Dynamics in Games: Algorithms and Learning

Sylvain Sorin
sylvain.sorin@imj-prg.fr

Institut de Mathématiques Jussieu-PRG
Sorbonne Université, UPMC - Paris 6
CNRS UMR 7586

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Abstract

Game theory studies interactions between agents with specific aims, be they rational actors, genes, or computers. This course is intended to provide the main mathematical concepts and tools used in game theory with a particular focus on their connections to learning and convex optimization. The first part of the course deals with the basic notions: value, (Nash and Wardrop) equilibria, correlated equilibria. We will give several dynamic proofs of the minmax theorem and describe the link with Blackwell's approachability. We will also study the connection with variational inequalities.

The second part will introduce no-regret properties in on-line learning and exhibit a family of unilateral procedures satisfying this property. When applied in a game framework we will study the consequences in terms of convergence (value, correlated equilibria). We will also compare discrete and continuous time approaches and their analog in convex optimization (projected gradient, mirror descent, dual averaging). Finally we will present the main tools of stochastic approximation that allow to deal with random trajectories generated by the players.

Part A

BASIC TOOLS AND RESULTS

A.1 Value and equilibria

This section deeply relies on the books:

Mertens J.-F., S. Sorin and S. Zamir (2015) *Repeated Games*, Cambridge University Press.

Laraki R., J. Renault and S. Sorin (2019) *Mathematical Foundations of Game Theory*, Springer.

Strategic games: introduction

Strategic games: notations

A strategic game G is defined by:
- a set I of players, (I will denote also the cardinal),
- a set S^i of strategies for each player $i \in I$,
- a mapping g from $S = \Pi_{i=1}^{I} S^i$ into $\mathbb{R}^I$.

$g^i(s)$ is the payoff of player i when the profile $s = (s^1, \cdots, s^I)$ is played.

Denote $s = (s^i, s^{-i})$ where s^{-i} is the vector $\{s^j; j \neq i\}$ and $S^{-i} = \Pi_{j \neq i} S^j$.

$\Delta(K)$= simplex on a finite set $K = \{x \in \mathbb{R}^K, x^k \geq 0, \sum_{k\in K} x^k = 1\}$,
= set of Borel probabilities on a topological space K (compact, metric).

Mixed extension of G:
$\Sigma^i = \Delta(S^i)$, $i \in I$ set of mixed strategies of player i, $\Sigma = \Pi_{i\in I}\Sigma^i$, multilinear extension of g to Σ (assuming Fubini):

$$g^j(\sigma) = \int_{\Pi_{i=1}^I S^i} g^j(s)\Pi_{i=1}^I d\sigma^i(s^i)$$

Strategic games: definitions

For $\varepsilon \geq 0$, the ($\varepsilon-$)best reply correspondence BR^i_ε of player i, from S^{-i} to S^i, is defined by:

$$BR^i_\varepsilon(s^{-i}) = \{s^i \in S^i : g^i(s^i, s^{-i}) \geq g^i(t^i, s^{-i}) - \varepsilon, \forall t^i \in S^i\}.$$

It associates to every profile of the opponents the set of $\varepsilon-$best replies of a player.

Write $BR : S \rightrightarrows S$, for the global best reply correspondence that maps $s \in S$ to $\prod_{i \in I} BR^i(s^{-i})$.

The extension of $BR : \Sigma \rightrightarrows \Sigma$ to the mixed extension of the game is straightforward.

Equilibrium

A Nash equilibrium (Nash, 1950 [33]) is a profile of strategies $s \in S$ where no player can gain by changing his strategy.

More generally, for $\varepsilon \geq 0$, an $\varepsilon-$equilibrium is a profile $s \in S$, such that for all i, $s^i \in BR^i_\varepsilon(s^{-i})$, which is:

$$g^i(t^i, s^{-i}) \leq g^i(s) + \varepsilon, \quad \forall t^i \in S^i, \quad \forall i.$$

Thus s is an equilibrium iff s is a fixed point of the BR correspondence:

$$s \in BR(s).$$

An equilibrium s is strict if $\{s\} = BR(s)$.

Alternatively, a profile t eliminates a profile s if there exists a player $i \in I$ with $g^i(t^i, s^{-i}) > g^i(s)$.
Let $E(t) \subset S$ be the set of profiles not eliminated by $t \in S$.

An equilibrium is then a profile in $\cap_{t \in S} E(t)$.

This formulation is in the spirit of an equilibrium being a “rational” rule of behavior.

