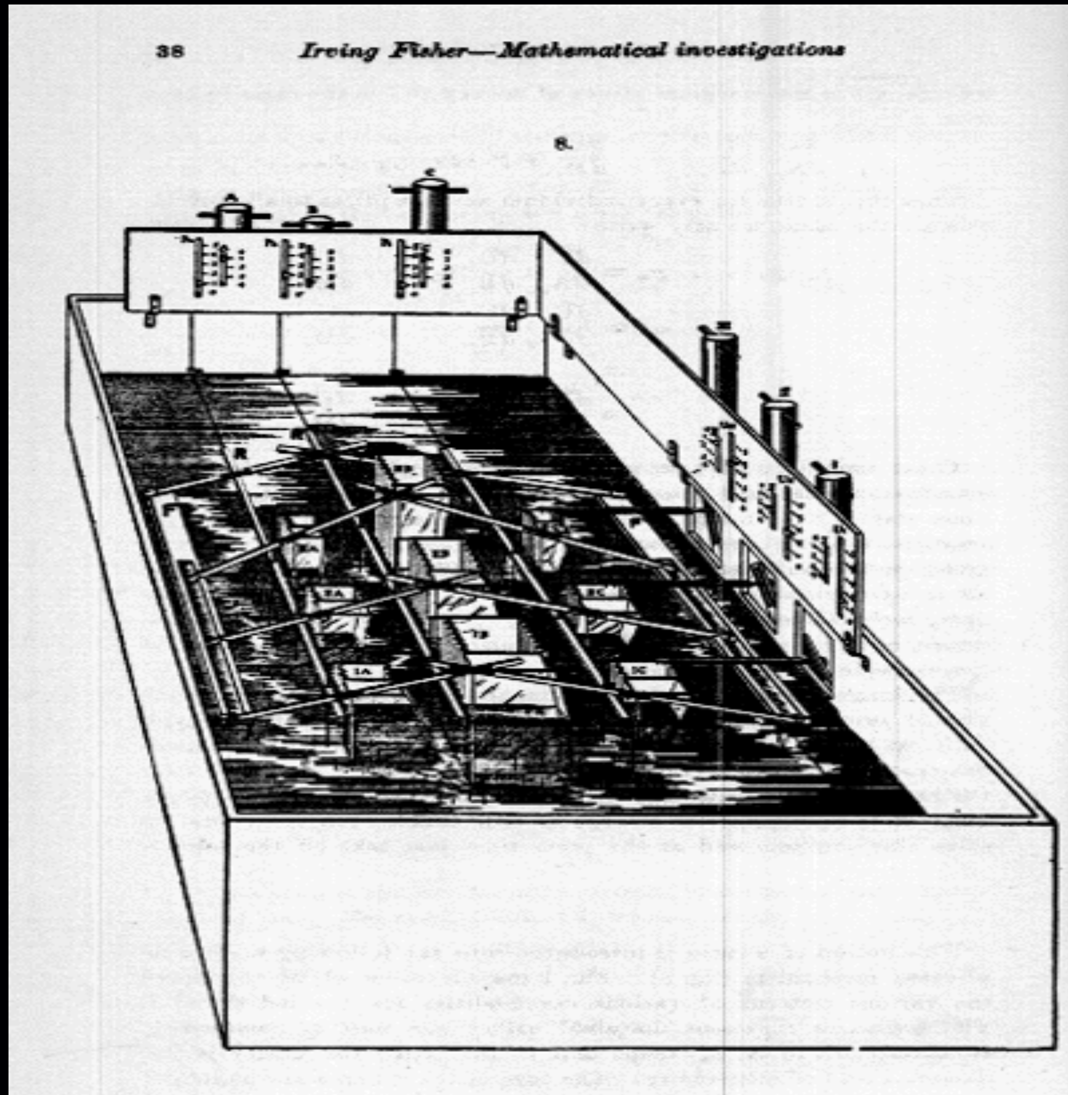
Econo
mics

equilibrium

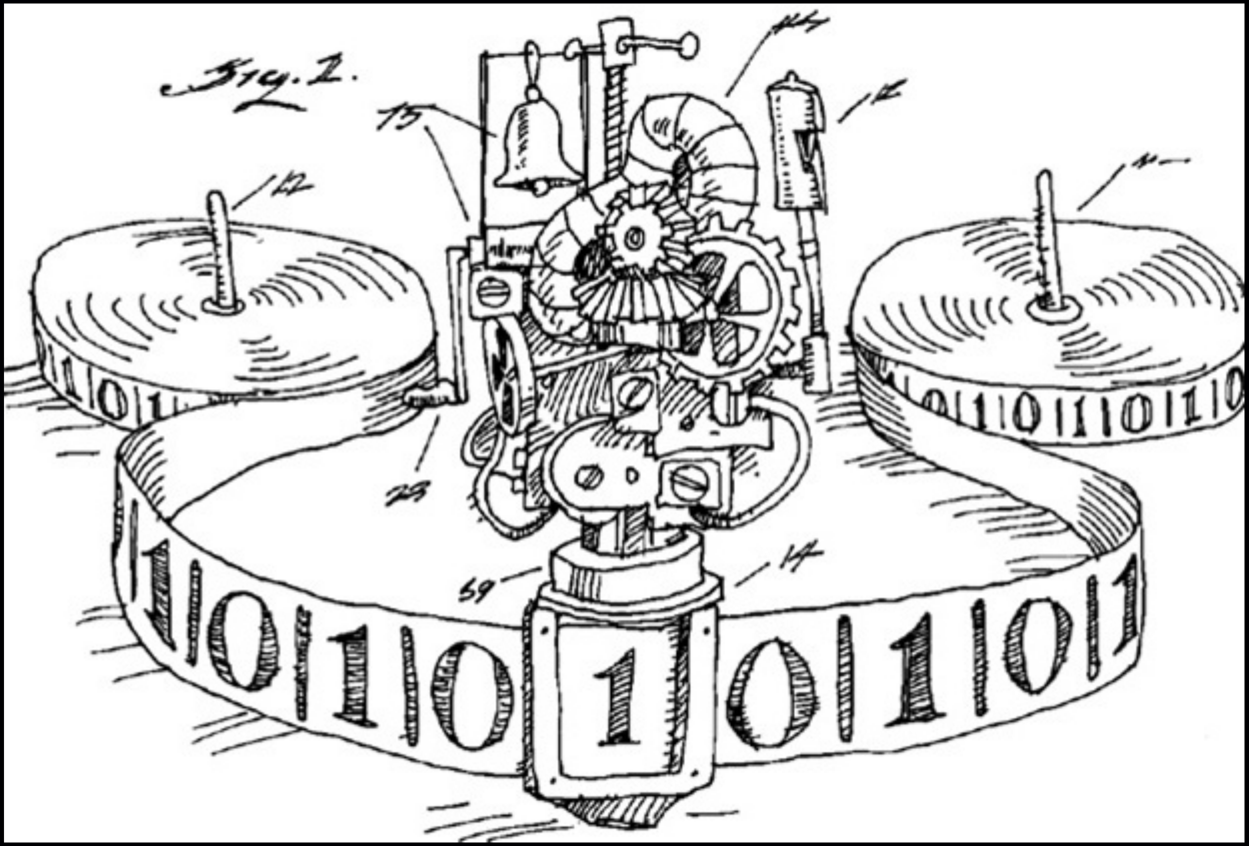
computation



How long to equilibrium?



[Irving Fisher 1891]: Hydraulic apparatus for calculating the equilibrium of a *3-person, 3-commodity* exchange economy.



How long to equilibrium?

Universality requires tractability

“If your laptop can’t find the equilibrium, how can they market?” **[Kamal Jain]**

von Neumann
1928

Nash 1950

Brouwer
1911

want to study the computational
features of these theorems

Menu

- Equilibria
- Existence proofs
  - Minimax
  - Nash
  - Brouwer
- Complexity of Equilibria
  - Total Search Problems in NP
  - PPAD
- The World Beyond

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Games and Equilibria

Dive \ Kick	Left (1/2)	Right (1/2)
Left (1/2)	1 , -1	-1 , 1
Right (1/2)	-1 , 1	1, -1

Penalty Shot Game

Equilibrium: A pair (x,y) of randomized strategies so that no player has incentive to *deviate* if the other does not.

$$x^T R y \geq x'^T R y, \forall x'$$
$$x^T C y \geq x^T C y', \forall y'$$

$\sum_{i,j} C_{ij} \cdot x_i \cdot y_j$

[von Neumann '28]: An equilibrium exists in every two-player zero-sum game ($R+C=0$)

+[Dantzig'40s]: …in fact, this follows from strong LP duality

+[Khachiyan'79]: …and is computable in polynomial-time.

Games and Equilibria

		2/5	3/5
	Kick / Dive	Left	Right
1/2	Left	2 , -1	-1 , 1
1/2	Right	-1 , 1	1, -1

Penalty Shot Game

Equilibrium: A pair (x,y) of randomized strategies so that no player has incentive to *deviate* if the other does not.

$$x^T R y \geq x'^T R y, \forall x'$$

$$x^T C y \geq x^T C y', \forall y'$$

$$\sum_{i,j} C_{ij} \cdot x_i \cdot y_j$$

[Nash '50/'51]: An equilibrium exists in every finite game.

- proof used Kakutani/Brouwer's fixed point theorem, and no constructive proof has been found in 70+ years
- same is true for economic equilibria

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von Neumann

The Minimax Theorem

- [von Neumann'28]: Suppose and are compact convex sets, and is a continuous function that is *convex-concave*, i.e. is convex for all fixed , and is concave for all fixed Then:

- In a zero-sum game, take
  - how much row pays column
- Then is an equilibrium, where

and

Presidential Elections

	Morality	Tax Cuts
Economy	+3, -3	-1, +1
Society	-2, +2	1, -1

Suppose **Clinton** announces strategy **(1/2,1/2)**.
What would Trump do?

A: focus on **Tax Cuts** with probability 1.

indeed against **(1/2, 1/2)** strategy "Morality" gives expected expected payoff -1/2 while "Tax Cuts" gives 0

Presidential Elections

	Morality	Tax Cuts
Economy	+3, -3	-1, +1
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More generally, suppose **Clinton** commits to strategy (x_1, x_2).
N.B.: Committing to a strategy in advance may not be optimal for Clinton since Trump may, in principle, exploit it.
How?

$$E[\text{“Morality”}] = -3x_1 + 2x_2$$
$$E[\text{“Tax Cuts”}] = x_1 - x_2$$

So Trump's payoff after best responding to (x_1, x_2) would be

$$\max(-3x_1 + 2x_2, x_1 - x_2),$$

resulting in the following payoff for Clinton:

$$-\max(-3x_1 + 2x_2, x_1 - x_2) = \min(3x_1 - 2x_2, -x_1 + x_2).$$

So the best strategy for Clinton to commit to is:

$$\boxed{(x_1, x_2) \in \operatorname{argmax} \min(3x_1 - 2x_2, -x_1 + x_2)}$$

Presidential Elections

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So the best strategy for Clinton to commit to is:

$$\boxed{(x_1, x_2) \in \text{argmax} \min(3x_1\text{-}2\, x_2,\ \text{-}x_1+x_2)}$$

To compute it Clinton writes the following Linear Program:

$$\begin{aligned} & \max z \\ \text{s.t.} \quad & 3x_1 - 2x_2 \geq z \\ & -x_1 + x_2 \geq z \\ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0. \end{aligned}$$

solution:

$z = 1/7$,

$(x_1, x_2)=(3/7,4/7)$

No matter what Trump does Clinton can guarantee 1/7 to himself by playing (3/7,4/7)

Presidential Elections

	Morality	Tax Cuts
Economy	+3, -3	-1, +1
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Conversely if Trump were forced to commit to a strategy (y_1, y_2) he would solve:

$$\begin{aligned} & \max w \\ \text{s.t.} \quad & -3y_1 + y_2 \geq w \\ & 2y_1 - y_2 \geq w \\ & y_1 + y_2 = 1 \\ & y_1, y_2 \geq 0 \end{aligned}$$

solution:

w = -1/7,

(y_1, y_2)=(2/7,5/7)

No matter what Clinton does Trump can guarantee -1/7 to himself by playing (2/7,5/7)

Presidential Elections "Miracle"

No matter what Trump does Clinton can guarantee 1/7 to himself by playing (3/7,4/7).

No matter what Clinton does Trump can guarantee -1/7 to himself by playing (2/7,5/7).

➔If Clinton plays (3/7,4/7) and Trump plays (2/7,5/7) then none of them can improve their payoff by changing their strategy (because their sum of irrevocable payoffs is 0 and the game is zero-sum).

➔I.e. (3/7,4/7) is best response to (2/7,5/7) and vice versa.

➔Hence they jointly comprise a **Nash equilibrium!**

Why is it a "Miracle"?

Because (3/7,4/7) was computed a priori for Clinton and (2/7,5/7) was computed a priori for Trump.

Nevertheless these strategies magically comprise a Nash equilibrium!

De-mystifying the "Miracle"

Clinton's LP

$$\begin{aligned} &\max z \\ \text{s.t.} \quad & 3x_1 - 2x_2 \geq z \\ & -x_1 + x_2 \geq z \\ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Trump's LP

$$\begin{aligned} &\max w \\ \text{s.t.} \quad & -3y_1 + y_2 \geq w \\ & 2y_1 - y_2 \geq w \\ & y_1 + y_2 = 1 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Why is it that the value of the left LP is **equal to minus** the value of the right LP?

De-mystifying the "Miracle"

Clinton's LP

$$\begin{aligned} &\max z \\ \text{s.t.} \quad & 3x_1 - 2x_2 \geq z \\ & -x_1 + x_2 \geq z \\ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Trump's LP

$$\begin{aligned} &\max -t \\ \text{s.t.} \quad & -3y_1 + y_2 \geq -t \\ & 2y_1 - y_2 \geq -t \\ & y_1 + y_2 = 1 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Why is it that the value of the left LP is **equal to minus** the value of the right LP?

De-mystifying the “Miracle”

Clinton’s LP

$$\begin{aligned} &\max z \\ \text{s.t.} \quad & 3x_1 - 2x_2 \geq z \\ & -x_1 + x_2 \geq z \\ & x_1 + x_2 = 1 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Trump’s LP

$$\begin{aligned} &\min t \\ \text{s.t.} \quad & -3y_1 + y_2 \geq -t \\ & 2y_1 - y_2 \geq -t \\ & y_1 + y_2 = 1 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Why is it that the value of the left LP is **equal to** the value of the right LP?

De-mystifying the "Miracle"

Clinton's LP

$$
\begin{aligned}
&\max z \\
\text{s.t.}\quad & 3x_1 - 2x_2 \geq z \\
& -x_1 + x_2 \geq z \\
& x_1 + x_2 = 1 \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

Trump's LP

$$
\begin{aligned}
&\min t \\
\text{s.t.}\quad & 3y_1 - y_2 \leq t \\
& -2y_1 + y_2 \leq t \\
& y_1 + y_2 = 1 \\
& y_1, y_2 \geq 0
\end{aligned}
$$

$$\min_x \max_y x^T C y = \max_y \min_x x^T C y$$

Why is it that the value of the left LP is **equal to** the value of the right LP?

Linear Programming Duality

➔ Left LP is DUAL to Right LP, hence they have equal values!

