Dynamics in Games: Algorithms and Learning

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PGMO Course
February 2021

Abstract

Game theory studies interactions between agents with specific aims, be they rational actors, genes, or computers. This course is intended to provide the main mathematical concepts and tools used in game theory with a particular focus on their connections to learning and convex optimization. The first part of the course deals with the basic notions: value, (Nash and Wardrop) equilibria, correlated equilibria. We will give several dynamic proofs of the minmax theorem and describe the link with Blackwell's approachability. We will also study the connection with variational inequalities.

The second part will introduce no-regret properties in on-line learning and exhibit a family of unilateral procedures satisfying this property. When applied in a game framework we will study the consequences in terms of convergence (value, correlated equilibria). We will also compare discrete and continuous time approaches and their analog in convex optimization (projected gradient, mirror descent, dual averaging). Finally we will present the main tools of stochastic approximation that allow to deal with random trajectories generated by the players.

Part B

ALGORITHMS AND LEARNING

B.2 No-regret procedures and stochastic approximation

This section relies in part on :

Lectures on Dynamics in Games, (2008) Université Paris 6, unpublished lectures notes.

Tutorial on learning, (2005) *Stochastic Methods in Game Theory*, IMS, NUS, Singapore.

No-regret (II) in learning, games and convex optimization

The purpose of this section is to underline the links between some no-regret algorithms used in on-line learning, games and convex optimization and to compare the continuous and discrete time versions.

1. Introduction

The general framework is as follows:
V is a normed vector space, finite dimensional, with dual V^* and duality map $\langle V^*|V\rangle$,
X is a non-empty compact convex subset of V.

We study properties of algorithms that associate to a process $\{u_t \in V^*, t \geq 0\}$, a procedure $\{x_t \in X, t \geq 0\}$, where x_t is function of the past $\{(x_s, u_s), 0 \leq s < t\}$.

The process corresponds to the observation, the procedure to the induced trajectory.

The adequation of $\{x_t\}$ to $\{u_t\}$ is measured by a regret function defined on X by:

$$R_t(x) = \int_0^t \langle u_s | x - x_s \rangle ds \tag{1}$$

and one will study procedures satisfying the no-regret condition:

$$\sup_{x \in X} R_t(x) \leq o(t). \tag{2}$$

Similarly in discrete time, given $\{u_m\}$ and $\{x_m\}$, $m \in \boldsymbol{N}$, with $\{x_m\}$ depending on $\{x_1, u_1, ..., x_{m-1}, u_{m-1}\}$, one defines:

$$R_n(x) = \sum_{m=1}^{n} \langle u_m | x - x_m \rangle$$

and requires:

$$\sup_{x \in X} R_n(x) \leq o(n). \tag{3}$$

A) We compare the performance of the algorithms in terms of the regret under three (increasing) assumptions:
(I) general case: $\{u_t\}$ is a bounded measurable process in V^*,
(II) closed form: $u_t = \phi(x_t)$ for a continuous vector field $\phi : X \to V^*$,
(III) convex gradient: $u_t = -\nabla f(x_t)$, f $\mathscr{C}^1$ convex function $: X \to \mathbb{R}$,
(with similar properties in discrete time).

B) We consider three different procedures:
a) Projected dynamics (PD),
b) Mirror descent (MD),
c) Dual averaging (DA).

C) We analyze the relations between the continuous and discrete time processes, in particular in terms of speed of convergence to 0 of the average regret.

D) We also study the convergence of the trajectories of $\{x_t\}$ or $\{x_n\}$ (in classes (II) and (III)).

Framework (I) corresponds to the usual model of on-line learning where the agent observes $\{u_s, s < t\}$ and chooses x_t.

Recall that if $V = \mathbb{R}^K$ and the agent selects a component $k_t \in K$ at random, then $X = \Delta(K)$, x_t is the law of k_t and the regret (II) is expressed in terms of conditional expectation.

Note that since no hypothesis is made on the process u_t, no prediction makes sense but the no-regret condition expresses a desirable a-posteriori property.

The notion of regret appears in Hannan, 1957 [27], Blackwell, 1956 [14] in a game theoretical set-up. Algorithms and properties are studied in this spirit in Foster and Vohra, 1993 [22], Fudenberg and Levine, 1995 [25], Foster and Vohra, 1999 [23], Hart and Mas-Colell, 2000 [29], Lehrer, 2003 [41] , Benaim, Hofbauer and Sorin, 2005 [11], Cesa-Bianchi and Lugosi, 2006 [18], Viossat and Zapechelnyuk, 2013 [82], ... among others.

Similar tools and properties occur in statistics and in the learning community: Vvok, 1990 [83], Cover, 1991 [20], Littlestone and Warmuth, 1994 [43], Freund and Shapire, 1999 [24], Auer, Cesa-Bianchi, Freund and Shapire, 2002 [4], Cesa-Bianchi and Lugosi, 2003 [17], Stoltz and Lugosi, 2005 [73], Kalai and Vempala, 2005 [36], ...

The next two frameworks (II) and (III), describe more specific cases where the observation u_t is a function of the action x_t.

Famework (II) *closed form* is relevant for game dynamics and variational inequalities.

Consider a strategic game with a finite set of players I, where the equilibrium set E is the set of solutions $x \in X$ of the following variational inequalities:

$$\langle \phi^i(x) | x^i - y^i \rangle \geq 0, \qquad \forall y^i \in X^i, \forall i \in I.$$

Here $X^i \subset V^i$ is the strategy set of player i, $X = \prod_i X^i$, and $\phi^i : X \to V^{i*}$ is her “evaluation" function.

For each player i, the reference process is $u^i_t = \phi^i(x_t)$ which, as a function of x_t, is determined by the behavior of all players.

In a concave $\mathscr{C}^1$ game with payoff F, the observation of player i is the gradient w.r.t. x_i of her payoff function F^i : $u_t^i = \phi^i(x_t) = \nabla_i F^i(x_t)$.

Note that the overall global dynamics of $\{x_t\}$ is generated by a family of unilateral procedures since for each i, x_t^i depends on (u^i, x^i) only. In particular for each player i, the knowledge of $\phi^j, j \neq i$ is not assumed.

Thus for each player individually the situation is like (I) *general case*, while the private observations of the players are linked via x_t.

We will analyze the consequences on the process $\{x_t\}$ assuming only that each player uses a procedure satisfying the no-regret condition (2) or (3).

Obviously the (global) algorithm associated to $g = \{g^i\}$ will also share the no-regret property since:

$$\int_0^t \langle g^i(x_s) | x^i - x_s^i \rangle ds \leq o(t), \qquad \forall x^i \in X^i, \qquad \forall i \in I$$

implies:

$$\int_0^t \langle g(x_s) | x - x_s \rangle ds \leq o(t), \qquad \forall x \in X,$$

but in addition it is ”decentralized" in the sense that x^i depends upon g^i only.

Framework (III) covers the case of convex optimization where the observation, after the choice x_t, is the gradient of the (unknown) convex function and $u_t = -\nabla f(x_t)$.

The research in this area is extremely active and very wide; it links basic optimization algorithms, Polyak, 1987 [59], Nemirovski and Yudin, 1983 [52], Nesterov, 2004 [54], to on-line procedures, see e.g. Zinkevich, 2003 [86].

Recent books include:

BUBECK S. (2015) [15] Convex optimization: Algorithms and complexity, *Fondations and Trends in Machine Learning*, **8**, 231-357.
HAZAN E. (2011) [31] The convex optimization approach to regret minimization, *Optimization for machine learning*, S. Sra, S. Nowozin, S. Wright eds, MIT Press, 287-303.
HAZAN E. (2015) [32] Introduction to Online Convex Optimization, *Fondations and Trends in Optimization*, **2**, 157-325.
HAZAN E. (2019) [33] Optimization for Machine Learning , https://arxiv.org/pdf/1909.03550.pdf.
SHALEV-SHWARTZ S. (2012) [65] Online Learning and Online Convex Optimization, *Foundations and Trends in Machine Learning*, **4**, 107-194.

Related algorithms have been developped in Operations Research (transportation, networks), see e.g. Dupuis and Nagurney, 1993 [21], Nagurney and Zhang, 1996 [51], Smith, 1984 [66].

Note that each community (learning, game theory, optimization) has its own terminology and point of view.

One of the aims of the current section is to clarify the relations between several approaches and results and unify the analysis.

In particular we will show that few basic principles are in use and we will underline the analogy between continuous and discrete time.

Section 1.2 is devoted to the *closed form* framework (II) and explores the links between no-regret, solutions of variational inequalities and convex optimization.

Section 1.3 deals with continuous time dynamics.
After introducing level functions, we describe the three algorithms (PD, MD, DA), prove that they satisfy the no-regret property and compare their performances.

Section 1.4 is the discrete time analog of Section 3.

Section 1.5 considers basically framework (III) under a regularity hypothesis on the convex function f.

Concluding comments are in Section 1.6.

2. Basic properties of the closed form

Assume that the procedure satisfies the no-regret property (2) or (3).

We describe here some relations with variational inequalities when the observation process has a *closed form* : $u = \phi(x)$.

Notation 1.1

$NE(\phi)$ is the set of solutions, in X, of the variational inequality:

$$\langle \phi(x) | y - x \rangle \leq 0, \qquad \forall y \in X. \tag{4}$$

If ϕ is the evaluation function in a game, $NE(\phi)$ corresponds to the set of equilibria.

A first property deals with convergent trajectories $\{x_t\}$.

Lemma 1.1

If ϕ is continuous and $x_s \to x$, then $x \in NE(\phi)$.

Proof:

Since $R_t(y) = \int_0^t \langle \phi(x_s) | y - x_s \rangle ds$:

$$\frac{R_t(y)}{t} \to \langle \phi(x) | y - x \rangle, \qquad \forall y \in X. \tag{5}$$

and $R_t(y) \leq o(t)$ implies $x \in NE(\phi)$. ∎

In particular, if x is a stationary point for the discrete or continuous time procedure, then $x \in NE(\phi)$.

Notation 1.2

$SE(\phi)$ is the set of solutions, in X, of the variational inequality:

$$\langle \phi(y) | y - x \rangle \leq 0, \qquad \forall y \in X. \tag{6}$$

Notice that $SE(\phi)$ is convex.

Clearly one has:

Lemma 1.2

If $x^* \in SE(\phi)$, then $R_t(x^*) \geq 0$ for all $t \geq 0$.

Recall:
ϕ is dissipative if it satisfies:

$$\langle \phi(x) - \phi(y) | x - y \rangle \leq 0, \qquad \forall x, y \in X. \tag{7}$$

If ϕ is dissipative, then :

$$NE(\phi) \subset SE(\phi)$$

and if ϕ is continuous the reverse inclusion is satisfied:

$$SE(\phi) \subset NE(\phi).$$

If $NE(\phi) = SE(\phi)$ we will also use the notation $E(\phi)$ for this set.