Zero-sum games: value

A two-person, zero-sum game is a game where $I = 2$, $S^1 = S$, $S^2 = T$ and given $f : S \times T \to \mathbb{R}$, the payoffs are $g^1 = -g^2 = f$.
The interests of the players are opposite: $g^1 + g^2 = 0$.
One introduces the following quantities:

$$\underline{v} = \sup_S \inf_T f(s,t) \qquad \overline{v} = \inf_T \sup_S f(s,t);$$

$\underline{v}$ is the largest amount that Player 1 can guarantee and a strategy $s \in S$ is $\varepsilon(\geq 0)$-optimal if:

$$f(s,t) \geq \underline{v} - \varepsilon, \quad \forall t \in T.$$

The game has a value v if: $\underline{v} = \overline{v} = v$.

The link between value and equilibria is as follows:

Proposition 1.1

Assume that the game has a value and that s, t *are* ε*-optimal. Then they form a* $2\varepsilon-$*equilibrium:*

$$f(s,t') + 2\varepsilon \geq f(s,t) \geq f(s',t) - 2\varepsilon, \quad \forall\, s', t' \in S \times T.$$

(For $\varepsilon = 0$*, this is a saddle point.)*

Minmax theorem 1: Finite case

The sets of pure strategies or actions (moves) $S = I$, $T = J$ are finite. The (payoff of the) game G is represented by a $I \times J$ matrix A, an element $x \in \Delta(I)$ corresponds to a row matrix (mixed strategy of player 1) and an element $y \in \Delta(J)$ to a column matrix (mixed strategy of player 2), so that the payoff is given by the bilinear form $f(x,y) = xAy$.

Theorem 1.1 (Von Neumann, 1928 [47])

Let A be a $I \times J$ real matrix.
There exist (x^*, y^*, v) in $\Delta(I) \times \Delta(J) \times \mathbb{R}$ such that :

$$x^*Ay \geq v, \qquad \forall y \in \Delta(J) \qquad \text{and} \qquad xAy^* \leq v, \ \forall x \in \Delta(I). \tag{1}$$

In other words, the mixed extension of a matrix game has a value (one also says that any finite zero-sum game has a value in mixed strategies) and both players have optimal strategies.

For an extension to coefficients in an ordered field, see Weyl, 1950 [49].

The real number v in the theorem is uniquely determined and corresponds to the value of the matrix A :

$$v = \max_{x\in\Delta(I)} \min_{y\in\Delta(J)} xAy = \min_{y\in\Delta(J)} \max_{x\in\Delta(I)} xAy.$$

It is also denoted by $\mathtt{val}(A)$.
As a mapping defined on matrices, from $\mathbb{R}^{I\times J}$ to $\mathbb{R}$, the operator $\mathtt{val}$ is positively homogeneous, monotonic (increasing) and non expensive :

$$|\mathtt{val}(A) - \mathtt{val}(B)| \leq \|A - B\|_\infty$$

These properties extend to the general framework of zero-sum games:

$$|\mathtt{val}(f) - \mathtt{val}(g)| \leq \|f - g\|_\infty.$$

Minmax theorem 2: Compact case

S and T are subsets of Hausdorff topological real vector spaces.

Theorem 1.2 (Sion, 1958 [41])

Let $G = (S, T, f)$ be a zero-sum game satisfying:
(i) S and T are convex,
(ii) S or T is compact,
(iii) for each t in T, $f(.,t)$ is quasi-concave and u.s.c. in s,
and for each s in S, $f(s,.)$ is quasi-convex and l.s.c. in t.

Then G has a value: $\sup_{s\in S}\inf_{t\in T} f(s,t) = \inf_{t\in T}\sup_{s\in S} f(s,t)$.

Moreover, if S (resp. T) is compact, the above suprema (resp. infima) are achieved, and the corresponding player has an optimal strategy.

The proof uses a finite dimentional version of the following intersection lemma (Berge (1966) [3, p. 172]).

Lemma 1.1

Let $C_1, \ldots, C_n$ be non-empty convex compact subsets of a Hausdorff topological real vector space. Assume:
1) the union $\bigcup_{i=1}^n C_i$ is convex,
2) for each $j = 1, \ldots, n$, the intersection $\bigcap_{i \neq j} C_i$ is non-empty.
Then the full intersection $\bigcap_{i=1}^n C_i$ is non-empty.

The proof is by induction and only uses the Hahn–Banach strict separation theorem.

See,e.g.:
MSZ, Section I.1
LRS, Chapter 3.