Moral of the Story

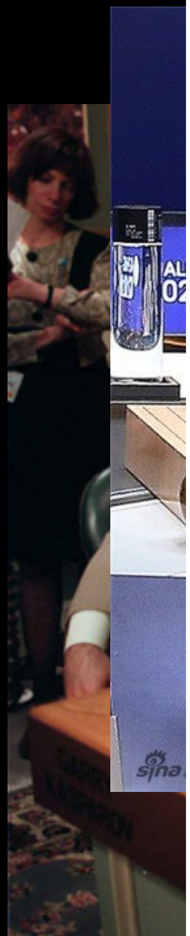
	Morality	Tax Cuts
Economy	+3, -3	-1, +1
Society	-2, +2	1, -1

Existence of a Nash equilibrium in the Presidential Election game follows from Strong Linear Programming duality.

This proof technique generalizes to any 2-player zero-sum game.

Allows us to efficiently (i.e. in polynomial-time) compute Nash equilibria in these games.

Moreover, a wide-class of distributed, online learning dynamics (namely no-regret learning) converge to equilibrium payoffs



All in: CMU's poker computer busts humans over 20-day competition

January 30, 2017 10:49 PM

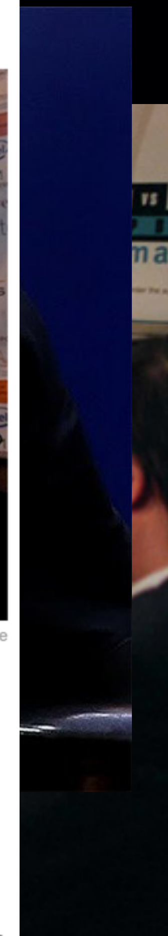
Darrell Sapp/Post-Gazette

Daniel McAulay of Scotland rubs his eyes Monday as he competes in the poker tournament against a Carnegie Mellon computer at the Rivers Casino on the North Shore.

By Sean D. Hamill / Pittsburgh Post-Gazette

The machines are taking over — at least in poker.

Though it had been a point conceded by the humans for the past week, Carnegie Mellon University's poker-playing computer, Libratus, on Monday finally, definitely and soundly defeated four of the world's best Heads-Up, No-Limit, Texas Hold 'em poker players by a resounding $1,766,250 in



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Brouwer

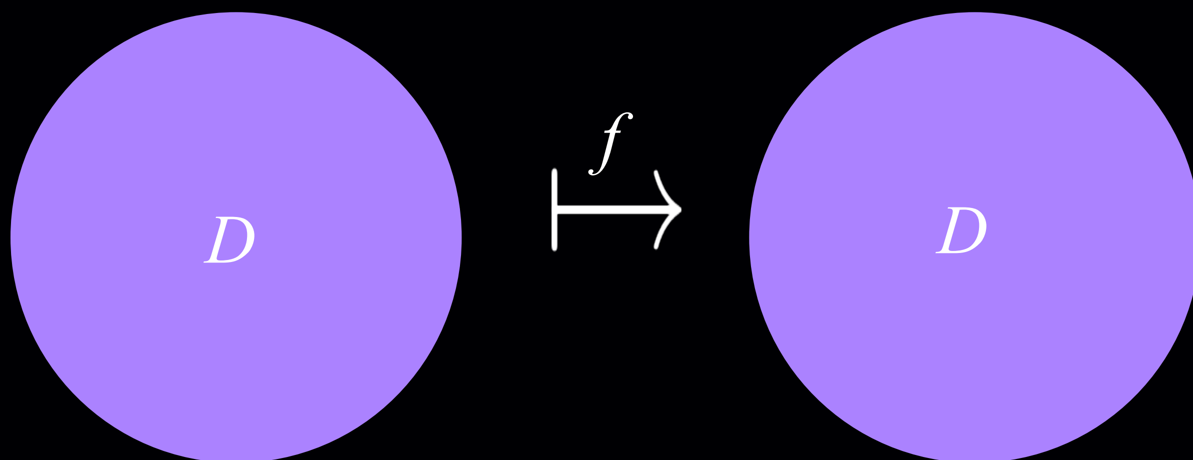
Brouwer's Fixed Point Theorem

[Brouwer 1910]: Let $f: D \longrightarrow D$ be a continuous function from a convex and compact subset D of the Euclidean space to itself.

Then there exists an $x \in D$ s.t. $x = f(x)$.

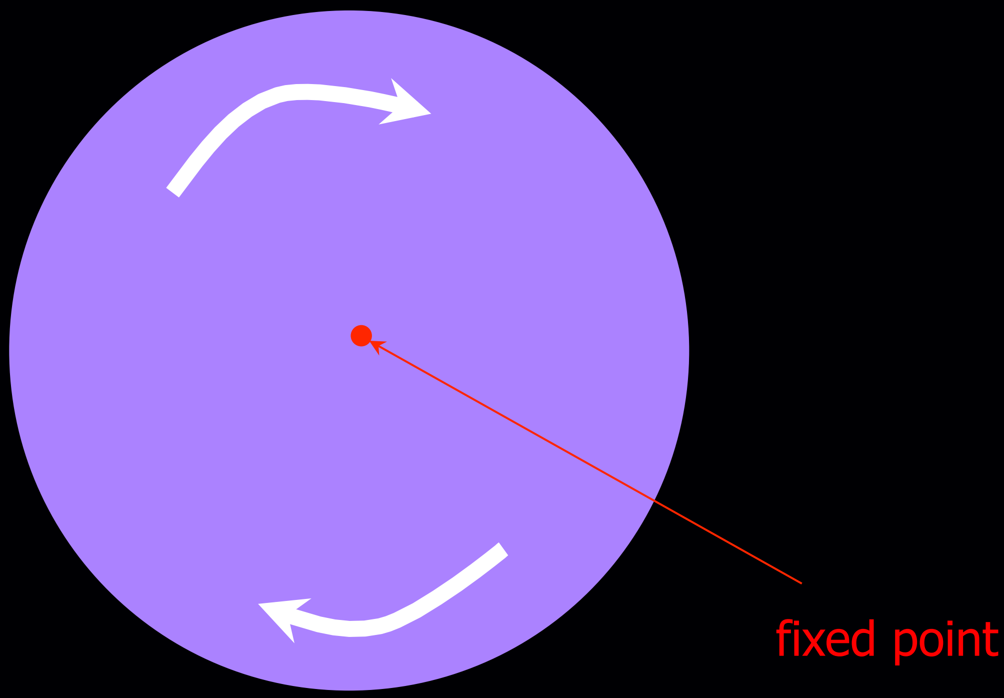
closed and bounded

Below we show a few examples, when D is the 2-dimensional disk.

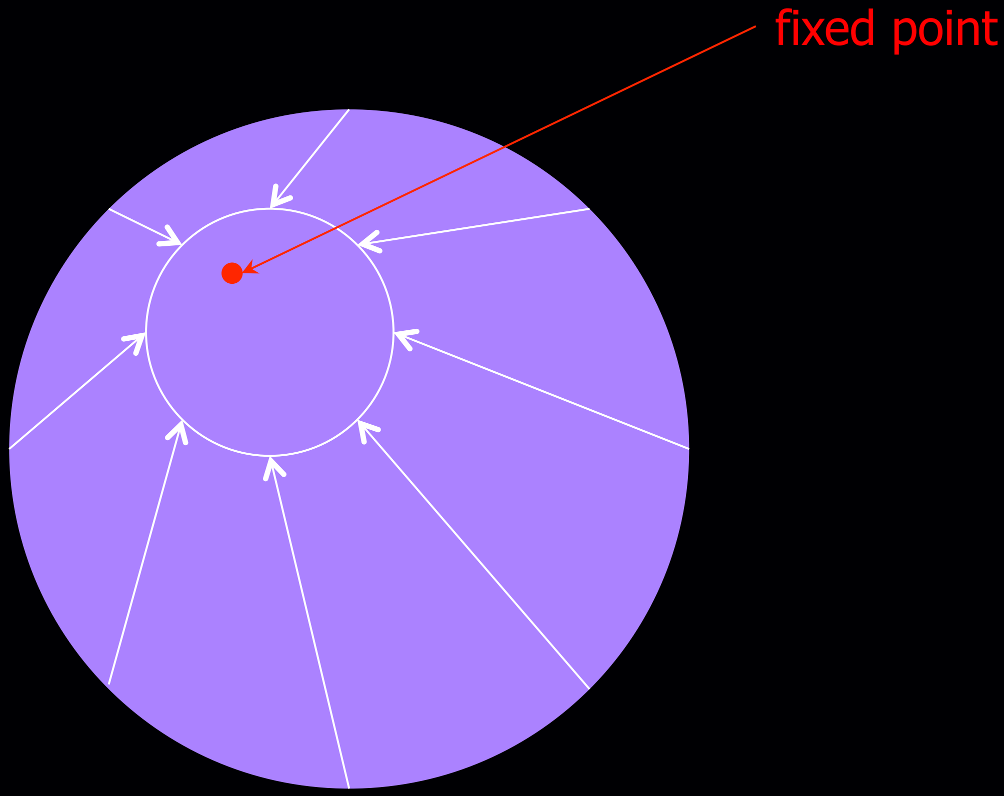


N.B. All conditions in the statement of the theorem are necessary.

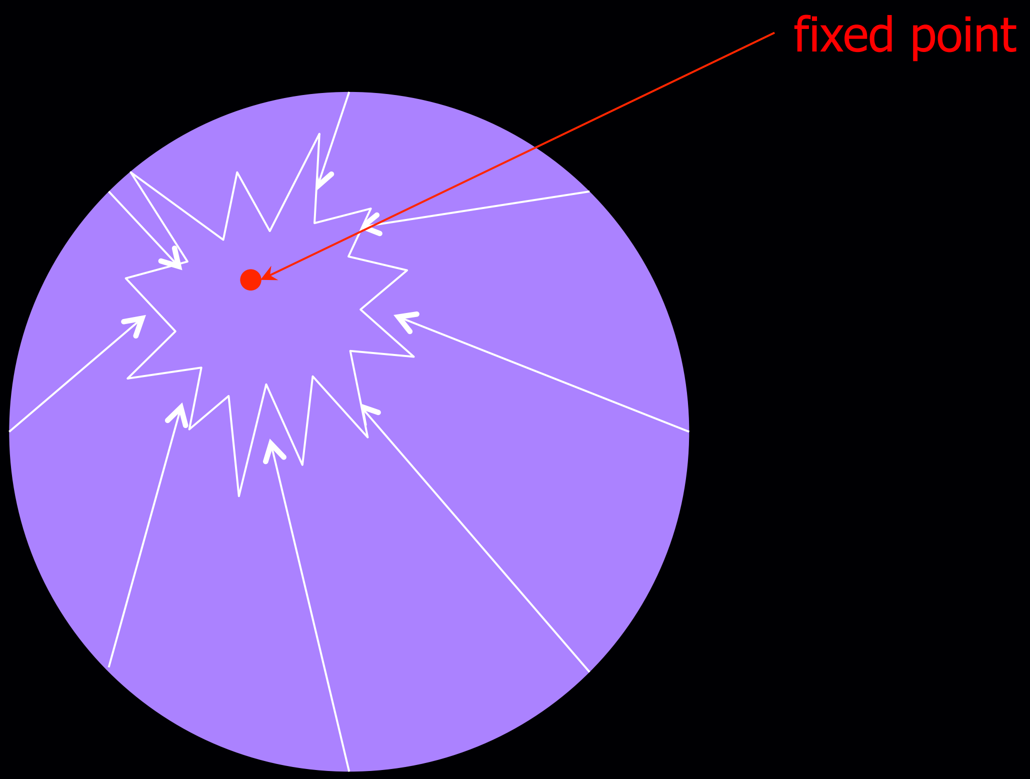
Brouwer's Fixed Point Theorem



Brouwer's Fixed Point Theorem



Brouwer's Fixed Point Theorem




Brouwer $\Rightarrow$ Nash

Visualizing Nash's Proof

Kick / Dive	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1, -1

Penalty Shot Game

$\longmapsto$

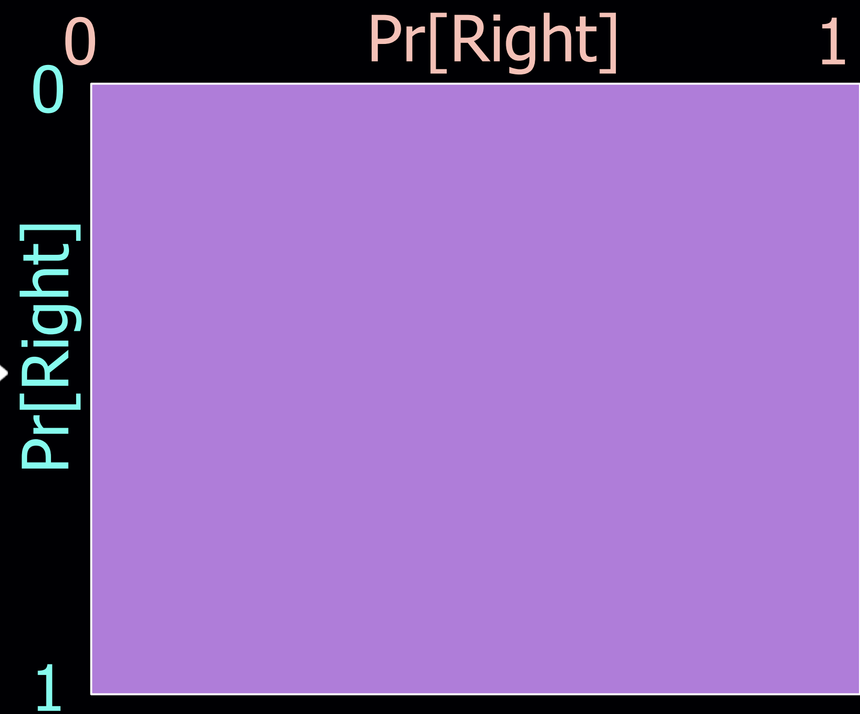
f: $[0,1]^2 \rightarrow [0,1]^2$, continuous
such that
fixed points $\equiv$ Nash eq.