Define the time average trajectories :

$$\bar{x}_t = \frac{1}{t}\int_0^t x_s ds \qquad \text{and} \qquad \bar{x}_n = \frac{1}{n}\sum_{m=1}^{n} x_m.$$

Lemma 1.3

If ϕ is dissipative, the accumulation points of $\{\bar{x}_t\}$ or $\{\bar{x}_n\}$ are in $SE(\phi)$.

Proof:

$$\frac{R_t(y)}{t} = \frac{1}{t}\int_0^t \langle \phi(x_s)|y - x_s\rangle \geq \frac{1}{t}\int_0^t \langle \phi(y)|y - x_s\rangle = \langle \phi(y)|y - \bar{x}_t\rangle.$$

Hence under (2) or (3) an accumulation point $\hat{x}$ of $\{\bar{x}_t\}$ will satisfy $\langle \phi(y)|y - \hat{x}\rangle \leq 0$. ∎

This result implies the non-emptiness of $SE(\phi)$ for dissipative ϕ.
In particular the minmax theorem (in the $\mathscr{C}^1$ case) follows from the existence of no-regret procedures.

Class (III): convex gradient.

Since $u_t = -\nabla f(x_t)$ with f $\mathscr{C}^1$ convex, this corresponds to a specific case of dissipative and continuous vector field ϕ, hence: $SE(\phi) = NE(\phi) = E = \mathrm{argmin}_X f$.

Use that:

$$\langle \nabla f(x_t) | y - x_t \rangle \leq f(y) - f(x_t)$$

to obtain with $u_t = -\nabla f(x_t)$ in (1):

$$\int_0^t [f(x_s) - f(y)]\, ds \leq \int_0^t \langle -\nabla f(x_s) | y - x_s \rangle ds = R_t(y)$$

which implies by Jensen's inequality:

$$f(\bar{x}_t) - f(y) \leq \frac{1}{t} \int_0^t [f(x_s) - f(y)]\, ds \leq \frac{R_t(y)}{t}. \tag{8}$$

Similarly in discrete time with $u_m = -\nabla f(x_m)$:

$$n[f(\bar{x}_n) - f(y)] \leq \sum_{m=1}^{n} f(x_m) - f(y) \leq \sum_{m=1}^{n} \langle \nabla f(x_m) | x_m - y \rangle = R_n(y).$$

In particular one obtains:

Lemma 1.4

i) The accumulation points of $\{\bar{x}_t\}$ *or* $\{\bar{x}_n\}$ *belong to* E*.*
ii) If $t \mapsto f(x_t)$ *(resp.* $n \mapsto f(x_n)$*) is decreasing, the accumulation points of* $\{x_t\}$ *or* $\{x_n\}$ *belong to* E*.*

Note that i) is a particular instance of Lemma 1.3.

3. Continuous time

We describe in this section three procedures in continuous time that satisfy the no-regret property. Their discrete time counterparts will be analyzed in the next section.

As usual, discrete time dynamics are easier to describe but their mathematical properties are more difficult to establish. This explain why we choose to start with the continuous time versions.

In addition a very useful tool in the form of a level function is available in this set-up and we start by analyzing it.

Definition 1.1

$P : \mathbb{R}^+ \times X \to \mathbb{R}^+$ is a level function (for $\{u_t, x_t\}$), if $P(t;y)$ is bounded and satisfies:

$$\langle u_t, x_t - y\rangle \geq \frac{d}{dt}P(t;y), \quad \forall t \in \mathbb{R}^+, \forall y \in X. \tag{9}$$

Lemma 1.5

If there exists a level function, R_t is upper bounded.
Hence: $R_t(y)/t \leq O(1/t)$.

Proof:

$$R_t(y) = \int_0^t \langle u_s | y - x_s\rangle ds \leq P(0;y) - P(t;y) \leq P(0;y).$$

■

Hence the existence of a level function implies the no-regret property.

Lemma 1.6

Consider class (II).
Assume $y^* \in SE(\phi)$, then $t \mapsto P(t;y^*)$ is decreasing.

Proof:

$$\frac{d}{dt}P(t;y^*) \leq \langle \phi(x_t), x_t - y^* \rangle \leq 0.$$

∎

Thus a level function evaluated at a point in $SE(g)$ is a weak Lyapounov function.

Lemma 1.7

Consider class (III).
If $\{x_t\}$ *is a descent procedure (meaning that* $\frac{d}{dt}f(x_t) \leq 0$*), then:*

$$E(t;y) = t(f(x_t) - f(y)) + P(t;y)$$

is decreasing, for all $y \in X$*.*

Proof:

$$\begin{aligned} \frac{d}{dt}E(t;y) &= f(x_t) - f(y) + t\frac{d}{dt}f(x_t) + \frac{d}{dt}P(t;y) \\ &\leq f(x_t) - f(y) + \langle \nabla f(x_t), y - x_t \rangle \\ &\leq 0 \end{aligned}$$

■

We recover the fact that the accumulation points of $\{x_t\}$ are in $E = \operatorname{argmin}_X f$ and that the speed of convergence of $f(x_t)$ to $\min f$ is $O(1/t)$.

3.2. Positive correlation

Given a dynamics, f decreases on trajectories if:

$$\frac{d}{dt}f(x_t) = \langle \nabla f(x_t) | \dot{x}_t \rangle \leq 0.$$

The analogous property for a vector field ϕ is:

$$\langle \phi(x_t) | \dot{x}_t \rangle \geq 0.$$

In the framework of games, a similar condition was described in discrete time as Myopic Adjustment Dynamics, Swinkels, 1993 [76] : if $x^i_{n+1} \neq x^i_n$ then $H^i(x^i_{n+1}, x^{-i}_n) > H^i(x^i_n, x^{-i}_n)$.

The corresponding property in continuous time corresponds to positive correlation, Sandholm, 2011 [61]:

$$\dot{x}^i_t \neq 0 \Longrightarrow \langle \phi^i(x_t), \dot{x}^i_t \rangle > 0.$$

The use of this notion in potential games is a follows:

Proposition 1.1

Consider a game $\Gamma(g)$ with potential function W.
If the dynamics satisfies positive correlation, then W is a strict Lyapunov function.
All ω-limit points are rest points.

Proof:

Let $V_t = W(x_t)$ for $t \geq 0$. Then:

$$\dot{V}_t = \langle \nabla W(x_t) | \dot{x}_t \rangle = \sum_{i \in I} \langle \nabla^i W(x_t) | \dot{x}_t^i \rangle = \sum_{i \in I} \mu^i(x) \langle \phi^i(x_t) | \dot{x}_t^i \rangle \geq 0.$$

since $\phi^i = \nabla^i g^i$. Moreover, $\langle \phi^i(x_t) | \dot{x}_t^i \rangle = 0$ holds for all i if and only if $\dot{x}_t = 0$.
One concludes by using Lyapunov's theorem (e.g. [34, Theorem 2.6.1]). ■ ■

This result is proved by Sandholm, 2001 [60] for his version of potential population game, see extensions in Benaim, Hofbauer and Sorin, 2005 [11].
A similar property for fictitious play in discrete time is established in Monderer and Shapley,1996 [49].

We will show that this property holds for the dynamics defined below.

We now introduce and study three dynamics:

- Projected dynamics (PD),
- Mirror descent (MD),
- Dual averaging (DA).

Ìn each case we first define the dynamics, then control the values of the regret by exhibiting a level function and finally study the trajectories for class (II) and (III).

3.3 Projected dynamics

We assume in this subsection that $V = V^*$ is an Euclidean space with scalar product denoted by $\langle , \rangle$.

Dynamics

The projected dynamics (PD), which is the continuous time analog of the generalization of the *Projected Gradient Descent*, Levitin and Polyak, 1966 [42], Polyak, 1987 [59], see also Section 4.1, is defined by $x_t \in X$ satisfying:

$$\langle u_t - \dot{x}_t, y - x_t \rangle \leq 0, \quad \forall y \in X. \tag{10}$$

which is:

$$\dot{x}_t = \Pi_{T_X(x_t)}(u_t) \tag{11}$$

since $T_X(x_t)$ is a cône.

Values

Let:

$$V(t;y) = \frac{1}{2}\|x_t - y\|^2, \quad y \in X. \tag{12}$$

Proposition 1.2

V is a level function.

Proof:
One has:

$$\frac{d}{dt}V(t;y) = \langle \dot{x}_t, x_t - y\rangle \leq \langle u_t, x_t - y\rangle$$

by (10). ∎

Thus the properties of section 3.1 hold.

- Consider class (II) : $u_t = \phi(x_t)$.

Proposition 1.3

Assume ϕ dissipative.
Then $\{\bar{x}_t\}$ converges to a point in $SE(\phi)$.

Proof:
- $\{\bar{x}_t\}$ is bounded hence has accumulation points.
- The limit points of $\{\bar{x}_t\}$ are in $SE(\phi)$ by Lemma 1.3.
- $\|x_t - y^*\|$ converges when $y^* \in SE(\phi)$ by Lemma 1.6.

Hence using Opial's lemma,1967 [57], which states:

> In an Hilbert space, if for any weak accumulation point $\hat{x}$ of $\{x_t\}$, (resp. $\{\bar{x}_t\}$), $\|x_t - \hat{x}\|$ has a limit as $t \to \infty$, then $\{x_t\}$ (resp. $\{\bar{x}_t\}$) weakly converges. (13)

it follows that $\bar{x}_t$ converges to a point in $SE(\phi)$. ∎

Lemma 1.8

Positive correlation holds.

Proof:

$$\langle \phi(x_t), \dot{x}_t \rangle = \|\dot{x}_t\|^2$$

since $\langle u_t - \dot{x}_t, \dot{x}_t \rangle = 0$ by (11) and Moreau's decomposition, Moreau, 1965 [50]. ∎

• Class (III) : $u_t = -\nabla f(x_t)$.

Lemma 1.9

i) $\{x_t\}$ *converges to a point in* E*.*
ii) $f(x_t)$ *decreases to* $\min f$ *with speed* $O(1/t)$*.*

Proof:
i) Both Lemma 1.4 or Lemma 1.7, and Lemma 1.8 imply that the accumulation points of $\{x_t\}$ are in E.
Then using Lemma 1.6, Opial's Lemma (13) applies.
ii) Follows from lemma 1.7. ∎

We assume in this subsection that V is an Hilbert space with scalar product $\langle , \rangle$ and that $X \subset V$ is non-empty convex and closed.

The results of Section 2 and 3.1 extend while dealing with weak accumulation points.

- Consider class (II)

Lemma 1.10

Assume $SE(\phi) \neq \emptyset$. Then the trajectory $\{x_t\}$ is bounded.