Proof (of Theorem 1.2):
Assume S compact.
Suppose by contradiction that G has no value. Then there exists a real number v such that

$$\sup_{s\in S}\inf_{t\in T} f(s,t) < v < \inf_{t\in T}\sup_{s\in S} f(s,t).$$

1) We first reduce the problem to the case where S and T are polytopes.
Define for each t in T the set $S_t = \{s \in S, g(s,t) < v\}$. The family $(S_t)_{t\in T}$ is an open covering of the compact set S, so there exists a finite subset T_0 of T such that $S = \bigcup_{t\in T_0} S_t$. Let $\hat{T} = \mathrm{co}(T_0)$ which is compact and satisfies:

$$\max_{s\in S}\inf_{t\in \hat{T}} f(s,t) < v < \inf_{t\in \hat{T}}\sup_{s\in S} f(s,t).$$

Proceed similarly with the strategy space of player 1: the family $(\hat{T}_s = \{t \in \hat{T}, f(s,t) > v\})_{s \in S}$ being an open covering of $\hat{T}$, there exists a finite subset S_0 of S such that :

$$\forall s \in \hat{S} = \mathrm{co}(S_0), \exists t \in T_0, \quad f(s,t) < v,$$

$$\forall t \in \hat{T} = \mathrm{co}(T_0), \exists s \in S_0, \quad f(s,t) > v.$$

Assume that (S_0, T_0) is a minimal pair for inclusion satisfying this property: if necessary drop elements from S_0 and/or T_0.
2) $\forall s \in S_0$, let $A_s = \{t \in \hat{T}; f(s,t) \leq v\}$ which is non-empty convex and compact. Note that $\bigcap_{s \in S_0} A_s = \emptyset$ and by minimality of S_0, $\bigcap_{s \in S_0 \setminus \{s_0\}} A_s \neq \emptyset$ for each s_0 in S_0.
By the intersection lemma, the union $\bigcup_{s \in S_0} A_s$ is thus not convex.
Hence there exists a t^* in $\hat{T} \setminus \bigcup_{s \in S_0} A_s$, so that $f(s,t^*) > v, \forall s \in S_0$. By quasi-concavity of $f(\cdot, t^*)$, the inequality $f(s,t^*) > v$ also holds for each $s \in \hat{S}$.
Similarly, there exists $s^* \in \hat{S}$ such that $f(s^*, t) < v$ for each $t \in \hat{T}$.
Considering $f(s^*, t^*)$ gives the required contradiction. ■

Theorem 1.3 (Mixed extension)

Let $G = (S, T, f)$ be a zero-sum game such that:
(i) S and T are compact Hausdorff topological spaces,
(ii) for each t in T, $f(.,t)$ is u.s.c., and for each s in S, $f(s,.)$ is l.s.c.
(iii) f is bounded and mesurable with respect to the product Borel σ-algebra $\mathscr{B}_S \otimes \mathscr{B}_T$.

Then the mixed extension $(\Delta(S), \Delta(T), f)$ of G has a value. Each player has a mixed optimal strategy, and for each $\varepsilon > 0$ each player has an ε-optimal strategy with finite support.

Recall the earlier result:

Theorem 1.4 (Ville, 1938 [46])

Let $I = J = [0,1]$ and f be a real-valued continuous function on $I \times J$. The mixed extension $(\Delta(I), \Delta(J), f)$ has a value and each player has an optimal strategy.

This is the first proof of the minmax theorem using a separation (Hahn-Banach) argument.

See:
MSZ, Section I.1
LRS, Chapter 3.

Minmax principle

The next example, due to Aumann and Maschler, 1968 [2], shows the difference between an analysis in terms of (maxmin/minmax) optimal strategies or of equilibria.

	L	R
T	$(2,0)$	$(0,1)$
B	$(0,1)$	$(1,0)$

Considering only the payoff of player 1, this defines a zero-sum game with value $V_1 = 2/3$ and optimal strategy for player 1: $\bar{x} = (1/3, 2/3)$.
The dual parameters are $V_2 = 1/2$ and $\bar{y} = (1/2, 1/2)$ for player 2.
On the other hand the game has a single equilibrium:
$x^* = (1/2, 1/2), y^* = (1/3, 2/3)$ with payoff $E = (2/3, 1/2)$.
Note that for player 1 the equilibrium payoff is equal to his value (2/3) but that the equilibrium strategy x^* does not guarantee it, while $\bar{x}$ does. A similar statement holds for player 2.
However the strategies $(\bar{x}, \bar{y})$ are not in equilibrium.