Visualizing Nash's Proof

Kick / Dive	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1, -1

Penalty Shot Game

$\longmapsto$



Visualizing Nash's Proof

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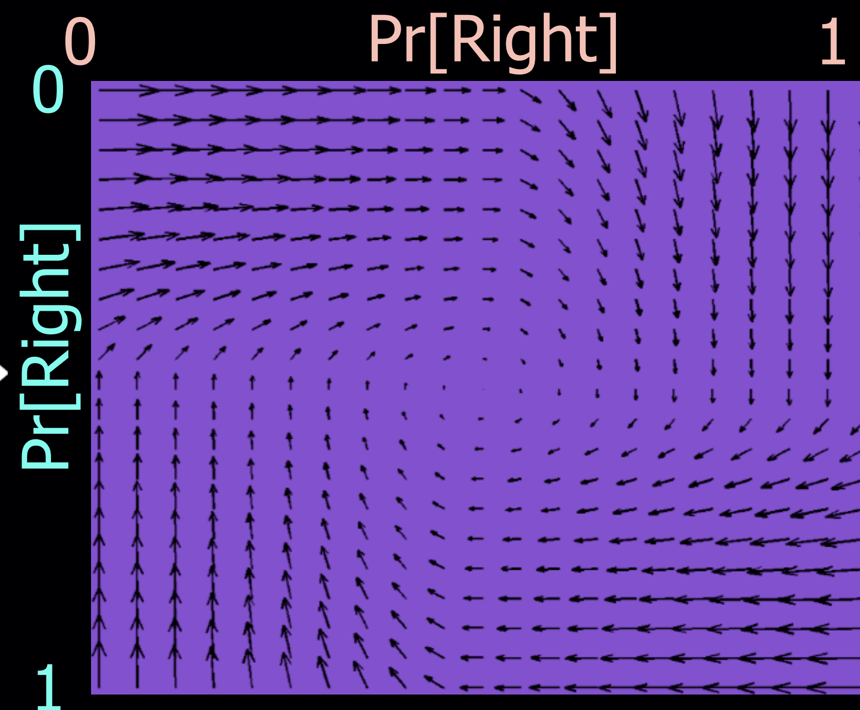

Visualizing Nash's Proof

Kick / Dive	Left	Right
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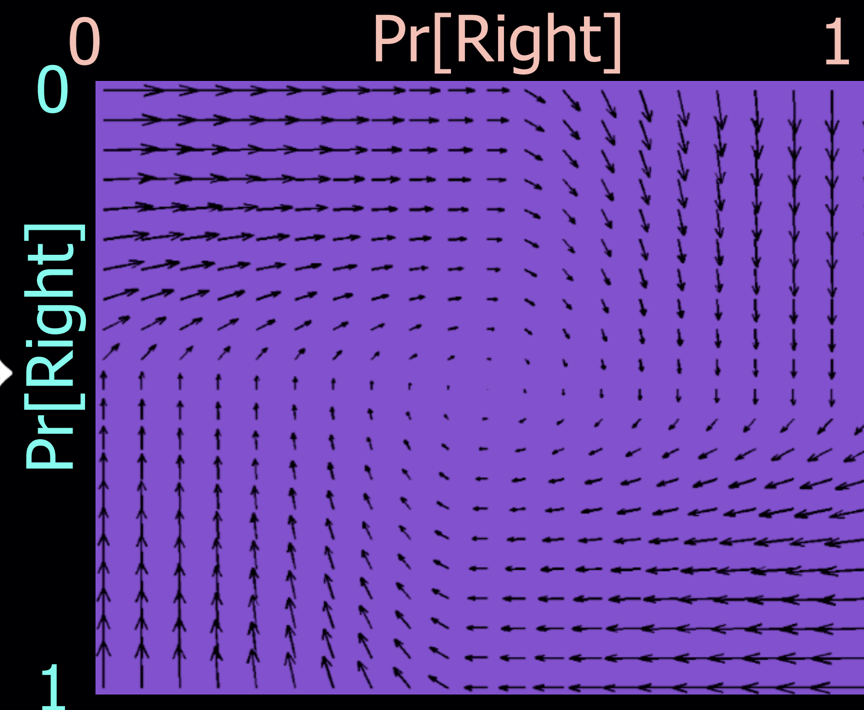
Penalty Shot Game

$\mapsto$

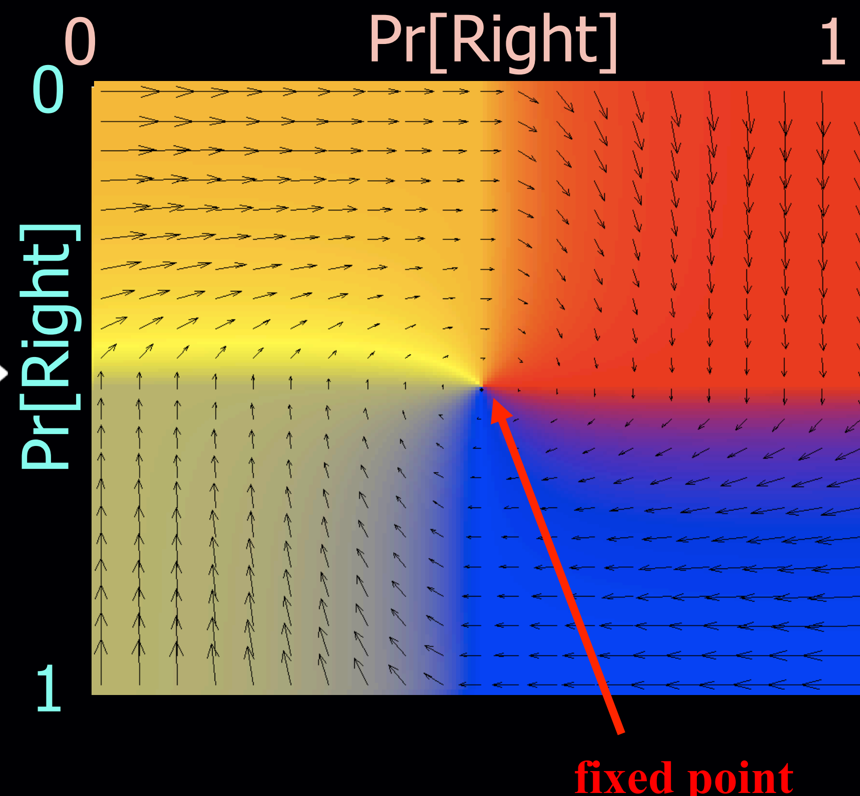


Visualizing Nash's Proof

		½	½
	Kick / Dive	Left	Right
½	Left	1 , -1	-1 , 1
½	Right	-1 , 1	1, -1

Penalty Shot Game

$\mapsto$



Real proof: on the board

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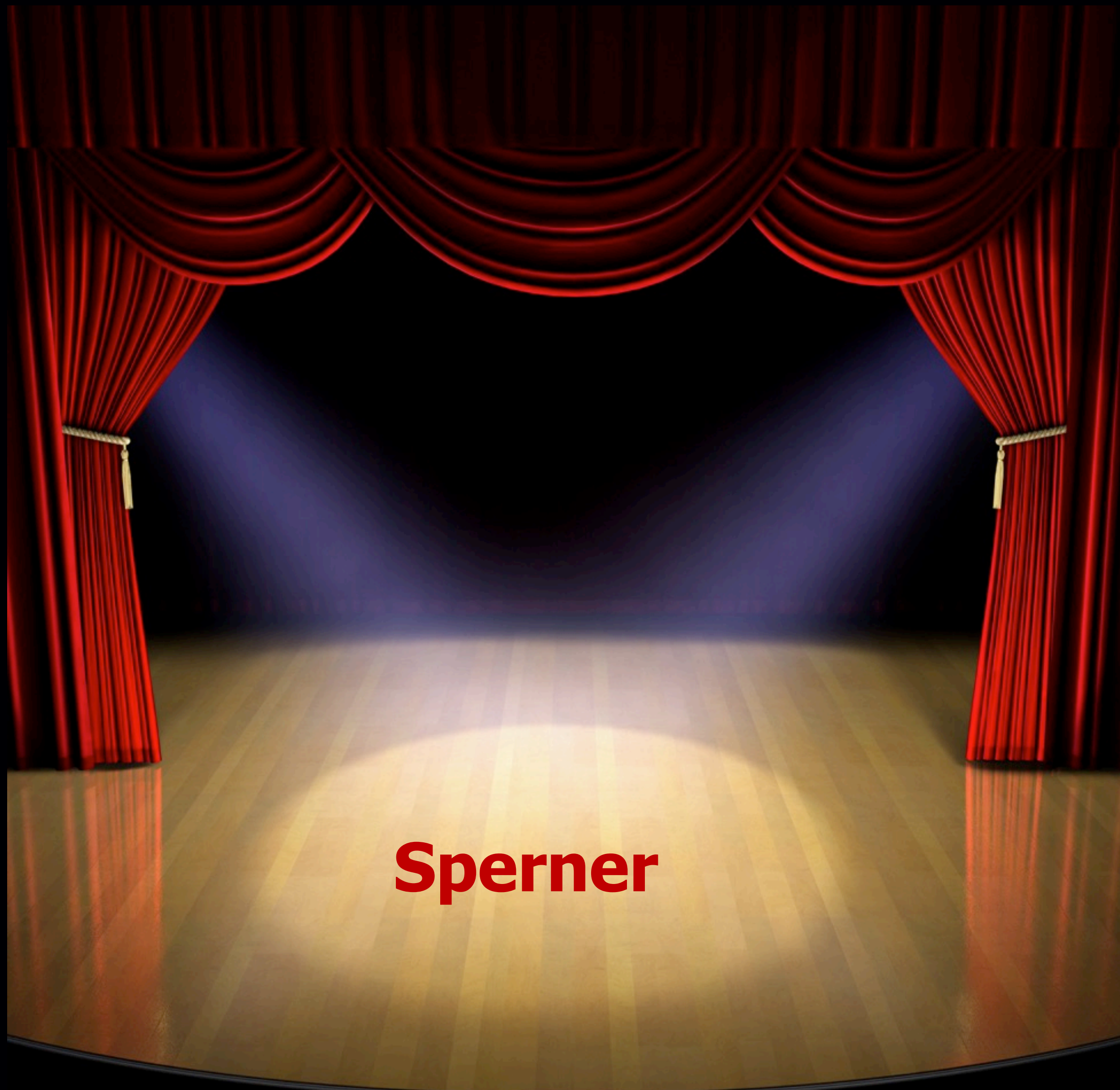
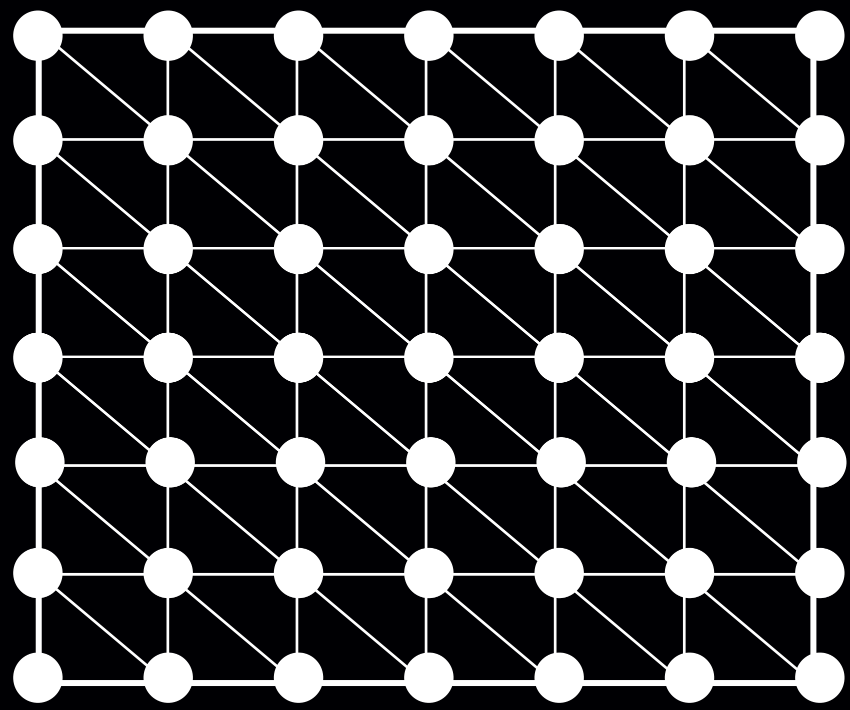

Sperner

Sperner's Lemma (2-d)



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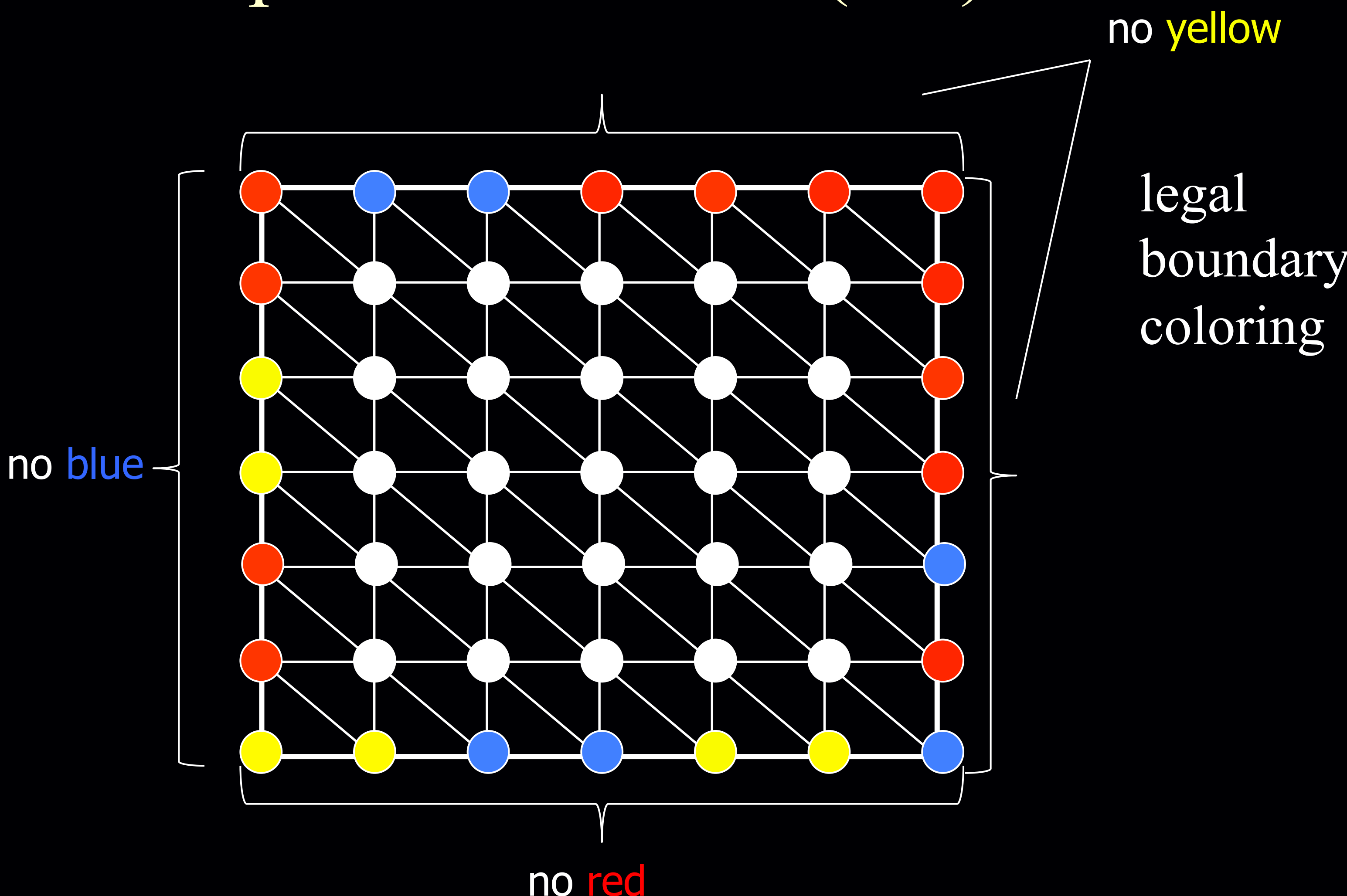
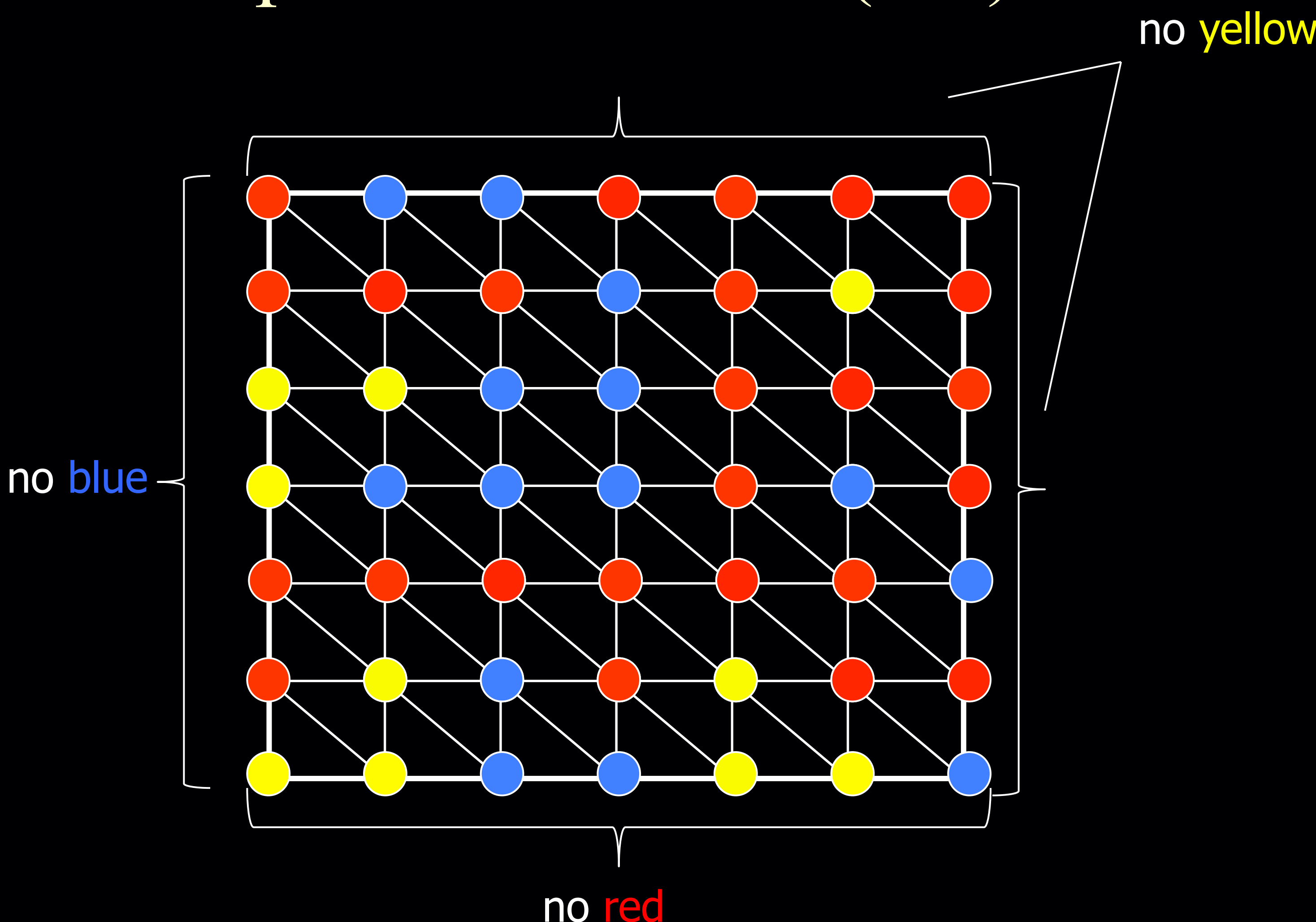