Proof:
By lemma 1.6, for $y^* \in SE(\phi)$, $V(t; y^*)$ is decreasing. ∎

In particular this implies the following convergence result:

Proposition 1.4

Assume $SE(\phi) \neq \emptyset$ and ϕ dissipative.
Then $\{\bar{x}_t\}$ converges weakly to a point in $SE(\phi)$.

Proof:
- $\{\bar{x}_t\}$ is bounded by Lemma 1.10 hence has weak accumulation points.
- The weak limit points of $\{\bar{x}_t\}$ are in $SE(\phi)$ by Lemmapro1.3.
- $\|x_t - y^*\|$ converges when $y^* \in SE(\phi)$ by Lemma 1.6.

Hence by Opial's lemma (13), $\bar{x}_t$ converges weakly to a point in $SE(g\phi)$. ■

Recall that if g is dissipative and X is bounded, $SE(\phi) \neq \emptyset$ by Lemma 1.3.

• Consider class (III).

Proposition 1.5

Assume $E \neq \emptyset$.
i) $\{x_t\}$ weakly converges to a point in E.
ii) $f(x_t)$ decreases to $\min f$ with speed $O(1/t)$.

Proof:
i) - $\{x_t\}$ is bounded by Lemma 1.10 hence has weak accumulation points.
- Both Lemma 1.4 or Lemma 1.7, and Lemma 1.8 imply that the weak accumulation points of $\{x_t\}$ are in E. Then using Lemma 1.6, Opial's Lemma (13) applies.
ii) Follows from lemma 1.7. ■

Note that if X is bounded, $E \neq \emptyset$ by Lemma 1.3.

3.4 Mirror descent : differential/incremental approach

We study here the continuous version of the extension of the *mirror descent algorithm* studied in convex optimization, Nemirovski and Yudin, 1983 [52], Beck and Teboulle, 2003 [6], see also section 4.2.

The assumptions are:
H is a strictly convex, $\mathscr{C}^1$ function from V to $\mathbb{R} \cup \{+\infty\}$.
$X \subset V$ is nonempty, compact, convex and $X \subset \mathrm{dom} H$.

The continuous time procedure miror descent (MD) satisfies $x_t \in X$ with:

$$\langle u_t - \frac{d}{dt} \nabla H(x_t) | y - x_t \rangle \leq 0, \quad \forall y \in X. \tag{14}$$

The previous analysis of Section 2.2 corresponds to the Euclidean case with:

$$H(x) = \frac{1}{2} \|x\|^2.$$

Recall that the Bregman distance associated to H is:

$$D_H(y,x) = H(y) - H(x) - \langle \nabla H(x) | y - x \rangle \, (\geq 0). \tag{15}$$

Values

The use of the Bregman distance is the following:

Proposition 1.6

$P(t;y) = D_H(y,x_t)$ *is a level function.*

Proof:

Note the relation:

$$\frac{d}{dt} D_H(y,x_t) = -\langle \nabla H(x_t) | \dot{x}_t \rangle - \frac{d}{dt} \langle \nabla H(x_t) | y - x_t \rangle = \langle \frac{d}{dt} \nabla H(x_t) | x_t - y \rangle \tag{16}$$

so that (14) implies:

$$\frac{d}{dt} D_H(y,x_t) \leq \langle u_t | x_t - y \rangle. \tag{17}$$

■

Hence the properties of section 3.1 apply.

Interior trajectory

The use of a specfic function H adapted to X, with $\mathrm{dom}\, H = X$, H $\mathscr{C}^2$ on $int\, X$ and $\|\nabla H(x)\| \to +\infty$ as $x \to \partial X$ allows to produce a trajectory that remains in $int\, X$.
In this case (14) leads to an equality:

$$\frac{d}{dt}\nabla H(x_t) = u_t \tag{18}$$

thus:

$$\nabla H(x_t) = \int_0^t u_s ds \tag{19}$$

or, with H^* being the Fenchel conjugate of H:

$$x_t = \nabla H^*(\int_0^t u_s ds) \tag{20}$$

and then:

$$\dot{x}_t = \nabla^2 H(x_t)^{-1} u_t. \tag{21}$$

$\nabla^2 H(x)$ induces a Riemannian metric, see Alvarez, Bolte and Brahic, 2004 [1], Mertikopoulos and Sandholm, 2018 [47].

Lemma 1.11

Positive correlation holds.

Proof :

$$\langle \phi(x_t) | \dot{x}_t \rangle = \langle \phi(x_t) | \nabla^2 H(x_t)^{-1} \phi(x_t) \rangle \geq 0.$$

■

Consider now class (III).

By Lemma 1.4 the accumulation points of $\{x_t\}$ are in E.

To prove convergence one introduces the following :
Hypothesis [H1]: if $z^k \to y^* \in S$ then $D_H(y^*, z^k) \to 0$.
For example H is L-smooth (see e.g. Nesterov, 2004 [54] Section 1.2.2.) and then:

$$0 \leq D_H(x,y) \leq \frac{L}{2}\|x-y\|^2.$$

Hypothesis [H2]: if $D_H(y^*, z^k) \to 0, y^* \in S$ then $z^k \to y^*$.
For example H is β-strongly convex (see e.g. Nesterov, 2004 [54] Section 2.1.3.) and then:

$$D_H(x,y) \geq \frac{\beta}{2}\|x-y\|^2.$$

Proposition 1.7

Consider class (III). If H is smooth and strongly convex, $\{x_t\}$ converges weakly to some $x^* \in E$.

Proof:

Let x^* be an accumulation point of $\{x_t\}$. Then $x^* \in E$ by Lemma 1.4 and thus $D_H(x^*, x_t)$ is decreasing by Lemma 1.6 and Proposition 1.6. Since this sequence is decreasing to 0 on a subsequence $x_{t_k} \to x^*$ by [H1], it is decreasing to 0, hence by [H2] $x_t \to x^*$. ■

3.5. Dual averaging: integral/cumulative approach

We consider here the continuous version of the extension of *dual averaging*, Nesterov, 2009 [55], see also section 4.3.

We follow the analysis and results in Kwon and Mertikopoulos, 2017 [39].

Dynamics

The assumptions are :
h *is a bounded strictly convex l.s.c. function from* V *to* $\mathbb{R} \cup \{+\infty\}$ *with* $\operatorname{dom} h = X \neq \emptyset$ *convex compact.*
Let $h^*(w) = \sup_{x \in V} \langle w | x \rangle - h(x)$ *be the Fenchel conjugate.* h^* *is differentiable since* h *is strictly convex.*

Introduce :

$$U_t = \int_0^t u_s ds$$

and let the dual averaging (DA) dynamics be defined by:

$$x_t = argmax\{\langle U_t | x \rangle - h(x); x \in V\} = argmax\{\langle U_t | x \rangle - h(x); x \in X\}.$$

The dynamics can be written as:

$$x_t = \nabla h^*(U_t) \in X. \tag{22}$$

Values

Define, for $y \in X$:

$$W(t;y) = h^*(U_t) - \langle U_t | y \rangle + h(y) \tag{23}$$

which corresponds to the Fenchel coupling between the cumulative input U_t and a reference point y.

Proposition 1.8

W is a level function.

Proof:
$W(t;y) \geq 0$ by Fenchel inequality.
Use that:

$$\frac{d}{dt} h^*(U_t) = \langle u_t | \nabla h^*(U_t) \rangle = \langle u_t | x_t \rangle \tag{24}$$

by (22), thus:

$$\frac{d}{dt} W(t;y) = \langle u_t | x_t - y \rangle.$$

In particular one has:

$$R_t(y) = \int_0^t \langle u_s | y - x_s \rangle ds = W(0;y) - W(t;y) \leq [-\inf_X h + h(y)] \leq r_X(h) \quad (25)$$

with $r_X(h) = \sup_X h(x) - \inf_X h(x)$.

Note that due to the integral formulation of the dynamics (x_t as a function of U_t rather than $\dot{x}_t$ as a function of u_t) the level function is expressed through the dual space, however properties of section 3.1 applies as well.

Lemma 1.12

Positive correlation holds.

Proof:

$$\langle \phi(x_t) | \dot{x}_t \rangle = \langle \phi(x_t) | \nabla^2 h^*(U_t)(u_t) \rangle$$

with $u_t = \phi(x_t)$. ∎

Hence in class (III), using Lemma (1.4) the accumulation points of $\{x_t\}$ are in E.

Remark

In the interior smooth case both dynamics and level functions of sections 3.3 (MD) and 3.4 (DA) are the same, since one has:

$$x_t = \nabla h^*(U_t), \qquad \nabla h(x_t) = U_t, \qquad h^*(U_t) + h(x_t) = \langle U_t | x_t \rangle$$

and

$$\begin{aligned} D_h(y, x_t) &= h(y) - h(x_t) - \langle \nabla h(x_t) | y - x_t \rangle \\ &= h(y) + h^*(U_t) - \langle U_t | x_t \rangle - \langle \nabla h(x_t) | y - x_t \rangle \\ &= h(y) + h^*(U_t) - \langle U_t | y \rangle \end{aligned}$$

3.5 Comments on the continuous dynamics framework

1) One obtains the existence of a level function and same speed of convergence of the no-regret quantities in classes (I), (II) or (III) : $O(\frac{1}{t})$, which is optimal Nesterov, 2004 [54].
2) Hence by Section 2 the accumulation point of the average $\{\bar{x}_t\}$ in class (II) with ϕ dissipative are in $SE(\phi)$.
3) In addition weak convergence of the average $\{\bar{x}_t\}$ holds in class (II) with ϕ dissipative, under (PD), via Opial's lemma.
The linear aspect of the derivative of the level function seems crucial to obtain this property.
4) Similarly weak convergence of $\{x_t\}$ in case (III) holds for (PD), and (MD) with adapted penalization function H.
5) The accumulation points of $\{x_t\}$ are in E in case (III) under (DA) .

6) For vector fields ϕ with potential W, $W(x_t)$ is decreasing in (PD) and (DA), and under conditions on H for (MD).
7) In the framework of games the function:

$$h(x) = \sum_{p \in S} x^p Log x^p$$

defined on the simplex $X = \Delta(S)$ leads (via (MD) or (DA)) to the replicator dynamics on $int\, X$, Taylor and Jonker, 1978 [77], Hofbauer and Sigmund, 1998 [34], Sorin, 2009 [67], 2020 [69].
The corresponding Riemannian metric is introduced in Shahshahani, 1979 [64].
Recall that $h(x) = \frac{1}{2}\|x^2\|$ leads to the local/direct projection dynamics, for a comparison, see Lahkar and Sandholm, 2008 [40], Sandholm, Dokumaci and Lahkar, 2008 [63].

Note that the replicator dynamics is the continuous version of the multiplicative weight algorithm, Littlestone and Warmuth, 1994 [43], Vovk, 1990 [79], Sorin, 2009 [67], 2020 [69].

4. Discrete time: general case

We consider now discrete time algorithms.

Remark that the dynamics depends on an additional parameter, the *step size*.

4.1. Projected dynamics

Recall that V is Euclidean and let $m(X)$ denote the diameter of X.
Assumption:
$\|u_m\| \leq M, \forall m \in \boldsymbol{N}$.

Dynamics

The standard discrete dynamics (PD) (*Gradient projection method* in class (III), Levitin and Polyak, 1966 [42], Polyak, 1987 [59]) is given by:

$$x_{m+1} = argmax_X \{\langle u_m, x\rangle - \frac{1}{2\eta_m}\|x - x_m\|^2\}, \quad (26)$$

with $\eta_m > 0$ decreasing, which corresponds to:

$$x_{m+1} = \Pi_X[x_m + \eta_m u_m]. \quad (27)$$

The variational characterization is $x_{m+1} \in X$ satisfying :

$$\langle x_m + \eta_m u_m - x_{m+1}, y - x_{m+1} \rangle \leq 0, \quad \forall y \in X, \tag{28}$$

which is :

$$\langle u_m - \frac{x_{m+1} - x_m}{\eta_m}, y - x_{m+1} \rangle \leq 0, \quad \forall y \in X,$$

and leads to the continuous time equation (10) as $\eta_m \to 0$.

Recall that :

$$R_n(x) = \sum_{m=1}^{n} \langle u_m | x - x_m \rangle.$$

Proposition 1.9

$$R_n(x) \le \frac{1}{2\eta_n} m(X)^2 + \frac{M^2}{2} \sum_{m=1}^{n} \eta_m.$$

Hence with $\eta_n = 1/\sqrt{n}$:

$$R_n(x) \le O(\sqrt{n}).$$

Proof:
Let $x_{m+1} = \Pi_X(y_{m+1})$ with $y_{m+1} = x_m + \eta_m u_m$.
So that:

$$\begin{aligned} 2\eta_m \langle u_m, x - x_m \rangle &= 2\langle y_{m+1} - x_m, x - x_m \rangle \\ &= \|y_{m+1} - x_m\|^2 + \|x - x_m\|^2 - \|x - y_{m+1}\|^2. \end{aligned}$$

Note that for $x \in X$, (28) implies $\|y_{m+1} - x\|^2 \geq \|x_{m+1} - x\|^2$ hence:

$$2\eta_m \langle u_m, x - x_m \rangle \leq \eta_m^2 \|u_m\|^2 + \|x - x_m\|^2 - \|x - x_{m+1}\|^2$$

which is the discrete analog of Proposition 1.2 .
Thus:

$$\langle u_m, x - x_m \rangle \leq \frac{\eta_m}{2} \|u_m\|^2 + \frac{1}{2\eta_m} [\|x - x_m\|^2 - \|x - x_{m+1}\|^2].$$

It follows that:

$$\begin{aligned} R_n(x) &= \sum_{m=1}^{n} \langle u_m, x - x_m \rangle \\ &\leq \frac{1}{2\eta_1} \|x - x_1\|^2 - \frac{1}{2\eta_n} \|x - x_{n+1}\|^2 \\ &\quad + \sum_{m=2}^{n} [\frac{1}{2\eta_m} - \frac{1}{2\eta_{m-1}}] \|x - x_m\|^2 + \frac{M^2}{2} \sum_{m=1}^{n} \eta_m. \end{aligned}$$

Thus, with $m(X)$ being the diameter of X:

$$R_n(x) \leq \frac{1}{2\eta_n} m(X)^2 + \frac{M^2}{2} \sum_{m=1}^{n} \eta_m,$$

and the choice of $\eta_m = \frac{1}{\sqrt{m}}$ gives:

$$R_n(x) \leq \sqrt{n}[m(X)^2 + M^2].$$

■

Trajectories

Consider class (II).

Lemma 1.13

For $x^* \in SE(\phi)$, $\|x_m - x^*\|$ converges if $\{\eta_n\} \in \ell^2$.

Proof:

If $x^* \in SE(\phi)$ then:

$$\|x^* - x_{m+1}\|^2 \leq \|x^* - (x_m + \eta_m \phi(x_m))\|^2 \leq \eta_m^2 \|\phi(x_m)\|^2 + \|x^* - x_m\|^2$$

so that $\|x_m - x^*\|$ converges if $\{\eta_n\} \in \ell^2$. ■

Lemma 1.14

If ϕ is dissipative and $\{\eta_n\} \in \ell^2$, $\{\bar{x}_n\}$ converges to a point in $SE(\phi)$.

Proof:

The limit points of $\{\bar{x}_n\}$ are in $SE(\phi)$ by Lemma 1.3, hence by Lemma 1.13 and Opial's Lemma (13), $\{\bar{x}_n\}$ converges to a point in $SE(\phi)$. ∎

4.2. Mirror descent

Assumptions:
a) H *is a* $\mathscr{C}^1$ *function from* V *to* $\mathbb{R} \cup \{+\infty\}$*,* L*-strongly convex for some norm* $\|.\|$ *on* $V = \mathbb{R}^n$ *and* $X \subset \mathrm{dom}\, H$*,*
b) $\|u_n\|_* \leq M, \forall n \in$ ***N****.*

Dynamics

The classical discrete mirror descent algorithm (MD), introduced for class (III) in Nemirovski and Yudin, 1983 [52], see also Beck and Teboulle, 2003 [6], is given by (recall (15)):

$$x_{m+1} = argmax_X \{\langle u_m | x \rangle - \frac{1}{\eta_m} D_H(x, x_m)\}. \tag{29}$$

The variational expression takes the form, $x_m \in X$ and :

$$\langle \nabla H(x_m) + \eta_m u_m - \nabla H(x_{m+1}) | x - x_{m+1} \rangle \leq 0, \quad \forall x \in X. \tag{30}$$

Note that $D_H(x,y)$ plays the rôle of $\frac{1}{2}\|x-y\|^2$ in (26).

We will use the identity:

$$D_H(x,z) - D_H(x,y) - D_H(y,z) = \langle \nabla H(y) - \nabla H(z) | x - y \rangle \qquad (31)$$

which is a direct consequence of the definition of D_H.

Proposition 1.10

Let the step size $\eta_n = \frac{1}{\sqrt{n}}$, then:

$$R_n(x) \leq O(\sqrt{n}).$$

Proof:

$$\begin{aligned}
\langle \eta_n u_n, x - x_n \rangle &= \langle \eta_n u_n, x - x_{n+1} \rangle + \langle \eta_n u_n, x_{n+1} - x_n \rangle \\
&\leq \langle \nabla H(x_{n+1}) - \nabla H(x_n) | x - x_{n+1} \rangle + \langle \eta_n u_n, x_{n+1} - x_n \rangle \\
&= D_H(x, x_n) - D_H(x, x_{n+1}) - D_H(x_{n+1}, x_n) \\
&\quad + \langle \eta_n u_n, x_{n+1} - x_n \rangle \qquad (32)
\end{aligned}$$

by using (31).

Now, H is L strongly convex, hence:

$$D_H(x_{n+1}, x_n) \geq \frac{L}{2} \|x_{n+1} - x_n\|^2 \tag{33}$$

and moreover :

$$\langle \eta_n u_n | x_{n+1} - x_n \rangle - \frac{L}{2} \|x_n - x_{n+1}\|^2 \leq M \eta_n \|x_n - x_{n+1}\| - \frac{L}{2} \|x_n - x_{n+1}\|^2 \leq \frac{(\eta_n M)^2}{2L}$$

so that one obtains the analogous of Proposition 1.6:

$$\langle \eta_n u_n, x - x_n \rangle \leq D_H(x, x_n) - D_H(x, x_{n+1}) + \frac{(\eta_n M)^2}{2L}.$$

Summing one obtains:

$$R_n(x) \leq \sum_m [D_H(x, x_m)(\frac{1}{\eta_{m+1}} - \frac{1}{\eta_m}) + \eta_m \frac{M^2}{2L}]. \tag{34}$$

Hence the bound like in Proposition 1.9. ■

Consider class (II).

Lemma 1.15

For $x^* \in SE(\phi)$, $D_H(x^*, x_n)$ converges if $\{\eta_n\} \in \ell^2$.

Proof:

Start with (32) for $x^* \in SE(\phi)$ and use $D_H \geq 0$ to get:

$$D_H(x^*, x_{m+1}) \leq D_H(x^*, x_m) - \langle \eta_m u_m | x_m - x_{m+1} \rangle \qquad (35)$$

thus it remains to control $\langle \eta_m u_m | x_m - x_{m+1} \rangle$.

H being L strongly convex implies:

$$\langle \nabla H(x_m) - \nabla H(x_{m+1}) | x_m - x_{m+1} \rangle \geq L \|x_{m+1} - x_m\|^2$$

but one has by (30):

$$\begin{aligned} \langle \nabla H(x_m) - \nabla H(x_{m+1}) | x_m - x_{m+1} \rangle &\leq \langle -\eta_m u_m | x_m - x_{m+1} \rangle \\ &\leq \|\eta_m u_m\|_* \|x_m - x_{m+1}\|. \end{aligned}$$

It follows that:

$$\|x_m - x_{m+1}\| \leq \frac{1}{L} \|\eta_m u_m\|_*$$

hence:

$$\langle -\eta_m u_m | x_m - x_{m+1} \rangle \leq \frac{1}{L} \|\eta_m u_m\|_*^2.$$

Altogether this implies from (35) that $D_H(x^*, x_m)$ converges if $\{\eta_n\} \in \ell^2$. ■

Notice that there is no monotonicity property.

4.3. Dual averaging

Assumptions

a) h is a l.s.c. function from V to $\mathbb{R} \cup \{+\infty\}$, L-strongly convex for some norm $\|.\|$ on $V = \mathbb{R}^n$, with $\operatorname{dom} h = X$.