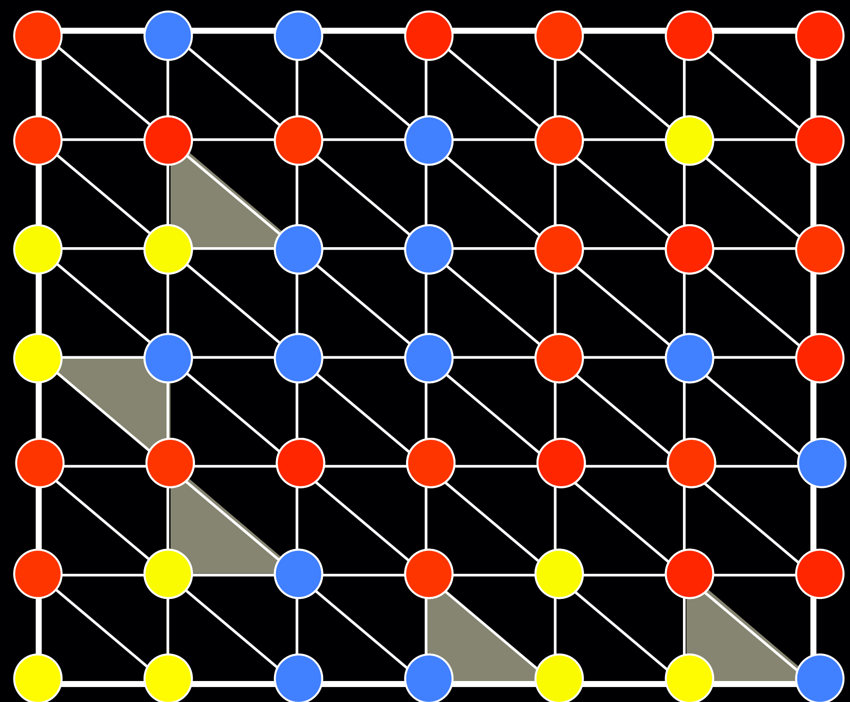
[Sperner 1928]: Color the boundary using three colors in a legal way.

Sperner's Lemma (2-d)



[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Sperner's Lemma (2-d)



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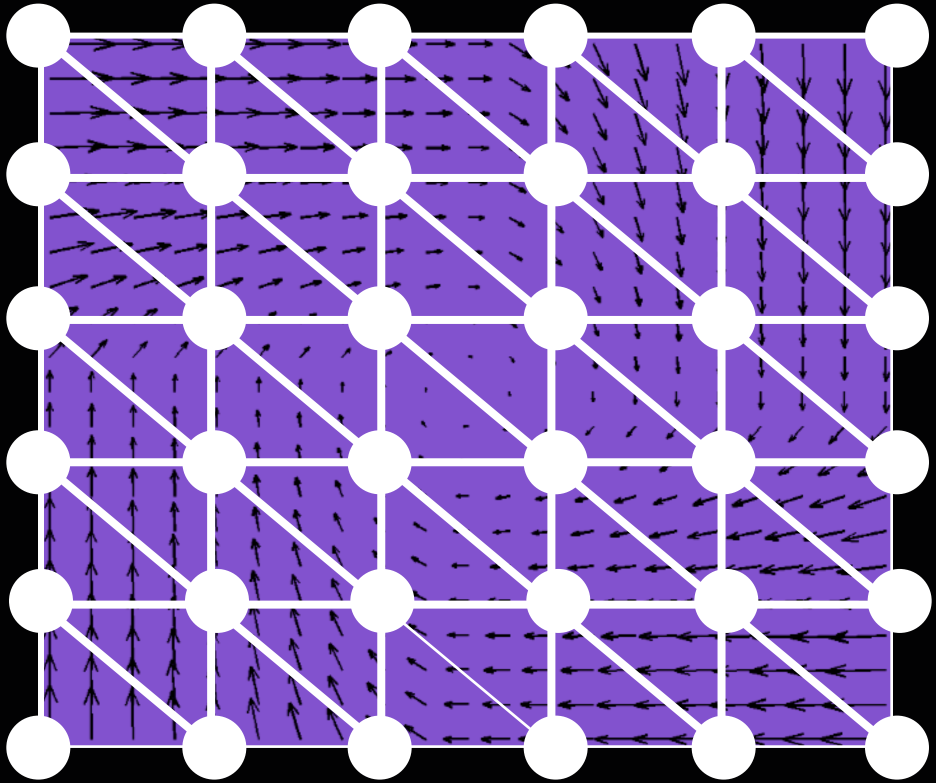
Sperner ⇒ Brouwer

Sperner ⇒ Brouwer (High-Level)

Given $f : [0,1]^2 \rightarrow [0,1]^2$

1. For all ε, existence of approximate fixed point $|f(x)-x| < \varepsilon$, can be shown via Sperner's lemma.
2. Then use compactness.

For 1: Triangulate $[0,1]^2$;

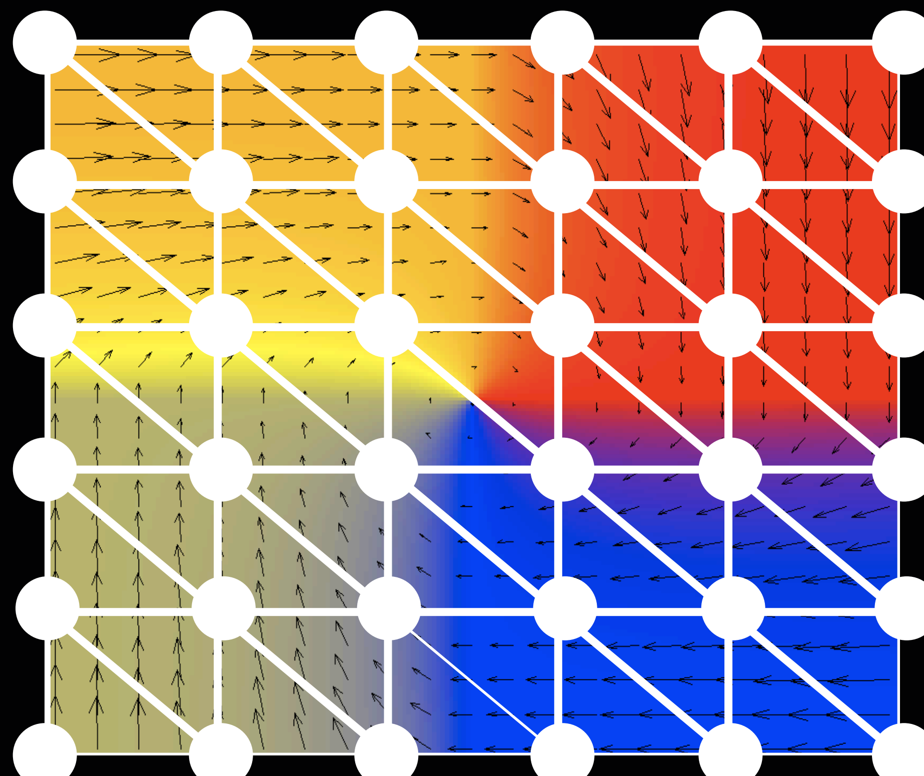


Sperner ⇒ Brouwer (High-Level)

Given $f : [0,1]^2 \to [0,1]^2$

1. For all ε, existence of approximate fixed point $|f(x)-x| < \varepsilon$, can be shown via Sperner's lemma.
2. Then use compactness.

For 1: Triangulate $[0,1]^2$; then color points according to the direction of $f(x)-x$;



Sperner ⇒ Brouwer (High-Level)

Given $f : [0,1]^2 \rightarrow [0,1]^2$

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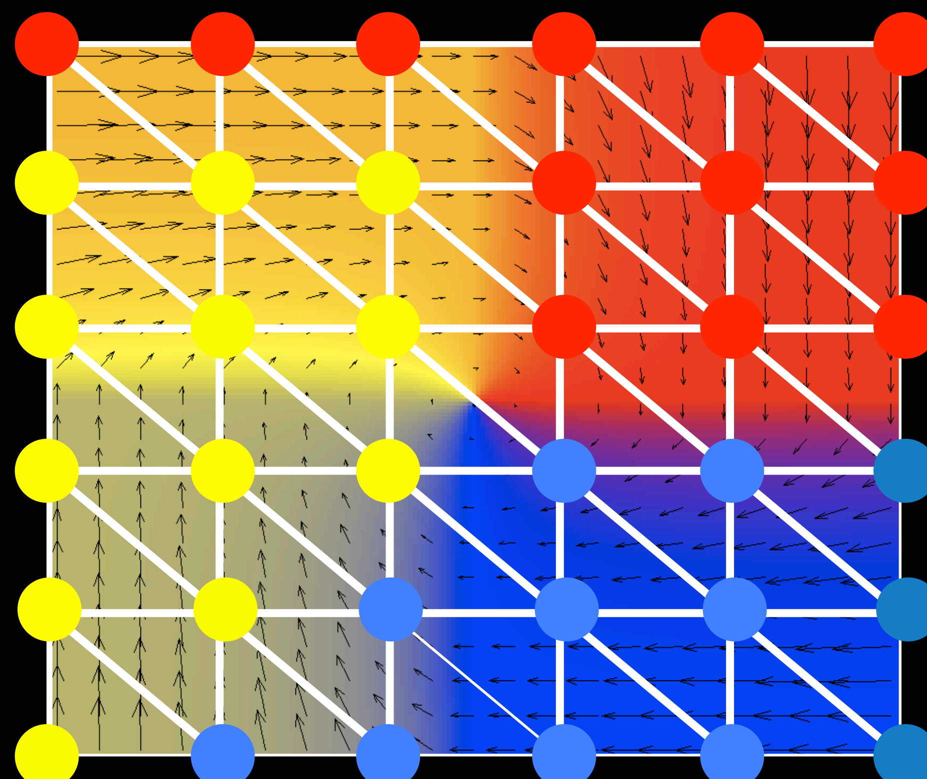
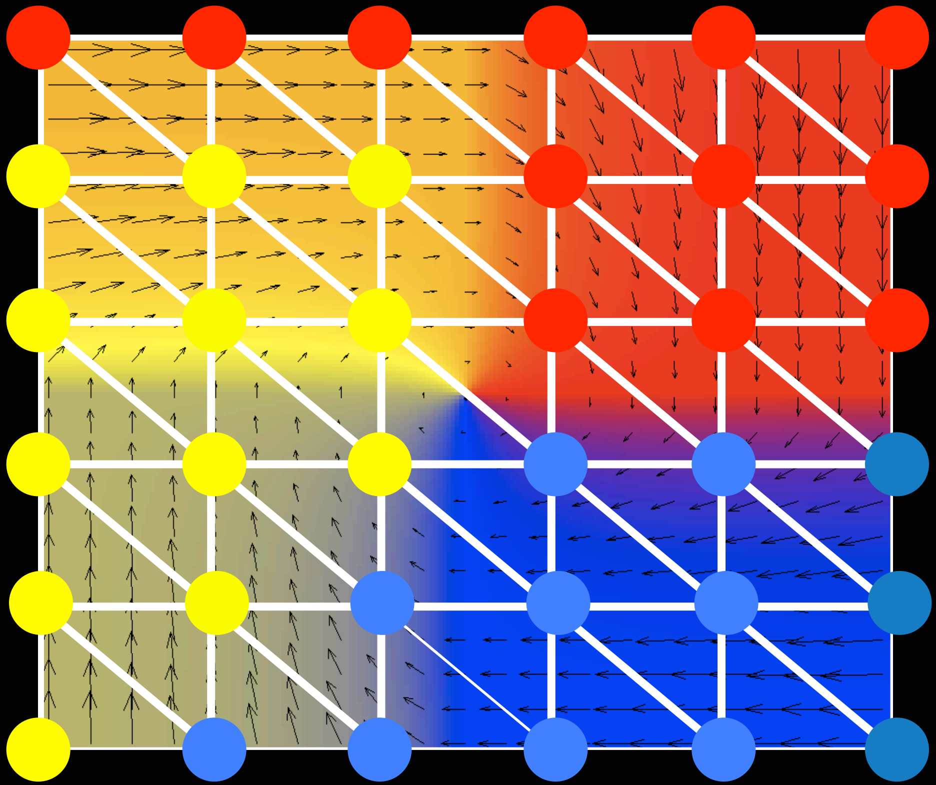


Sperner ⇒ Brouwer (High-Level)

Given $f : [0,1]^2 \rightarrow [0,1]^2$

1. For all ε, existence of approximate fixed point $|f(x)-x| < \varepsilon$, can be shown via Sperner's lemma.
2. Then use compactness.