*b) $\|u_m\|_* \leq M, \forall n \in$ **N**.*

Dynamics

We follow the formulation in Nesterov, 2009 [55]. The starting point is again a maximization property:

$$x_{m+1} = argmax_X \{\langle U_m | x \rangle - \frac{1}{\eta_m} h(x)\}, \tag{36}$$

with $U_m = \sum_{k=1}^m u_k$ and where $\{\eta_m\}$ is decreasing.
Note that there is an explicit form without using a variational formulation. The dual averaging algorithm is given by:

$$\boxed{x_{m+1} = \nabla h^*(\eta_m U_m).} \tag{37}$$

A direct proof, see Xiao, 2010 [84] or a discrete approximation of (22), see Kwon and Mertikopoulos, 2017 [39], allows to obtain:

Proposition 1.11

$$R_n(x) = \sum_{m=1}^{n} \langle u_m | x - x_m \rangle \leq \frac{r_X(h)}{\eta_n} + \frac{\sum_{m=1}^{n} \eta_{m-1} \|u_m\|_*^2}{2L}. \tag{38}$$

Proof:
Fenchel inequality:

$$\langle \eta_n U_n | x \rangle \leq h^*(\eta_n U_n) + h(x)$$

implies:

$$\langle U_n | x \rangle \leq \frac{h^*(0)}{\eta_0} + \sum_{m=1}^{n} \left(\frac{h^*(\eta_m U_m)}{\eta_m} - \frac{h^*(\eta_{m-1} U_{m-1})}{\eta_{m-1}}\right) + \frac{\max_X h}{\eta_n}. \tag{39}$$

Now:

$$
\begin{aligned}
\frac{h^*(\eta_m U_m)}{\eta_m} &= \sup_X [\langle U_m | x \rangle - \frac{h(x)}{\eta_m}] \\
&\leq \sup_X [\langle U_m | x \rangle - \frac{h(x)}{\eta_{m-1}}] + \sup_X [-\frac{h(x)}{\eta_m} + \frac{h(x)}{\eta_{m-1}}] \\
&= \frac{h^*(\eta_{m-1} U_m)}{\eta_{m-1}} + (\frac{1}{\eta_{m-1}} - \frac{1}{\eta_m}) \min_X h,
\end{aligned}
$$

so that replacing in (39) gives:

$$
\begin{aligned}
\langle U_n | x \rangle &\leq \frac{h^*(0)}{\eta_0} + \sum_{m=1}^{n} \frac{1}{\eta_{m-1}} [h^*(\eta_{m-1} U_m) - h^*(\eta_{m-1} U_{m-1})] \\
&\quad + \min_X h (\frac{1}{\eta_0} - \frac{1}{\eta_n}) + \frac{\max_X h}{\eta_n}. \qquad (40)
\end{aligned}
$$

h is L strongly convex for $\|.\|$, so that h^* is $1/L$ smooth for $\|.\|_*$ hence:

$$
\begin{aligned}
& h^*(\eta_{m-1} U_m) - h^*(\eta_{m-1} U_{m-1}) - \langle \eta_{m-1} U_m - \eta_{m-1} U_{m-1} | \nabla h^*(\eta_{m-1} U_{m-1}) \\
= \; & h^*(\eta_{m-1} U_m) - h^*(\eta_{m-1} U_{m-1}) - \langle \eta_{m-1} u_m | x_m \rangle \\
\leq \; & \frac{\eta_{m-1}^2}{2L} \|u_m\|_*^2.
\end{aligned}
$$

This leads to a property similar to Proposition 1.8 since:

$$\begin{aligned}
\langle u_m | x - x_m \rangle &\leq \langle u_m | x \rangle + \frac{1}{\eta_{m-1}} [h^*(\eta_{m-1} U_{m-1}) - h^*(\eta_{m-1} U_m)] + \frac{\eta_{m-1}}{2L} \|u_m\|_*^2 \\
&\leq \frac{1}{\eta_{m-1}} [h^*(\eta_{m-1} U_{m-1}) + h(x) - \langle \eta_{m-1} U_{m-1}, x \rangle] \\
&\quad - \frac{1}{\eta_m} [h^*(\eta_m U_m) + h(x) - \langle \eta_m U_m, x \rangle] \\
&\quad + \frac{\eta_{m-1}}{2L} \|u_m\|_*^2 + (\frac{1}{\eta_{m-1}} - \frac{1}{\eta_m}) \min_X h.
\end{aligned}$$

Now inserting in (40) gives:

$$\langle U_n | x \rangle - \sum_{m=1}^{n} \langle u_m | x_m \rangle \leq \frac{h^*(0)}{\eta_0} + \sum_{m=1}^{n} \frac{\eta_{m-1}}{2L} \|u_m\|_*^2 + (\frac{1}{\eta_0} - \frac{1}{\eta_n}) \min_X h + \frac{\max_X h}{\eta_n}$$

and recall that $h^*(0) = -\min_X h$. ■

Hence the convergence rate $O(\sqrt{n})$ with time varying parameters $\eta_n = 1/\sqrt{n}$.

4.4. Comments on the discrete dynamics framework

1) The three algorithms achieve the same bound $O(1/\sqrt{n})$ for the speed of convergence of the average regret, which is optimal already in class (III), Nesterov, 2004 [54] , using time varying step sizes $\eta_n = 1/\sqrt{n}$.
2) More precise properties concerning the trajectories are available only in the (PD) set-up. The results are similar to the ones in the continuous case, Section 3.2, if $\eta_n \in \ell^2$, for class (II).
(Compare to the analysis in Peypouquet and Sorin, 2010 [58] for dynamics induced by maximal monotone operators in discrete and continuous time.)
3) For vector fields ϕ with potential W one does not have the property $W(x_n)$ decreasing.

5. Discrete time: Regularity

This section deals mainly with class (III) *convex gadient*, where in addition f satisfies some regularity properties.

Recall that f is β smooth if:

$$|f(y)-f(x)-\langle \nabla f(x)|y-x\rangle| \leq \frac{\beta}{2}\|x-y\|^2. \tag{41}$$

Equivalently, ∇f is β-Lipschitz.

5.1. Projected dynamics

Assumption: f is β smooth.

The algorithm is like (28) with constant step size $\eta_m = 1/\beta$.

$$x_{m+1} = argmax_X\{\langle -\nabla f(x_m), x\rangle - \frac{1}{2\beta}\|x - x_m\|^2\}, \tag{42}$$

which gives:

$$\boxed{x_{m+1} = \Pi_X[x_m - \frac{1}{\beta}\nabla f(x_m)].} \tag{43}$$

The analysis in this section is standard, see e.g. Nesterov, 2004 [54].

Define $Tx = \Pi_X[x - \frac{1}{\beta}\nabla f(x)]$ and $v(x) = \beta(x - Tx)$ (which plays the role of $\nabla f(x)$, corresponding to $X = V$).
The projection property gives:

$$\langle x - \frac{1}{\beta}\nabla f(x) - Tx, y - Tx\rangle \leq 0, \quad \forall y \in X \tag{44}$$

so that:

$$\langle \nabla f(x), Tx - y\rangle \leq \langle v(x), Tx - y\rangle, \quad \forall y \in X. \tag{45}$$

Now one has, using f β-smooth, convex:

$$\begin{aligned} f(Tx) - f(y) &= f(Tx) - f(x) + f(x) - f(y), \\ &\leq \langle \nabla f(x), Tx - x\rangle + \frac{\beta}{2}\|Tx - x\|^2 + \langle \nabla f(x), x - y\rangle, \\ &= \langle \nabla f(x), Tx - y\rangle + \frac{1}{2\beta}\|v(x)\|^2, \\ &\leq \langle v(x), Tx - y\rangle + \frac{1}{2\beta}\|v(x)\|^2, \quad \forall y \in X \qquad \text{by } (45) \end{aligned}$$

hence:

$$f(Tx) - f(y) \leq \langle v(x), x - y\rangle - \frac{1}{2\beta}\|v(x)\|^2, \quad \forall y \in X. \tag{46}$$

The following property is crucial and shows the difference with the general non-smooth case.

Lemma 1.16 (Descent lemma)

$$f(x_{m+1}) - f(x_m) \leq -\frac{1}{2\beta}\|v(x_m)\|^2 = -\frac{\beta}{2}\|x_{m+1} - x_m\|^2. \tag{47}$$

Proof:
By the previous inequality (46) with $x_m = x = y$. ■

In particular:

$$\frac{1}{2\beta}\sum_{m=1}^{n}\|v(x_m)\|^2 \leq f(x_1) - f(x_{n+1}) \leq f(x_1) - f^*$$

hence $\{\|v(x_n)\|\} \in \ell^2$.

Values

Proposition 1.12

$$f(x_n) - f^* \leq O(\frac{1}{n}).$$

Proof:
Consider the algorithm defined by $\{z_n\}$ and the process $\{v_n\}$ with $z_1 = x_1$, $v_n = -v(z_n)$ and $z_{n+1} = z_n + \eta v_n$ with $\eta = 1/\beta$.
Clearly $z_n = x_n$.
From section 3.1, Proposition 1.9 with η_m constant, one obtains:

$$R_n^v(y) = \sum_{m=1}^{n} \langle v_m, y - z_m \rangle \leq \frac{1}{2\eta} \|y - z_1\|^2 + \frac{\eta}{2} \sum_{m=1}^{n} \|v_m\|^2 \quad (48)$$

which is bounded since $\{\|v(x_n)\|\} \in \ell^2$. This implies, using $f(x_n)$ decreasing and (46):

$$n[f(x_{n+1}) - f(y)] \leq R_n^v(y) - \frac{1}{2\beta} \| \sum_{m=1}^{n} \|v_m\|^2 = \frac{\beta}{2} \|y - x_1\|^2.$$

Lemma 1.17

For $y^* \in S$, $\|x_n - y^*\|$ is decreasing.

Proof:

From (46) one has:

$$0 \leq f(x_{n+1}) - f(y^*) \leq \langle v(x_n), x_n - y^* \rangle - \frac{1}{2\beta}\|v(x_n)\|^2$$

hence :

$$\langle v(x_n), x_n - y^* \rangle \geq \frac{1}{2\beta}\|v(x_n)\|^2.$$

So that:

$$\begin{aligned}
\|x_{n+1} - y^*\|^2 &= \|x_{n+1} - x_n\|^2 + \|x_n - y^*\|^2 + 2\langle x_{n+1} - x_n, x_n - y^* \rangle \\
&= \frac{1}{\beta^2}\|v(x_n)\|^2 + \|x_n - y^*\|^2 + 2\langle -\frac{1}{\beta}v(x_n), x_n - y^* \rangle \\
&\leq \|x_n - y^*\|^2.
\end{aligned}$$

Proposition 1.13

$\{x_n\}$ *converges to a point in* E*.*

Proof:

Since $f(x_n)$ decreases, the accumulation points of $\{x_n\}$ are in E and by the previous Lemma 1.17, Opial's Lemma (13) applies.

∎

5.2. Mirror descent

We initially assume only that H and f are $\mathscr{C}^1$.
The next analysis follows Bauschke, Bolte and Teboulle, 2017 [5].

The main assumption in this section is the existence of a constant $L > 0$ such that (recall that D_H is the Bregman distance (15)):

$$(A) \qquad\qquad LD_H - D_f \geq 0$$

which is equivalent to : $LH - f$ convex.

Note that if H is strongly convex and f is smooth (not assumed convex), there exists L such that (A) holds. However f is not required to be smooth.
(A similar pre-oder on convex functions appears in Nguyen, 2017 [56]).

Recall the procedure (30) with constant step size λ:

$$\langle \lambda \nabla f(x_n) + \nabla H(x_{n+1}) - \nabla H(x_n) | x - x_{n+1} \rangle \geq 0, \quad \forall x \in X \quad (49)$$

and the identity:

$$D_H(x,z) - D_H(x,y) - D_H(y,z) = \langle \nabla H(y) - \nabla H(z) | x - y \rangle. \quad (50)$$

Lemma 1.18

Let $2\lambda L = 1$*, then :*

$$\begin{aligned}\lambda[f(x_{n+1}) - f(y)] \quad &\leq \quad D_H(y, x_n) - D_H(y, x_{n+1}) - \frac{1}{2} D_H(x_{n+1}, x_n) \\ &\quad -\lambda D_f(y, x_n), \forall y \in X. \end{aligned} \tag{51}$$

Proof :

Since:

$$D_f(x,z) = D_f(y,z) + f(x) - f(y) - \langle \nabla f(z) | x - y \rangle, \tag{52}$$

one has, by (A):

$$f(x) \leq f(y) + \langle \nabla f(z) | x - y \rangle + L D_H(x,z) - D_f(y,z).$$

Let $x = x_{n+1}, z = x_n$, so that:

$$f(x_{n+1}) - f(y) \leq \langle \nabla f(x_n) | x_{n+1} - y \rangle + L D_H(x_{n+1}, x_n) - D_f(y, x_n)$$

hence by (49):

$$\begin{aligned}\lambda[f(x_{n+1})-f(y)] &\leq \langle \nabla H(x_{n+1})-\nabla H(x_n)|y-x_{n+1}\rangle \\ &\quad +\lambda L D_H(x_{n+1},x_n)-\lambda D_f(y,x_n).\end{aligned}$$

Use then (50):

$$\begin{aligned}\lambda[f(x_{n+1})-f(y)] &\leq D_H(y,x_n)-D_H(y,x_{n+1})-D_H(x_{n+1},x_n) \\ &\quad +\lambda L D_H(x_{n+1},x_n)-\lambda D_f(y,x_n).\end{aligned}$$

Hence the result with $2\lambda L = 1$. ∎

Proposition 1.14

Assume H convex . Then:
1) $f(x_n)$ is decreasing.
2) $\sum D_H(x_{n+1}, x_n) < +\infty$.
Assume f convex. Then :
3)

$$f(x_n) - f(y) \leq \frac{2L}{n} D_H(y, x_1), \quad \forall y \in X.$$

Proof:
1) and 2) Take $y = x_n$ in (51).
3) Use $f(x_n)$ decreasing, $D_f \geq 0$ and the telescoping sum in (51).

■

Proposition 1.15

Assume f convex.
1) $y \in E$ implies: $D_H(y, x_n)$ decreases.
2) Assume:
$[H1] : x^k \to x^* \in E \Rightarrow D_H(x^*, x^k) \to 0$,
$[H2] : x^* \in E, D_H(x^*, x^k) \to 0 \Rightarrow x^k \to x^*$,
then $\{x_n\}$ converges to a point in E.

Proof:
1) follows from (51).
2) By the previous Proposition 1.14, the accumulation points of $\{x_n\}$ are in E. Let thus $x_{n_k} \to x^* \in E$.
By $H1$, $D_H(x^*, x_{n_k}) \to 0$ then by 1) $D_H(x^*, x_n) \to 0$. Now use $H2$. ∎

Compare with the proof of Proposition 1.7.

Note that the result is more precise than in the continuous case, Section 3.3, where there was no decreasing property (for general H).

Let us finally mention the very recent result due to Bui and Combettes, 2020 [16] Theorem 3.9., where the use of variable metrics H_n allows to reach $f(x_n) - f^* = o(1/n)$.

5.3.Dual averaging

We follow the analysis in Lu, Freund and Nesterov, 2018 [44].
Recall that we consider class (III) : f convex and $\mathscr{C}^1$.
As in the previous Section 4.2. the main hypothesis is the existence of $L > 0$ with:

$$(A) \qquad\qquad LD_h - D_f \geq 0.$$

where $h : V \to \mathbb{R} \cup \{+\infty\}$ is l.s.c. with $\operatorname{dom} h = X$.
Let $x_0 = argmin_X h(x)$ and assume $h(x_0) = 0$.

Dynamics

Define :

$$G_k(x) = \sum_{i=0}^{k-1} [\langle \nabla f(x_i), x - x_i \rangle + f(x_i)] + Lh(x) \tag{53}$$

and as in (36), (with as usual $u_k = -\nabla f(x_k)$ and $U_m = \sum_{k=1}^{m} u_k$):

$$\boxed{x_k = argmin_X G_k(x) = argmax_X \{\langle U_{k-1}, x \rangle - Lh(x)\}} \tag{54}$$

Proposition 1.16

$$f(\bar{x}_k) - f(x) \leq \frac{L}{k} h(x), \tag{55}$$

$$\min_{i=0,\ldots,k} f(x_i) - f(x) \leq \frac{L}{k} h(x). \tag{56}$$

Proof:
By definition:

$$G_{k+1}(x_{k+1}) = G_k(x_{k+1}) + \langle \nabla f(x_k), x_{k+1} - x_k \rangle + f(x_k). \tag{57}$$

Note that (A) implies that each $G_k - f$ is convex hence:

$$G_k(x_{k+1}) - f(x_{k+1}) \geq G_k(x_k) - f(x_k) + \langle \partial G_k(x_k) - \nabla f(x_k), x_{k+1} - x_k \rangle. \tag{58}$$

Thus one has:

$$G_{k+1}(x_{k+1}) \geq f(x_{k+1}) + G_k(x_k) + \langle \partial G_k(x_k), x_{k+1} - x_k \rangle \tag{59}$$

but there exists $u_k \in \partial G_k(x_k)$ with :

$$\langle u_k, x - x_k \rangle \geq 0, \quad \forall x \in X$$

by the choice of x_k.
Finally, with $g_{k+1} = G_{k+1}(x_{k+1})$, one obtains:

$$g_{k+1} \geq f(x_{k+1}) + g_k, \tag{60}$$

which implies, using f convex, thus $G_k(x) \leq kf(x) + Lh(x)$ by (53), that:

$$\sum_{i=0}^{k+1} f(x_i) \leq g_{k+1} \leq (k+1)f(x) + Lh(x) \tag{61}$$

hence the result. ■

5.4. Comments on the regular case

1) In the three cases (PD), (MD) and (DA) the speed of convergence of the values is $O(1/n)$ and the algorithms use a constant step parameter.
2) Using (PD) with f smooth implies $f(x_n)$ decreasing and the convergence of $\{x_n\}$.
3) The approach in Section 5.2 shows that similar results can be obtained using (MD) without assuming f with Lipschitz gradient if the regularization function H is adapted to f : condition (A).
4) Analogous results for the values are much simpler to obtain in the (DA) framework. However the properties concern the value at the average $f(\bar{x}_n)$ and no result is available on the trajectories.

6. Concluding remarks

For the three dynamics (PG), (MD) and (DA) 1), 2) and 3) below holds:
1) In continuous time the speed of convergence of the average regret to 0, of the order $O(1/t)$ is not better in the general gradient convex case than in on-line learning.
2) In discrete time the speed of convergence of the average regret to 0, of the order $O(1/\sqrt{n})$ is not better in the general gradient convex case than in on-line learning.
3) Adding a smoothness hypothesis on the convex function does not change the convergence rate in continuous time but allow a better convergence in discrete time from $O(1/\sqrt{n})$ to $O(1/n)$.

4) A similar phenomena appears with the so-called acceleration procedures following Nesterov, 1983 [53].

In the continuous time case a second order ODE leads to a speed of convergence $f(x_t) - f(x^*) \leq O(\frac{1}{t^2})$ with no further hypothesis on f, see Su, Boyd and Candes, 2014 [74], 2016 [75], Krichene, Bayen and Bartlett, 2015 [37], 2016 [38], Attouch, Chbani, Peypouquet and Redont, 2018 [2].

To obtain a similar property in discrete time, namely $f(x_n) - f(x^*) \leq O(\frac{1}{n^2})$ one has to assume f smooth, Chambolle and Dossal, 2015[19], Attouch, Chbani, Peypouquet, Redont 2018 [2].

The same remark apply to the convergence of the trajectory, where the smooth hypothesis on f is needed in discrete time and not in continuous time.

5) Concerning the link between discrete and continuous time dynamics, there are no direct results of the form: no-regret property in continuous time imply no-regret property in discrete time but analogy of the tools used and ad-hoc choice of the stage parameters, see Sorin, 2009 [67], Kwon and Mertikopoulos, 2017 [39].

6) The Hilbert framework for (PD) allows to obtain convergence results on the trajectories. The two other algorithms are more flexible and can achieve better explicit speed of convergence of the values by choosing an adequate norm, see the discussion in Bauschke, Bolte and Teboulle, 2017 [5]. For (MD), specific regularization functions H can also lead to convergence of the trajectories. (DA) is much simpler to implement due to its integral formulation. However no convergence properties of the trajectories are in general available.

7) In the framework of games, positive results are obtained in the class of dissipative games. Accumulation points of the average trajectory are equilibria.

8) Obviously potential games with a concave potential P will share the properties of class (III) since basically one can replace for each i, $\langle \phi^i(x), y^i - x^i \rangle$ by $\langle \nabla_i P(x), y^i - x^i \rangle$.

1. Stochastic approximation

We summarize here results from Benaïm, Hofbauer and Sorin, 2005 [11], following the approach for ODE by Benaïm, 1996 [7], 1999 [8], Benaïm and Hirsch, 1996 [10].

1.1. Differential inclusions

Let F be a correspondence from $\mathbb{R}^m$ to itself, u.s.c. with compact convex non empty values satisfying, for some $c > 0$:

$$\sup_{z \in F(x)} \| z\| \leq c(1 + \|x\|), \quad \forall x \in \mathbb{R}^m.$$

Consider the differential inclusion, see Aubin and Cellina, 1984[3]:

$$\dot{\mathbf{x}} \in F(\mathbf{x}). \qquad (I)$$

It induces a set-valued dynamical system $\{\Phi_t\}_{t \in \mathbb{R}}$ defined by

$$\Phi_t(x) = \{\mathbf{x}(t) : \mathbf{x} \text{ is a solution to } (I) \text{ with } \mathbf{x}(0) = x\}.$$

We also write $\mathbf{x}(t) = \varphi_t(x)$ and define $\Phi_A(B) = \cup_{t \in A,\, x \in B} \Phi_t(x)$.

1.2. Attractors

Definition 2.1

1) C is invariant if for any $x \in C$ there exists a complete solution: $\varphi_t(x) \in C$ for all $t \in \mathbb{R}$.
2) C is attracting if it is compact and there exist a neighborhood U, $\varepsilon_0 > 0$ and a map $T : (0, \varepsilon_0) \to \mathbb{R}^+$ such that: for any $y \in U$, any solution φ, $\varphi_t(y) \in C^\varepsilon$ for all $t \geq T(\varepsilon)$, i.e.

$$\Phi_{[T(\varepsilon),+\infty)}(U) \subset C^\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

U is a uniform basin of attraction of C and we write $(C;U)$ for the couple.
3) C is an attractor if it is attracting and invariant.
4) The ω-limit set of C is defined by

$$\omega_\Phi(C) = \cap_{s\geq 0}\overline{\cup_{y\in C} \cup_{t\geq s} \Phi_t(y)} = \cap_{s\geq 0}\overline{\Phi_{[s,+\infty)}}(C). \qquad (62)$$

5) Given a closed invariant set L, the induced set-valued dynamical system is denoted by Φ^L. L is attractor free if Φ^L has no proper attractor.