For 1: Triangulate $[0,1]^2$; then color points according to the direction of $f(x)$-x; then apply Sperner.



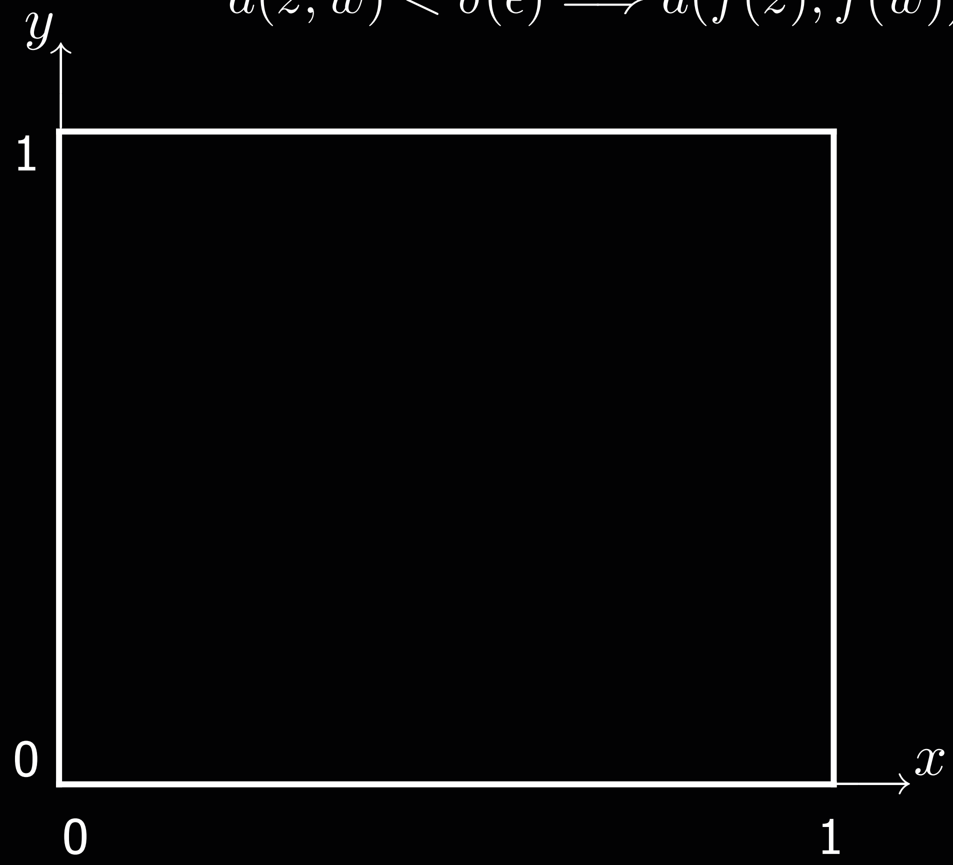
2D-Brouwer on the Square

say d is the ℓ_∞ norm

Suppose f: $[0,1]^2 \rightarrow [0,1]^2$, continuous

→ must be uniformly continuous (by the Heine-Cantor theorem)

$$\forall \epsilon > 0, \;\exists \delta(\epsilon) > 0, s.t.$$
$$d(z,w) < \delta(\epsilon) \implies d(f(z), f(w)) < \epsilon$$



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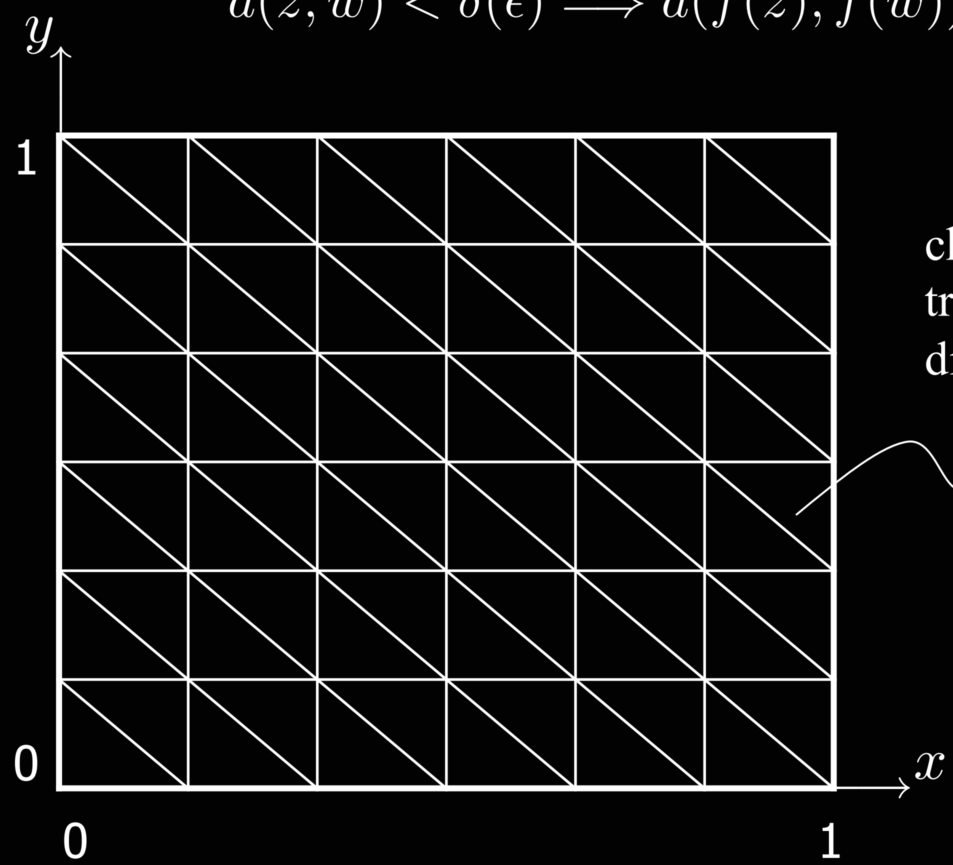

choose some ϵ and triangulate so that the diameter of cells is

$$\delta < \delta(\epsilon)$$

2D-Brouwer on the Square

say d is the ℓ_∞ norm

Suppose f: $[0,1]^2 \to [0,1]^2$, continuous

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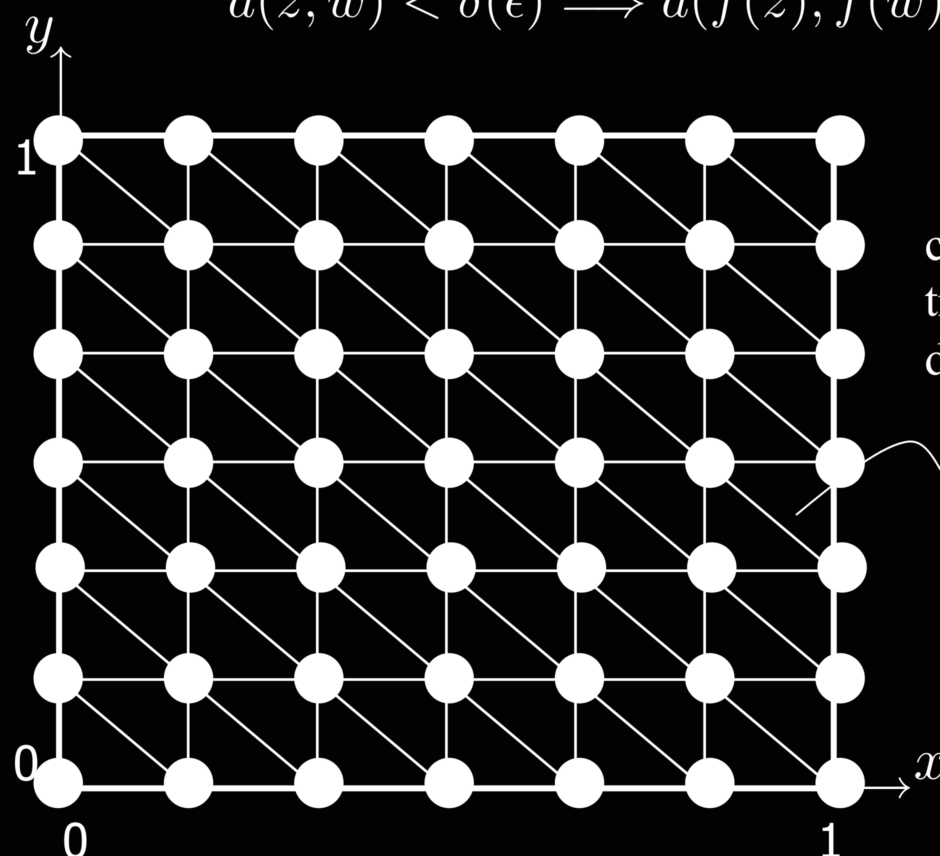
$$\forall \epsilon > 0, \ \exists \delta(\epsilon) > 0, s.t.$$

$$d(z,w) < \delta(\epsilon) \implies d(f(z), f(w)) < \epsilon$$

color the nodes of the triangulation according to the direction of

$$f(x) - x$$



choose some ϵ and triangulate so that the diameter of cells is

$$\delta < \delta(\epsilon)$$

2D-Brouwer on the Square

say d is the ℓ_∞ norm

Suppose f: $[0,1]^2 \rightarrow [0,1]^2$, continuous

→ must be uniformly continuous (by the Heine-Cantor theorem)

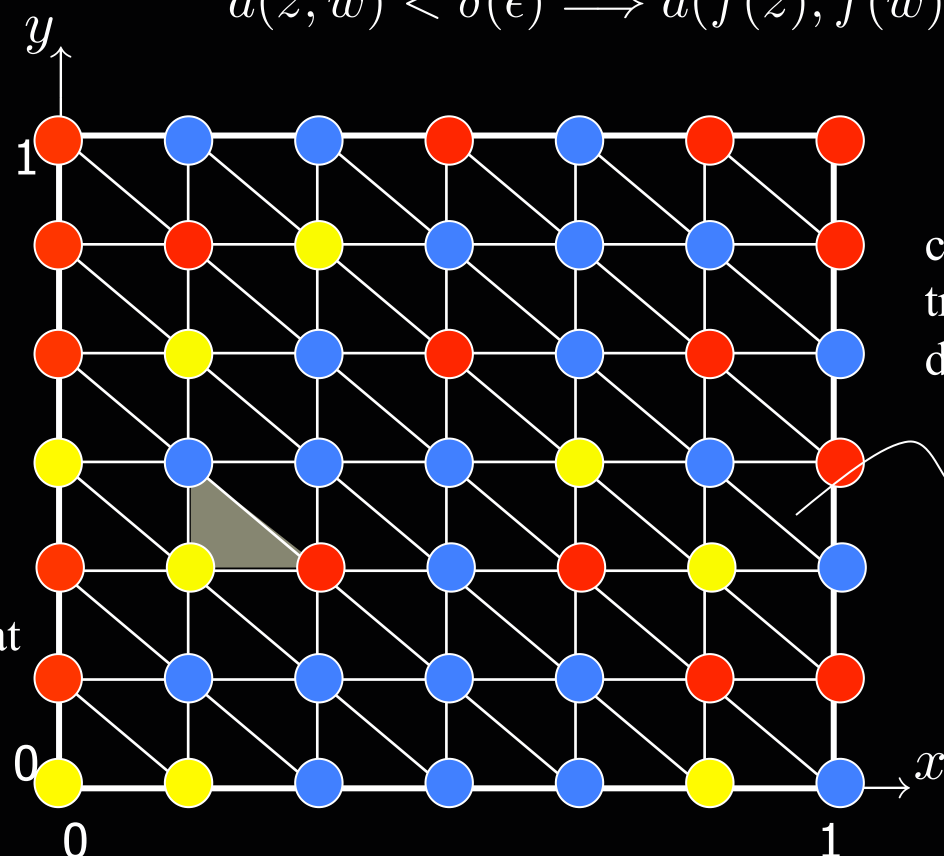
$$\forall \epsilon > 0, \ \exists \delta(\epsilon) > 0, s.t.$$

$$d(z,w) < \delta(\epsilon) \Longrightarrow d(f(z), f(w)) < \epsilon$$

color the nodes of the triangulation according to the direction of

$$f(x) - x$$

(tie-break at the boundary angles, so that the resulting coloring respects the boundary conditions required by Sperner's lemma)



choose some ϵ and triangulate so that the diameter of cells is

$$\delta < \delta(\epsilon)$$

find a trichromatic triangle, guaranteed by Sperner

2D-Brouwer on the Square

say d is the ℓ_∞ norm

Suppose f: $[0,1]^2 \rightarrow [0,1]^2$, continuous

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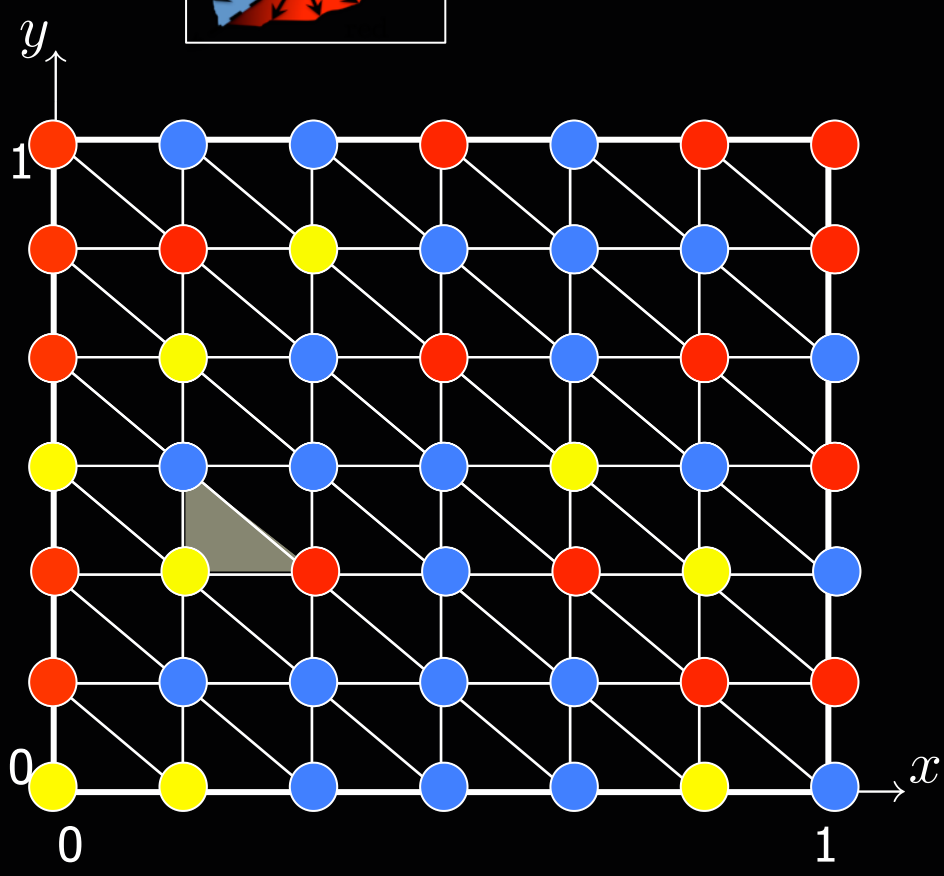