1.3. Lyapounov functions

We describe here practical criteria for attractors.

Proposition 2.1

Let A be a compact set, U be a relatively compact neighborhood of A and V a function from $\overline{U}$ to $\mathbb{R}^+$.

Assume:

i) $\Phi_t(U) \subset U$ for all $t \geq 0$.

ii) $V^{-1}(0) = A$

iii) V is continuous and strictly decreasing on trajectories on $\overline{U} \setminus A$:

$$V(x) > V(y), \quad \forall x \in U \setminus A, \forall y \in \Phi_t(x), \quad \forall t > 0.$$

Then:

a) A is Lyapounov stable and $(A;U)$ is attracting.

b) $(B;U)$ is an attractor for some $B \subset A$.

Definition 2.2

A real continuous function V *on* U *open in* $\mathbb{R}^m$ *is a* *Lyapunov function* *for* (A,U), $A \subset U$ *if :*
$V(y) < V(x)$ *for all* $x \in U \setminus A, y \in \Phi_t(x), t > 0,$
$V(y) \leq V(x)$ *for all* $x \in A, y \in \Phi_t(x)$ *and* $t \geq 0.$

Proposition 2.2

Suppose V *is a Lyapunov function for* (A,U).
Assume that $V(A)$ *has empty interior.*
Let $L \subset U$ *be non empty, compact, invariant and attractor free.*
Then L *is contained in* A *and* $V_{|L}$ *is constant.*

1.4. Asymptotic pseudo-trajectories

Definition 2.3

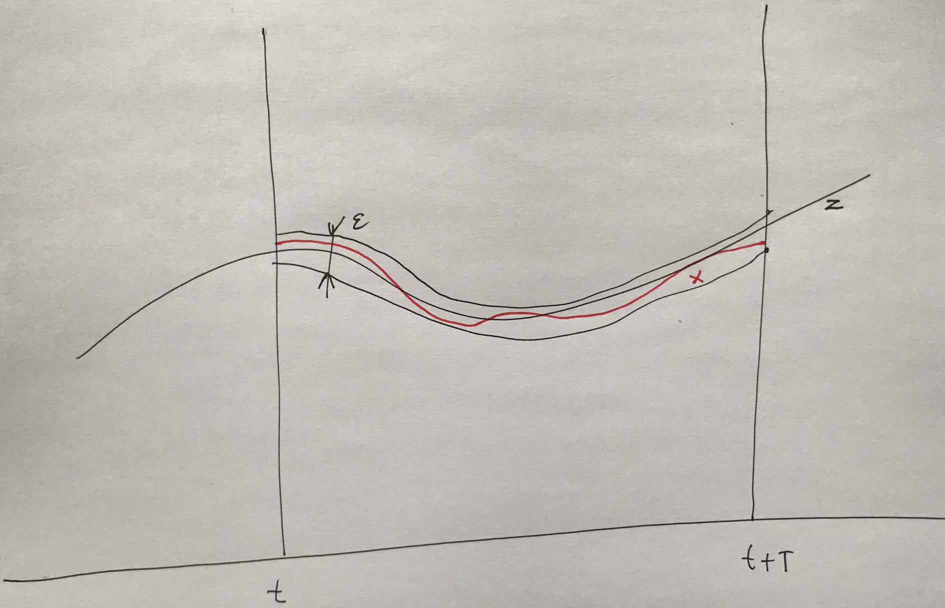
A continuous function $\mathbf{z} : \mathbb{R}^+ \to \mathbb{R}^m$ *is an asymptotic pseudo-trajectory (APT) for* (I) *if for all* T*:*

$$\lim_{t \to \infty} \inf_{\mathbf{x} \in S_{\mathbf{z}(t)}} \sup_{0 \leq s \leq T} \|\mathbf{z}(t+s) - \mathbf{x}(s)\| = 0, \quad (63)$$

where S_x *denotes the set of solutions of* (I) *starting from* x *at 0.*

In other words, for each fixed T, the curve: $s \to \mathbf{z}(t+s)$ from $[0,T]$ to $\mathbb{R}^m$ shadows some trajectory for (I) of the point $\mathbf{z}(t)$ over the interval $[0,T]$ with arbitrary accuracy, for sufficiently large t.

$\forall \varepsilon > 0, T \quad \exists t_0, \; t \geq t_0$



Let:

$$L(\mathbf{z}) = \bigcap_{t \geq 0} \overline{\{\mathbf{z}(s) : s \geq t\}}$$

be the limit set.

Theorem 2.1

Let $\mathbf{z}$ be a bounded APT of (I).
Then $L(\mathbf{z})$ is (internally chain transitive, hence) compact, invariant and attractor free.

1.5. Perturbed solutions

Definition 2.4

A continuous function $\mathbf{y} : \mathbb{R}_+ = [0,\infty) \to \mathbb{R}^m$ *is a perturbed solution to* (I) *if it satisfies the following set of conditions* (II)*:*

i) $\mathbf{y}$ *is absolutely continuous.*

ii) There exists a locally integrable function $t \mapsto U(t)$ *such that* $\lim_{t\to\infty} \sup_{0\le v\le T} \left\| \int_t^{t+v} U(s)\,ds \right\| = 0$*, for all* $T > 0$*.*

iii)

$$\dot{\mathbf{y}}(t) \in F^{\delta(t)}(\mathbf{y}(t)) + U(t),$$

for almost every $t > 0$*, for some function* $\delta : [0,\infty) \to \mathbb{R}$ *with* $\delta(t) \to 0$ *as* $t \to \infty$*.*

Here $F^{\delta}(x) := \{y \in \mathbb{R}^m : \exists z : \|z - x\| < \delta,\ d(y, F(z)) < \delta\}$.

The purpose is to investigate the long-term behavior of $\mathbf{y}$ and to describe its limit set $L(\mathbf{y})$ in terms of the dynamics induced by F.

Theorem 2.2

Any bounded solution $\mathbf{y}$ of (II) is an APT of (I).

A natural class of perturbed solutions to F arises from certain stochastic approximation processes.

Definition 2.5

A discrete time process $\{x_n\}$ *with values in* $\mathbb{R}^m$ *is a* (γ, U) *discrete stochastic approximation for* (I) *if it verifies a recursion of the form:*

$$x_{n+1} - x_n \in \gamma_{n+1}[F(x_n) + U_{n+1}], \qquad (III)$$

where the characteristics $\{\gamma_n\}$ *and* $\{U_n\}$ *satisfy:*
i) $\{\gamma_n\}_{n\geq 1}$ *is a sequence of nonnegative numbers such that:*

$$\sum_n \gamma_n = \infty, \qquad \lim_{n\to\infty} \gamma_n = 0;$$

ii) $U_n \in \mathbb{R}^m$ *are (deterministic or random) perturbations.*

To such a process is naturally associated a continuous time interpolated (random) process $\mathbf{w}$, denoted (IV), as usual: $t_n = \sum_{m=1}^{n} \gamma_m$, $\mathbf{w}(t_n) = x_n$ and $\mathbf{w}$ is linear on $[t_n, t_{n+1}]$.

1.6. From interpolated process to perturbed solutions

The next result gives sufficient conditions on the characteristics of the discrete process (III) for its interpolation (IV) to be a perturbed solution (II).

Proposition 2.3

Assume that:

$$(*) \quad \forall T > 0, \quad \lim_{n\to\infty} \sup \left\{ \left\| \sum_{i=n}^{k-1} \gamma_{i+1} U_{i+1} \right\| : k = n+1, \ldots, m(\tau_n + T) \right\} = 0,$$

where $\tau_n = \sum_{i=1}^{n} \gamma_i$ *and* $m(t) = \sup\{k \geq 0 : t \geq \tau_k\}$;

$(**)$ $\sup_n \|x_n\| = M < \infty$.

Then the interpolated process $\mathbf{w}$ *is a perturbed solution of* (I).

We describe now sufficient conditions for condition $(*)$ to hold.
Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\{\mathscr{F}_n\}_{n\geq 0}$ a filtration of $\mathscr{F}$.

A stochastic process $\{x_n\}$ satisfies the Robbins–Monro condition if:
i) $\{\gamma_n\}$ is a deterministic sequence.
ii) $\{U_n\}$ is *adapted* to $\{\mathscr{F}_n\}$,
iii) $\mathbf{E}(U_{n+1} \mid \mathscr{F}_n) = 0$.

Proposition 2.4

Let $\{x_n\}$ given by (III) be a Robbins–Monro process. Suppose that for some $q \geq 2$

$$\sup_n \mathbf{E}(\|U_n\|^q) < \infty \quad \textit{and} \quad \sum_n \gamma_n^{1+q/2} < \infty.$$

Then assumption $(*)$ holds with probability 1.

Remark
Typical applications are
i) U_n uniformly bounded in L^2 and $\gamma_n = \frac{1}{n}$,
ii) U_n uniformly bounded and $\gamma_n = o(\frac{1}{\log n})$.

1.7. Main result

Consider a random discrete process defined on a compact subset of $\mathbb{R}^K$ and satisfying the differential inclusion:

$$Y_n - Y_{n-1} \in a_n[T(Y_{n-1}) + W_n]$$

where:
i) T is an u.s.c. correspondence with compact convex values,
ii) $a_n \geq 0$, $\sum_n a_n = +\infty$, $\sum_n a_n^2 < +\infty$,
iii) $E(W_n | Y_1, ..., Y_{n-1}) = 0$, W_n uniformly bounded in L^2.

Theorem 2.3

The set of accumulation points of $\{Y_n\}$ is almost surely a compact set, invariant and attractor free for the dynamical system defined by the differential inclusion:

$$\dot{Y} \in T(Y).$$

A typical application is the case where:

$$Y_n - Y_{n-1} \in a_n \mathbf{T}(Y_{n-1})$$

with $\mathbf{T}$ random, where one writes:

$$Y_n - Y_{n-1} \in a_n [E[\mathbf{T}(Y_{n-1})|Y_1, ..., Y_{n-1}]$$

$$+(\mathbf{T}(Y_{n-1}) - E[\mathbf{T}(Y_{n-1})|Y_1, ..., Y_{n-1}])].$$

2. Applications

We mainly follow Benaïm M., J. Hofbauer and S. Sorin, 2006 [12].