Claim: If z^{Y} is the yellow corner of a trichromatic triangle, then

$$|f(z^{\mathrm{Y}}) - z^{\mathrm{Y}}|_\infty < \epsilon + \delta.$$

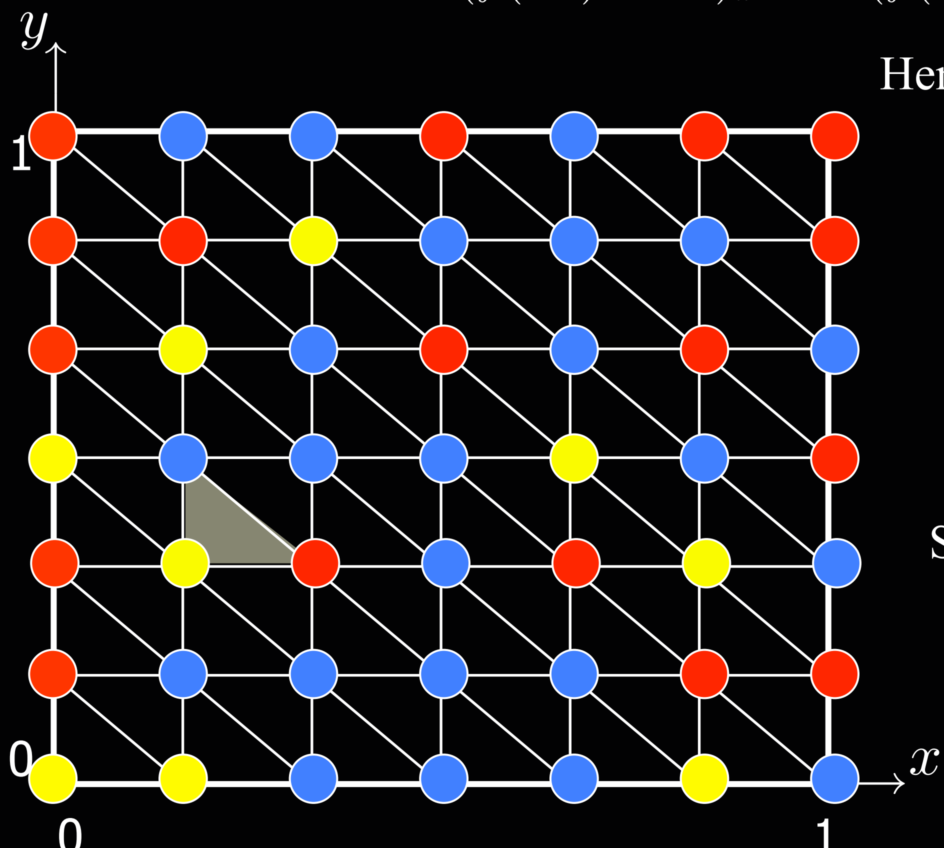
Proof of Claim

Claim: If z^{Y} is the yellow corner of a trichromatic triangle, then $|f(z^{\mathrm{Y}}) - z^{\mathrm{Y}}|_\infty < \epsilon + \delta$.

Proof: Let z^{Y}, z^{R}, z^{B} be the yellow/red/blue corners of a trichromatic triangle.

By the definition of the coloring, observe that the product of

$$(f(z^Y) - z^Y)_x \text{ and } (f(z^B) - z^B)_x \text{ is } \leq 0.$$



Hence:

$$\begin{aligned}
&|(f(z^Y) - z^Y)_x| \\
&\quad \leq |(f(z^Y) - z^Y)_x - (f(z^B) - z^B)_x| \\
&\quad \leq |(f(z^Y) - f(z^B))_x| + |(z^Y - z^B)_x| \\
&\quad \leq d(f(z^Y), f(z^B)) + d(z^Y, z^B) \\
&\quad \leq \epsilon + \delta.
\end{aligned}$$

Similarly, we can show:

$$|(f(z^Y) - z^Y)_y| \leq \epsilon + \delta.$$

$\square$

2D-Brouwer on the Square

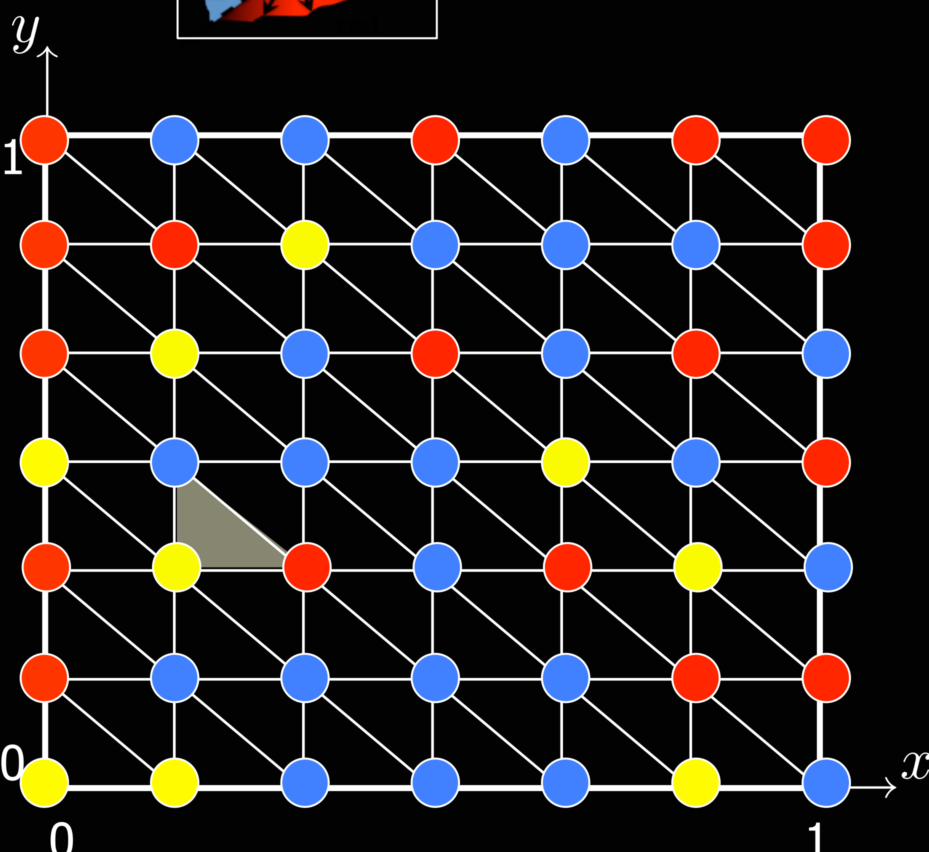
say d is the ℓ_∞ norm

Suppose f: $[0,1]^2 \to [0,1]^2$, continuous

$\hookrightarrow$ must be uniformly continuous (by the Heine-Cantor theorem)

$$\forall \epsilon > 0, \;\; \exists \delta(\epsilon) > 0, s.t.$$

$$d(z,w) < \delta(\epsilon) \Longrightarrow d(f(z), f(w)) < \epsilon$$



Claim: If z^{Y} is the yellow corner of a trichromatic triangle, then

$$|f(z^{\mathrm{Y}}) - z^{\mathrm{Y}}|_\infty < \epsilon + \delta.$$

Choosing $\delta = \min(\delta(\epsilon), \epsilon)$

$$|f(z^Y) - z^Y|_\infty < 2\epsilon.$$

2D-Brouwer on the Square

Finishing the proof of Brouwer's Theorem (Compactness):

- pick a sequence of epsilons: $\epsilon_i = 2^{-i}, i = 1, 2, \ldots$

- define a sequence of triangulations of diameter: $\delta_i = \min(\delta(\epsilon_i), \epsilon_i), i = 1, 2, \ldots$

- pick a trichromatic triangle in each triangulation, and call its yellow corner $z_i^{\mathrm{Y}}, i = 1, 2, \ldots$

- by compactness, this sequence has a converging subsequence $w_i, \quad i = 1, 2, \ldots$ with limit point w^*

Claim: $f(w^*) = w^*.$

Proof: Define the function $g(x) = d(f(x), x)$. Clearly, g is continuous since $d(\cdot, \cdot)$ is continuous and so is f. It follows from continuity that

$$g(w_i) \longrightarrow g(w^*), \text{ as } i \to +\infty.$$

But $0 \le g(w_i) \le 2^{-i+1}$. Hence, $g(w_i) \longrightarrow 0$. It follows that $g(w^*) = 0$.

Therefore, $d(f(w^*), w^*) = 0 \implies f(w^*) = w^*.$ $\blacksquare$

Summary

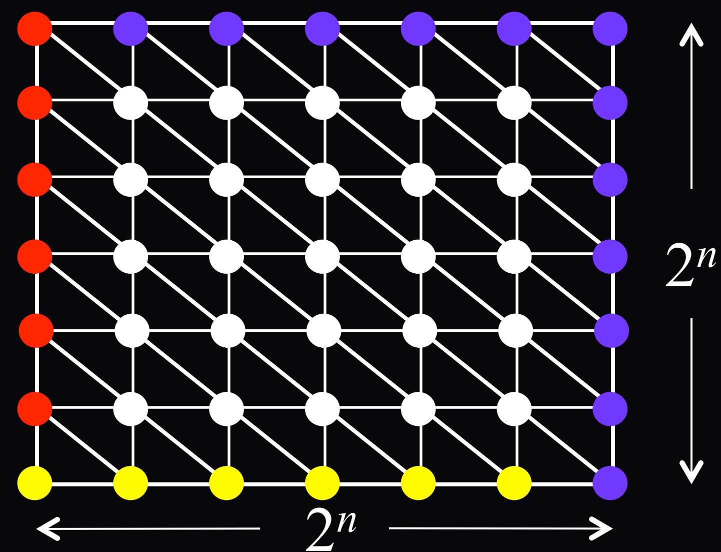
Sperner $\Rightarrow$ Brouwer $\Rightarrow$ Nash

Harder Easier

SPERNER

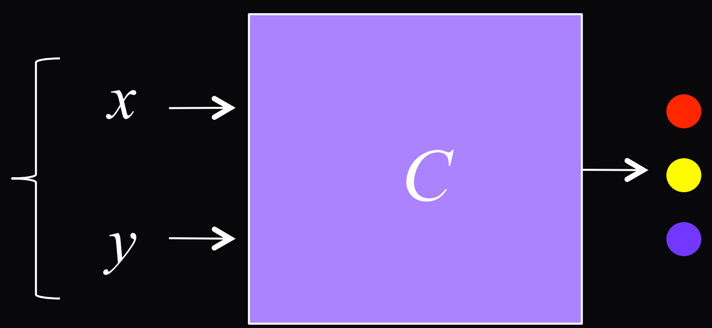
INPUT:

(i) n: specifies the size of a grid



(ii) Suppose boundary has standard coloring, and colors of internal vertices are given by a circuit:

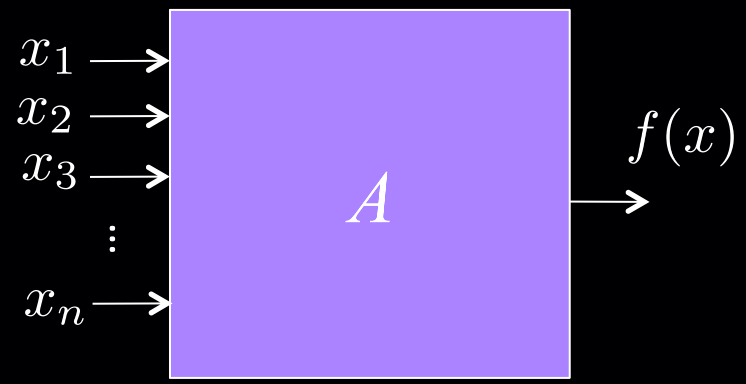
input: the coordinates of a point *(n bits each)*



OUTPUT: A tri-chromatic triangle.

BROUWER

INPUT: a. an algorithm A that evaluates a function $f: [0,1]^n \rightarrow [0,1]^n$:



b. an approximation requirement ϵ;

c. a Lipschitz constant c that the function is claimed to satisfy.

BROUWER: Find x such that $|f(x) - x| < \epsilon$

OR a pair of points x, y violating the Lipschitz constraint, i.e.

$$|f(x) - f(y)| > c|x - y|$$

OR a point that is mapped outside of $[0,1]^n$.

NASH

INPUT: (i) A Game defined by

- the number of players n;
- an enumeration of the strategy set S_p of every player $p = 1,\ldots, n$;
- the utility function $u_p : S \longrightarrow \mathbb{R}$ of every player.

(ii) An approximation requirement ε

OUTPUT: An ε-Nash equilibrium of the game.

i.e. the expected payoff of every player is within additive ε from the optimal expected payoff given the others' strategies

Intense effort for equilibrium algorithms following Nash's work:

e.g. Kuhn '61, Mangasarian '64, Lemke-Howson '64, Rosenmüller '71, Wilson '71, Scarf '67, Eaves '72, Laan-Talman '79, and others…

Lemke-Howson: simplex-like, works with LCP formulation.
All these algorithm require worst-case exponential time
No efficient algorithm is known after 60+ years of research.

The Pavlovian reaction

- Is it **NP-complete** to find a Nash equilibrium?
  - and why should **you** care?
- **NP-completeness** is a standard complexity theoretic approach to prove that a problem is computationally intractable **[Cook'71, Karp'72]**.
  - established by showing that problem is computationally equivalent to the Boolean function satisfiability problem:
    - Given Boolean formula with AND, OR and NOT operations, can you set the variables to satisfy it, i.e. get 1 in the output?
    - e.g. $((\neg x_1) \vee x_2) \wedge (\neg x_2) \wedge (\neg x_1)$ can be satisfied by setting $x_1 = x_2 = 0$
    - but $((\neg x_1) \vee x_2) \wedge (\neg x_2) \wedge x_1$ cannot be satisfied
- If Nash is **NP-complete**, then we cannot compute Nash equilibria efficiently, so we're unable to predict player behavior in all games.
- Worse still, universality breaks down.
  - If the best algorithmic machinery is unable to find Nash equilibria, how

the Pavlovian reaction (cont.)

So: “Is it NP-complete to find a Nash equilibrium?”

two answers:

1. probably not, since a solution is guaranteed to exist…

2. it is NP-complete to find a “tiny” bit more info than “just” a Nash equilibrium; e.g., the following are NP-complete:

- find two Nash equilibria, if more than one exist
- find a Nash equilibrium whose third bit is one, if any

[Gilboa, Zemel ’89; Conitzer, Sandholm ’03]

But let us look into NP-completeness more formally

Menu

- **Equilibria**
- **Existence proofs**
  - **Minimax**
  - **Nash**
  - **Brouwer, Brouwer   Nash**
  - **Sperner, Sperner   Brouwer**
- Complexity of Equilibria
  - **Total Search Problems in NP**
  - PPAD
- The World Beyond

Function NP (FNP)

A *search problem* L is defined by a relation $R_L \subseteq \{0,1\}^* \times \{0,1\}^*$ such that $(x, y) \in R_L$ iff y is a solution to x

A search problem is called *total* iff $x.$ y such that $(x, y) \in R_L$.

A search problem $L \in$ FNP iff there exists a poly-time algorithm $A_L(\cdot, \cdot)$ and a polynomial function $p_L(\cdot)$ such that

(i) x, y: $A_L(x, y)=1 \Leftrightarrow (x, y) \in R_L$

(ii) x: $\exists y$ s.t. $(x, y) \in R_L \Rightarrow \exists z$ with $|z| \leq p_L(|x|)$ s.t. $(x, z) \in R_L$

TFNP = $\{L \in \text{FNP} \mid L \text{ is total}\}$

SPERNER, NASH, BROUWER $\in$ FNP.

FNP-completeness

A search problem $L \in$ FNP, associated with A_L and p_L, is ***poly-time (Karp) reducible*** to another problem $L' \in$ FNP, associated with $A_{L'}$ and $p_{L'}$, iff there exist efficiently computable functions f, g such that

(i) $f : \{0,1\}^* \rightarrow \{0,1\}^*$ maps inputs x to L into inputs $f(x)$ to L'

(ii)

$$x,y:\ A_{L'}(f(x), y)=1 \quad \Rightarrow \quad A_L(x, g(y))=1$$

$$x:\ A_{L'}(f(x), y)=0,\ \forall\, y \Rightarrow \quad A_L(x, y)=0,\ \forall\, y$$

can't Karp reduce SAT to SPERNER, NASH or BROUWER

A search problem L is ***FNP-complete*** iff

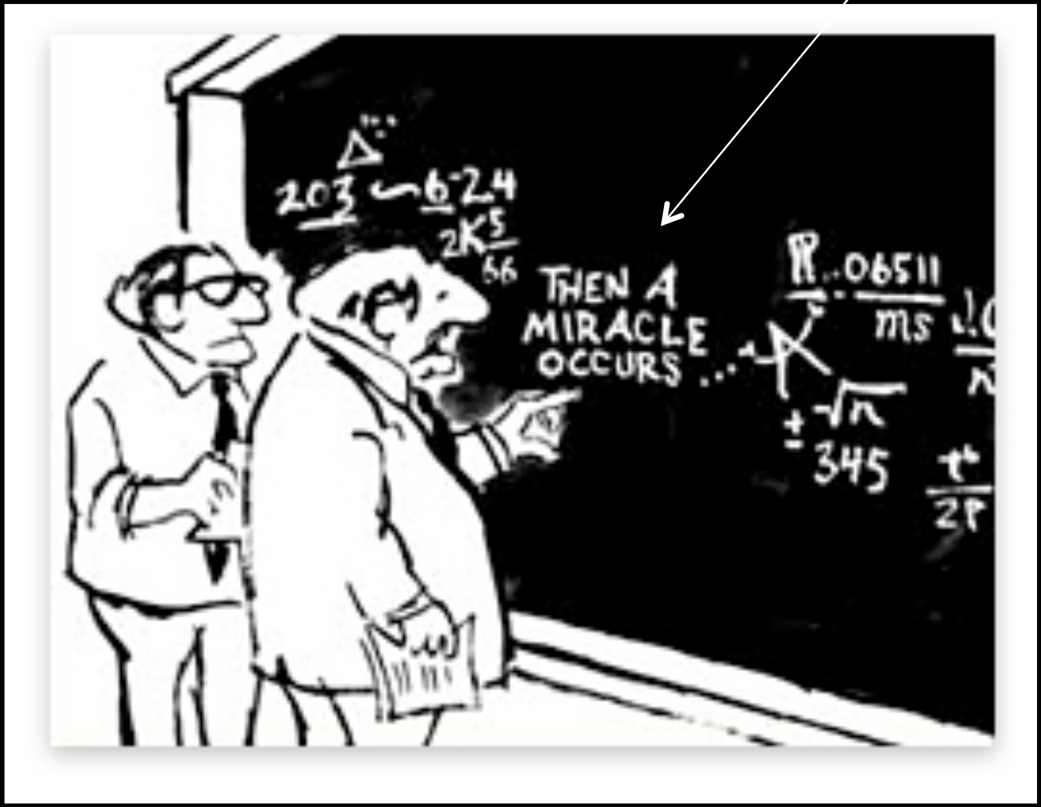
e.g. SAT

$L \in$ FNP

L' is poly-time reducible to L, for all $L' \in$ FNP

A Complexity Theory of Total Search Problems ?

??



A Complexity Theory of Total Search Problems ?

100-feet overview of our methodology:

1. identify the combinatorial argument of existence, responsible for making these problems total;

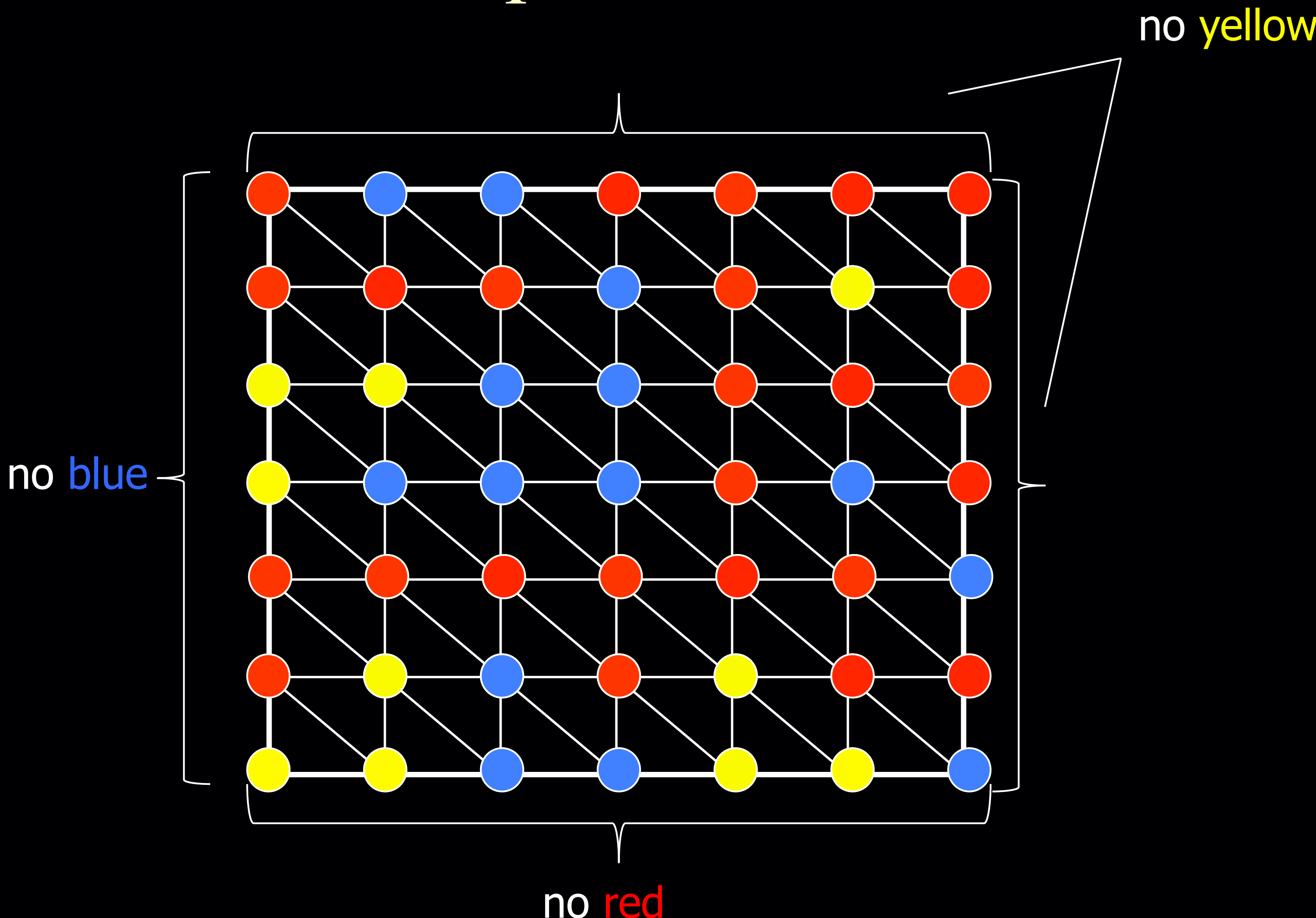
2. define a complexity class inspired by the argument of existence;

3. Litmus test: was complexity of underlying problem captured tightly?
   - prove completeness results

Menu

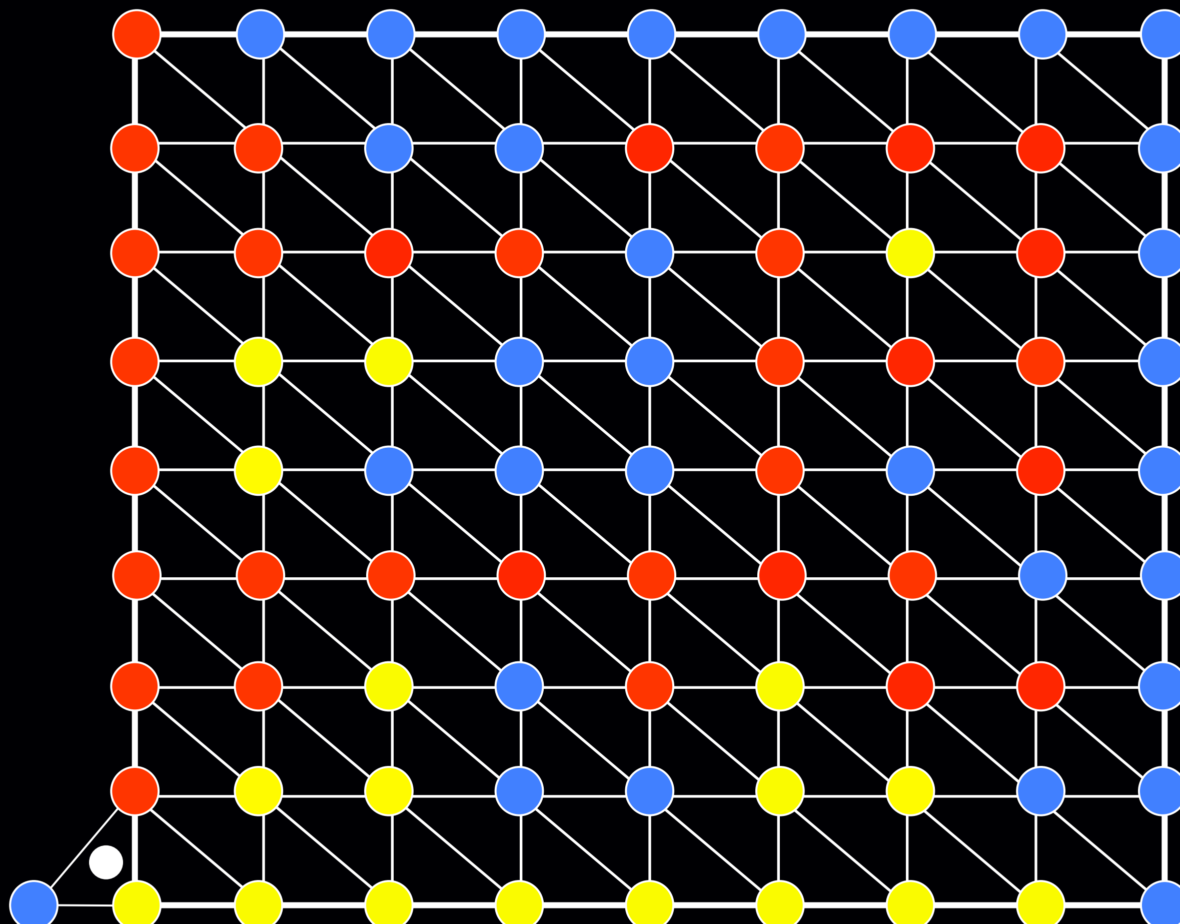
- **Equilibria**
- **Existence proofs**
  - **Minimax**
  - **Nash**
  - **Brouwer, Brouwer Nash**
  - **Sperner, Sperner Brouwer**
- Complexity of Equilibria
  - Total Search Problems in NP
  - **Proof of Sperner's Lemma**
  - PPAD
- The World Beyond

Proof of Sperner's Lemma



[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Sperner's Lemma



For convenience we introduce an outer boundary, that does not create new tri-chromatic triangles.

We also introduce an artificial tri-chromatic triangle.

Next we define a directed walk starting from the artificial tri-chromatic triangle.

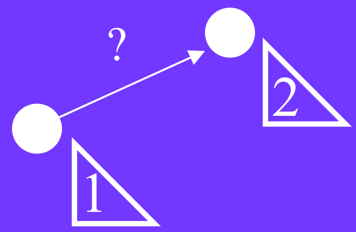
[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Sperner's Lemma

Space of Triangles

Transition Rule:

*If ∃ **red** - **yellow** door cross it with **red** on your left hand.*



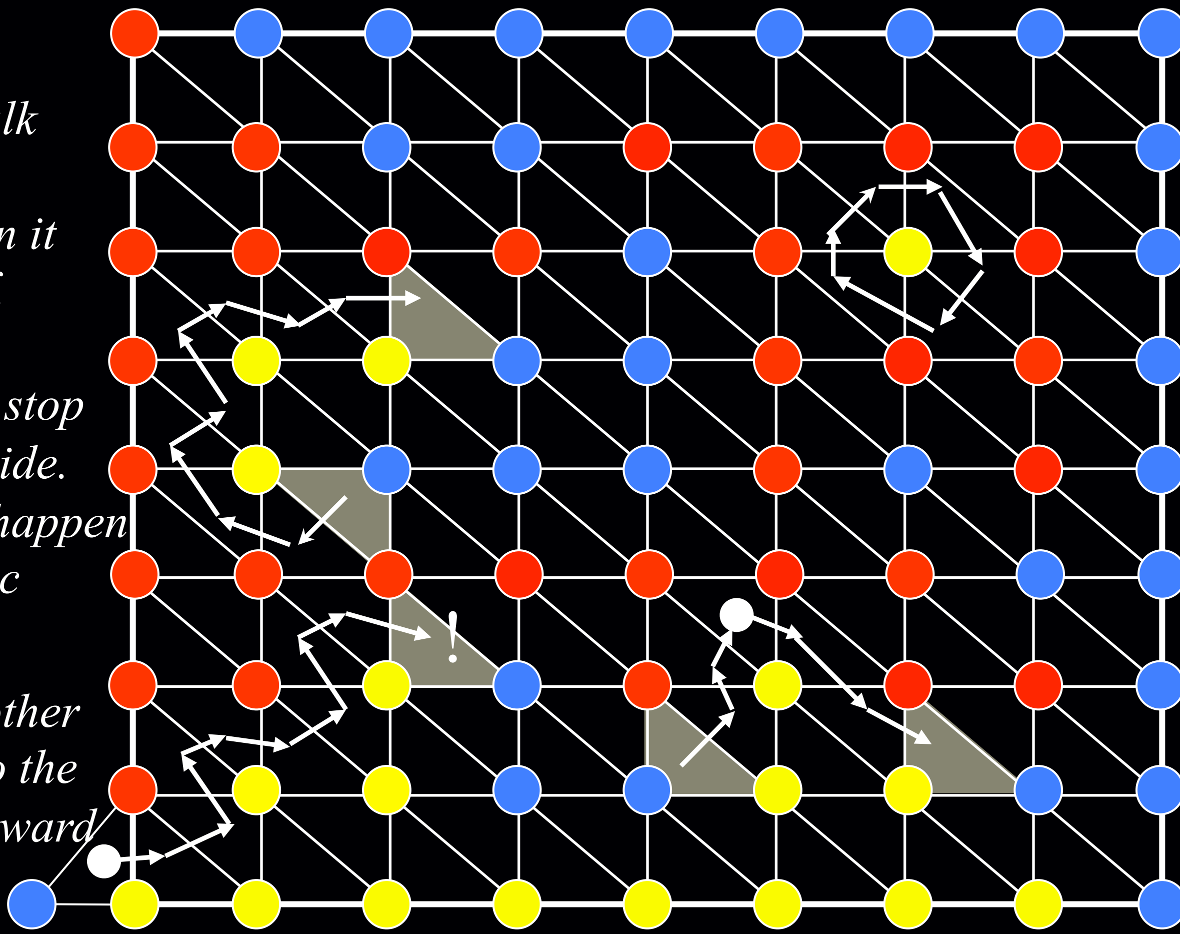
[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Sperner's Lemma

Claim: *The walk cannot exit the square, nor can it loop into itself.*

Hence, it must stop somewhere inside. This can only happen at tri-chromatic triangle…

Starting from other triangles we do the same going forward or backward.