2.1. Application 1: Fictitious Play for potential games

Proposition 2.5

Assume $G(NE(G))$ *with non empty interior. Then (DFP) converges to* $NE(G)$*.*

Proof:
Apply Proposition 2.2 to W with $A = NE(G)$ and $U = X$. ■

2.2. Application 2: No regret

Definition 2.6

P is a potential function for $D = \mathbb{R}^K_-$ if
(i) P is $\mathscr{C}^1$ from $\mathbb{R}^K$ to $\mathbb{R}^+$
(ii) $P(w) = 0$ iff $w \in D$
(iii) $\nabla P(w) \in \mathbb{R}^K_+$
(iv) $\langle \nabla P(w), w \rangle > 0, \forall w \notin D$.

Compare Hart and Mas Colell (2003).

Example: $P(w) = \sum_k ([w^k]^+)^2 = d(w, D)^2$.

a) External regret

Given a potential P for $D = \mathbb{R}^K_-$, the P-regret-based discrete procedure for player 1 is defined by:

$$\sigma(h_n) \div \nabla P(\overline{R}_n) \qquad \text{if} \quad \overline{R}_n \notin D \tag{64}$$

and arbitrarily otherwise.

The discrete dynamics associated to the average regret satisfies:

$$\overline{R}_{n+1} - \overline{R}_n = \frac{1}{n+1}(R_{n+1} - \overline{R}_n).$$

By the choice of σ, one has:

$$\langle \nabla P(\overline{R}_n), \mathsf{E}(R_{n+1}|h_n) \rangle = 0.$$

(recall $\langle x, \mathsf{E}_x(R(.,U)) \rangle = 0.$)
The continuous time version is expressed by the following differential inclusion in $\mathbb{R}^m$:

$$\dot{\mathbf{w}} \in N(\mathbf{w}) - \mathbf{w}, \tag{65}$$

where N is a correspondence that satisfies:

$$\langle \nabla P(\mathbf{w}), \mathbf{N}(\mathbf{w}) \rangle = \mathbf{0}.$$

Theorem 2.4

The potential P is a Lyapounov function associated to $D = \mathbb{R}^K_-$.
Hence, D contains a global attractor.

Proof :

For any solution $\mathbf{w}$, if $\mathbf{w}(t) \notin D$ then

$$\frac{d}{dt}P(\mathbf{w}(t)) = \langle \nabla P(\mathbf{w}(t)), \dot{\mathbf{w}}(t) \rangle$$

$$\in \langle \nabla P(\mathbf{w}(t)), N(\mathbf{w}(t)) - \mathbf{w}(t) \rangle = -\langle \nabla P(\mathbf{w}(t)), \mathbf{w}(t) \rangle < 0$$

■

Corollary 2.1

Any P-regret-based discrete dynamics satisfies external consistency.

Proof:

$D = \mathbb{R}^K_-$ contains an attractor whose basin of attraction contains the range $\mathscr{R}$ of R and the discrete process for $\bar{R}_n$ is a bounded DSA. ■

b) Internal regret

Definition 2.7

Given a potential Q for $M = \mathbb{R}_-^{K^2}$, a Q-regret-based discrete procedure for player 1 is a strategy σ satisfying:

$$\sigma(h_n) \in Inv[\nabla Q(\bar{S}_n)] \qquad \textit{if} \quad \bar{S}_n \notin M \tag{66}$$

and arbitrarly otherwise.

The discrete process of internal regret matrices is:

$$\bar{S}_{n+1} - \bar{S}_n = \frac{1}{n+1}[S_{n+1} - \bar{S}_n]. \tag{67}$$

with the property:

$$\langle \nabla Q(\bar{S}_n), \mathsf{E}(S_{n+1}|h_n) \rangle = 0.$$

(Recall $\langle A, \mathsf{E}_\mu(S(.,U)) \rangle = 0$.)

The corresponding continuous time procedure with $w \in \mathbb{R}^{K^2}$ is given by:

$$\dot{\mathbf{w}}(t) \in N(\mathbf{w}(t)) - \mathbf{w}(t) \tag{68}$$

and:

$$\langle \nabla Q(w), N(w) \rangle = 0.$$

Theorem 2.5

The previous continuous time process satisfy:

$$\mathbf{w}^{+}_{k\ell}(t) \rightarrow_{t\to\infty} 0.$$

Corollary 2.2

The discrete process (67) satisfy:

$$[\bar{S}_n^{k\ell}]^{+} \rightarrow_{t\to\infty} 0 \qquad a.s.$$

hence conditional consistency (internal no regret) holds.

2.3. Application 3: Consistency with smooth fictitious play

This procedure is based only on the previous observations and not on the moves of the predictor, hence the regret cannot be used, Fudenberg and Levine,1995 [25].

Definition 2.8

A smooth perturbation of the payoff $U \in \mathscr{U}$ *is a map*
$V^{\varepsilon}(x,U) = \langle x,U \rangle - \varepsilon\rho(x), \quad 0 < \varepsilon < \varepsilon_0$, *such that:*
(i) $\rho : X \to \mathbb{R}$ *is a* $\mathscr{C}^1$ *function with* $\|\rho\| \leq 1$,
(ii) $\text{argmax}_{x \in X} V^{\varepsilon}(.,U)$ *reduces to one point and defines a continuous map* $\mathbf{br}^{\varepsilon} : \mathscr{U} \to X$, *called a smooth best reply function,*
(iii) $D_1 V^{\varepsilon}(\mathbf{br}^{\varepsilon}(U),U).D\mathbf{br}^{\varepsilon}(U) = 0$
(for example $D_1 U^{\varepsilon}(.,U)$ *is 0 at* $\mathbf{br}^{\varepsilon}(U)$*).*

Recall that a typical example is obtained via the entropy function:

$$\rho(x) = \sum_k x_k \log x_k. \tag{69}$$

which leads to:

$$[\mathbf{br}^{\varepsilon}(U)]^k = \frac{\exp(U^k/\varepsilon)}{\sum_{j\in K}\exp(U^j/\varepsilon)}. \tag{70}$$

Le:t

$$W^{\varepsilon}(U) = \max_x V^{\varepsilon}(x,U) = V^{\varepsilon}(\mathbf{br}^{\varepsilon}(U),U).$$

Lemma 2.1 (Fudenberg and Levine, 1999 [26])

$$DW^{\varepsilon}(U) = \mathbf{br}^{\varepsilon}(U).$$

Let us first consider external consistency.

Definition 2.9

A smooth fictitious play strategy σ^ε *associated to the smooth best response function* $\mathbf{br}^\varepsilon$ *(in short a* SFP(ε) *strategy) is defined by (*$\overline{U}_n$ *is the average vector of payoffs up to stage* n*):*

$$\sigma^\varepsilon(h_n) = \mathbf{br}^\varepsilon(\overline{U}_n).$$

The corresponding discrete dynamics written in the spaces of both vectors and outcomes is:

$$\overline{U}_{n+1} - \overline{U}_n = \frac{1}{n+1}[U_{n+1} - \overline{U}_n], \tag{71}$$

$$\overline{\omega}_{n+1} - \overline{\omega}_n = \frac{1}{n+1}[\omega_{n+1} - \overline{\omega}_n], \tag{72}$$

with:

$$\mathsf{E}(\omega_{n+1}|h_n) = \langle \mathbf{br}^\varepsilon(\overline{U}_n), U_{n+1} \rangle. \tag{73}$$

Lemma 2.2

The process $(\overline{U}_n, \overline{\omega}_n)$ *is a Discrete Stochastic Approximation of the differential inclusion:*

$$(\dot{\mathbf{u}}, \dot{\omega}) \in \{(U - \mathbf{u}, \langle \mathbf{br}^{\varepsilon}(\mathbf{u}), U\rangle - \omega); U \in \mathscr{U}\}. \tag{74}$$

The main property of the continuous dynamics is given by:

Theorem 2.6

The set $\{(u, \omega) \in \mathscr{U} \times \mathbb{R} : W^{\varepsilon}(u) - \omega \leq \varepsilon\}$ *is a global attracting set for the continuous dynamics.*
In particular, for any $\eta > 0$*, there exists* $\bar{\varepsilon}$ *such that for* $\varepsilon \leq \bar{\varepsilon}$*,* $\limsup_{t\to\infty} W^{\varepsilon}(\mathbf{u}(t)) - \omega(t) \leq \eta$ *(i.e. continuous* SFP(ε) *satisfies* η*-consistency).*

Proof :
Let $q(t) = W^{\varepsilon}(\mathbf{u}(t)) - \omega(t)$.
Taking time derivative one obtains, using the previous Lemma:

$$\begin{aligned}
\dot{q}(t) &= DW^{\varepsilon}(\mathbf{u}(t)).\dot{\mathbf{u}}(t) - \dot{\omega}(t) \\
&= \langle \mathbf{br}^{\varepsilon}(\mathbf{u}(t)), \dot{\mathbf{u}}(t) \rangle - \dot{\omega}(t) \\
&= \langle \mathbf{br}^{\varepsilon}(\mathbf{u}(t)), U - \mathbf{u}(t) \rangle - (\langle \mathbf{br}^{\varepsilon}(\mathbf{u}(t)), U \rangle - \omega(t)) \\
&\leq -q(t) + \varepsilon.
\end{aligned}$$

Hence:

$$\dot{q}(t) + q(t) \leq \varepsilon,$$

so that $q(t) \leq \varepsilon + Me^{-t}$ for some constant M and the result follows. ■

Theorem 2.7

For any $\eta > 0$, there exists $\bar{\varepsilon}$ such that for $\varepsilon \leq \bar{\varepsilon}$, SFP$(\varepsilon)$ *is η-consistent.*

Proof:

The assertion follows from the previous result and the DSA property.

■

A similar result holds for internal no-regret procedures.

Benaim and Faure, 2013 [9] obtain consistency with vanishing perturbation $\varepsilon = n^{-a}, a < 1$.
Note that the corresponding process is non longer autonomous.

For the link with replicator dynamics and comparison with best reply procedures, see Hofbauer, Sorin and Viossat, 2009 [35].

Research directions:

a) Replicator dynamics and correlated equilibria, Viossat, 2007 [79], 2014 [80], 2015 [81],
b) Games on signals,
c) Subclasses of games and adapted equilibria,
d) Regularity and approximation discrete/continuous,
e) Asymptotic analysis: attractors vs set of fixed points.

Thanks : Institutions

IMPA, Rio, Brazil, 2006, 2009
CMM, Santiago, Chile, 2006, 2015,2017
GERAD, Montreal, Canada, 2011
AMS Short Courses Evolutionary Game Dynamics, New Orleans, USA., 2011
Summer School and Workshop on Dynamical Games and Applications, Valparaiso, Chili, 2012
Les Diablerets, Swiss Doctoral Program in Mathematics, 2012
Hausdorff Mathematical Research Institute, Bonn, Germany, 2013
IMS, NSU, Singapore, 2015
Courant Institute and Economics Dpt., NYU, NY, USA, 2017
IMCA, Lima, Peru, 2019

Thanks : People

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