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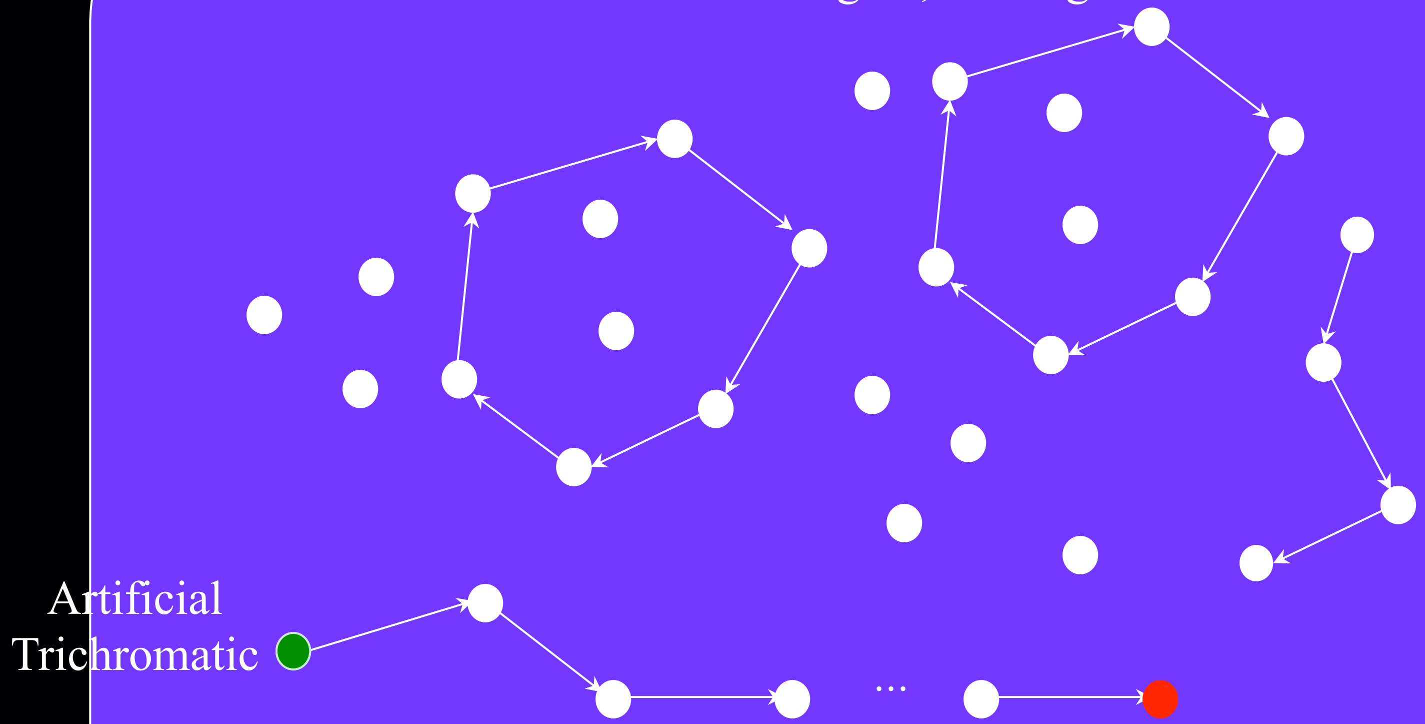
[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Structure of Proof:
A directed parity argument

Vertices of Graph ≡ Triangles
all vertices have in-degree, out-degree ≤ 1



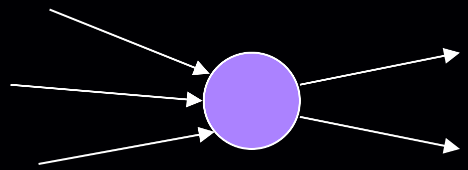
degree 1 vertices: trichromatic triangles
degree 2 vertices: no blue, non-trichromatic
degree 0 vertices: all other triangles

Proof: ∃ at least one trichromatic (artificial one) ➔ ∃ another trichromatic

The Non-Constructive Step

An easy parity lemma:

A directed graph with an unbalanced node (a node with indegree ≠ outdegree) must have another.



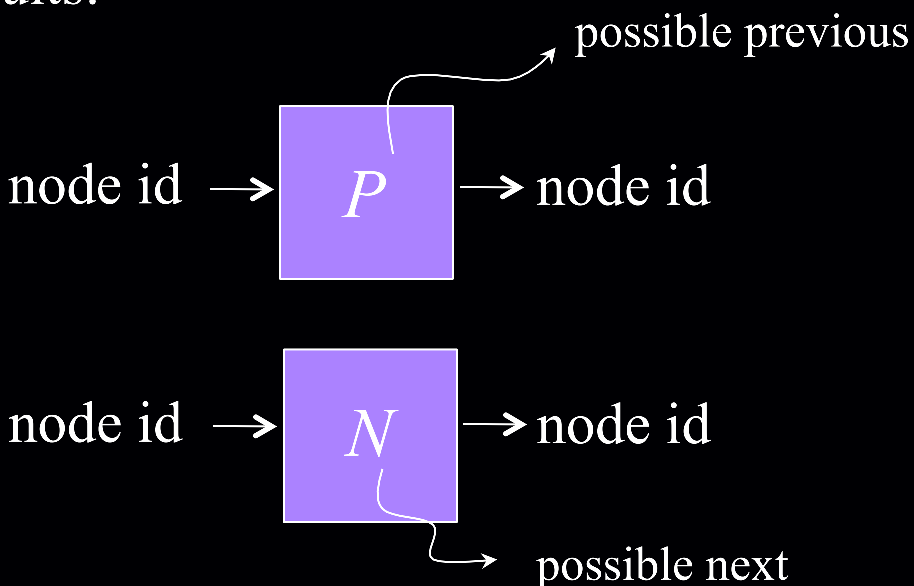
But, wait, why is this non-constructive?

Given a directed graph and an unbalanced node, isn't it trivial to find another unbalanced node?

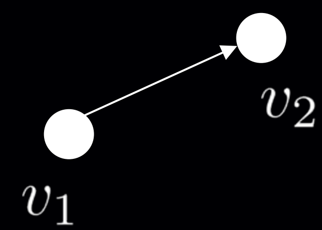
The graph can be exponentially large, but has succinct description…

The PPAD Class [Papadimitriou '94]

Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by two circuits:



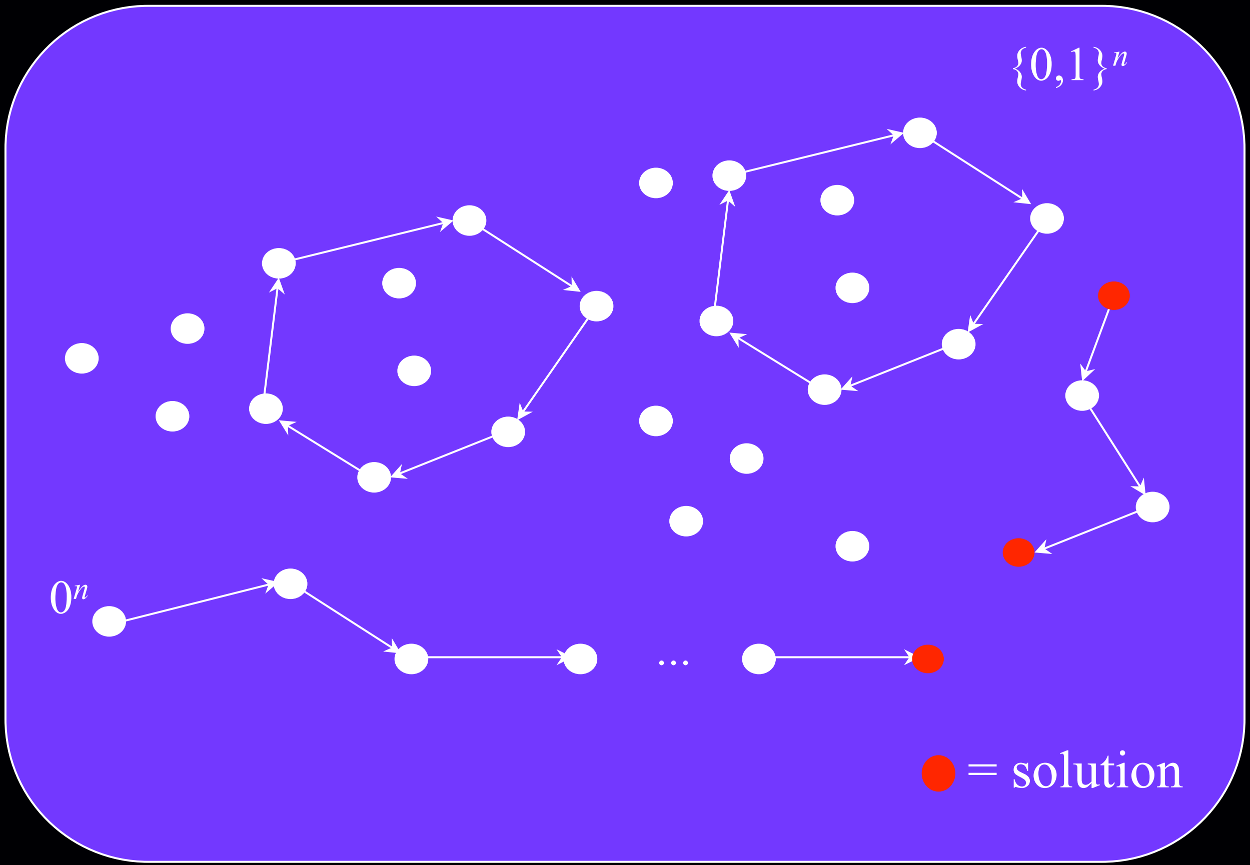
$$P(v_2) = v_1 \wedge N(v_1) = v_2$$

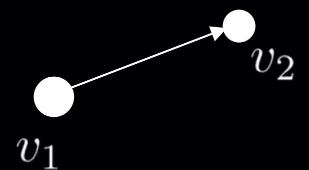


END OF THE LINE: Given P and N: If 0^n is an unbalanced node, find another unbalanced node. Otherwise output 0^n.

PPAD = { *Search problems in* FNP *reducible to* END OF THE LINE }

END OF THE LINE





$\Leftrightarrow \quad P(v_2) = v_1 \wedge N(v_1) = v_2$

Problems in PPAD

SPERNER $\in$ **PPAD** [Previous Slides]

BROUWER $\in$ **PPAD** [By Reduction to SPERNER-**Scarf '67]**

NASH $\in$ **PPAD** [By Nash's Proof, reducing to BROUWER]

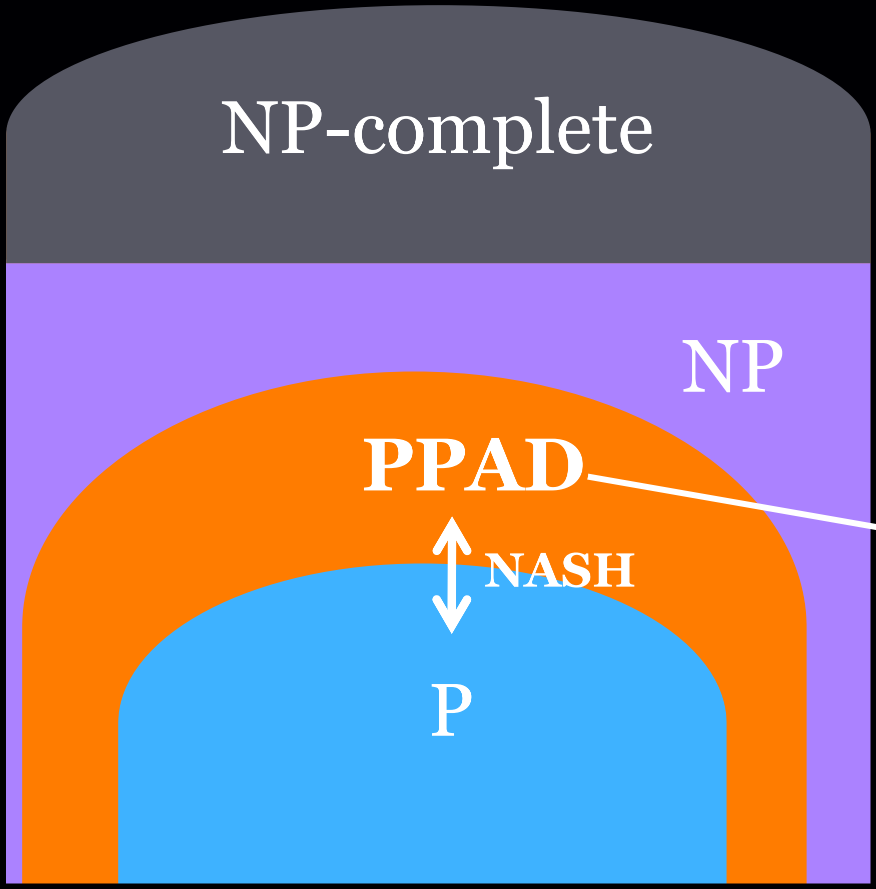
Litmus Test: Completeness

SPERNER is **PPAD**-Complete **[Papadimitriou '94; Chen-Deng '05** for 2d]

BROUWER is **PPAD**-Complete **[Papadimitriou '94]**

i.e. these problems are solvable via the directed parity argument and are at least as hard as any other problem in **PPAD**

Problems in PPAD



SPERNER, BROUWER are both PPAD-complete (i.e. as hard as any problem in **PPAD**)