Adding the optimal and equilibrium strategies gives the matrix:

	L	R	eq	op
T	$(2,0)$	$(0,1)$	$(2/3,\ .)$	$(1,1/2)$
B	$(0,1)$	$(1,0)$	$(2/3,1/3)$	$(\ .,1/2)$
eq	$(\ .,1/2)$	$(1/2,1/2)$	$(2/3,1/2)$	$(\ .,1/2)$
op	$(2/3,2/3)$	$(2/3,\ .)$	$(2/3,\ .)$	$(2/3,1/2)$

The next 5 sections provide proofs of the minmax theorem (finite case).

Proofs of minmax theorem

Minmax theorem via ODE

We follow Brown and von Neumann, 1950 [9]. A)

Lemma 2.1

Any real matrix B, $I \times I$, *antisymmetric* ($B = -{}^tB$), *has a value.*

Proof : Let $X = \Delta(I)$. It is equivalent to prove the non-emptiness of $X(B) = \{x \in X; Bx \leq 0\}$.
Let $K^i(x) = [e^i Bx]^+, i \in I, \overline{K}(x) = \sum_{i \in I} K^i(x)$ and consider the dynamical system on X defined by:

$$\dot{x}_t^i = K^i(x_t) - x_t^i \overline{K}(x_t), \quad i \in I. \tag{2}$$

Let $V(x) = \sum_{i \in I} K^i(x)^2$. The set of rest points of (2) is: $X(B) = V^{-1}(0)$ since $K^i(x) = x^i \overline{K}(x)$ gives $V(x) = \overline{K}(x) xBx = 0$.

Finally:

$$\frac{d}{dt}V(x_t) = 2\sum_i K^i(x_t)e^iB\dot{x}_t = 2[K(x_t)BK(x_t) - \{K(x_t)Bx_t\}\overline{K}(x_t)] = -2\overline{K}(x_t)V(x_t).$$

Hence $V(x_t)$ is strictly decreasing on the complement of $X(B)$. Compactness implies that the accumulation points of x_t are in $X(B)$ which is thus non empty. ■

B) We now deduce from Lemma 2.1 that any matrix A has a value. One can assume $A_{ij} > 0$ for all (i,j).

B.a) Following Gale, Kuhn and Tucker, 1950 [14], introduce the antisymmetric matrix B, of size $(I+J+1) \times (I+J+1)$ defined by:

$$B = \begin{pmatrix} 0 & A & -1 \\ -{}^tA & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Consider an optimal strategy $z = (x, y, t)$ for player 1 in the game B; then one checks easily that x and y (normalized) are optimal strategies for both players in the game A.

b) An alternative proof, Brown and von Neumann, 1950 [9], is to consider the $(I \times J) \times (I \times J)$ matrix C defined by:

$$C_{ij;i'j'} = A_{ij'} - A_{i'j}.$$

Hence each player plays in game C both as player 1 and 2 in the initial game A.
From an optimal strategy in game C one constructs optimal strategies for both players in game A. ∎

Replicator dynamics

We follow Hofbauer, 2018 [18].
Introduce the replicator dynamics, Taylor and Jonker,1978 [44], defined by the following equations, with $x_0 \in int(X), y_0 \in int(Y)$:

$$\dot{x}_t^i = x_t^i[e^i A y_t - x_t A y_t], \qquad \forall i \in I \tag{3}$$

$$\dot{y}_t^j = y_t^j[-x_t A e^j + x_t A y_t], \qquad \forall j \in J.$$

This defines trajectories in $X \times Y$ since $\frac{d}{dt}\sum_{i\in I} x_t^i = 0$ and $x_0^i > 0$ implies $x_t^i > 0, \forall t \geq 0, i \in I$.
Introduce the time average trajectory: $\bar{x}_T = \frac{1}{T}\int_0^T x_t dt$.
By integrating:

$$\frac{\dot{x}_t^i}{x_t^i} = e^i A y_t - x_t A y_t, \qquad \forall i \in I$$

one obtains:

$$\frac{1}{T}[\log x_T^i - \log x_0^i] = e^i A \bar{y}_T - \frac{1}{T}\int_0^T x_s A y_s ds \qquad \forall i \in I.$$

Consider a sequence $T_k \to \infty$ on which $(\bar{x}_{T_k}, \bar{y}_{T_k}, \frac{1}{T_k}\int_0^{T_k} x_s A y_s ds)$ converge to (x^*, y^*, w). Then:

$$e^i A y^* \leq w \leq x^* A e^j, \qquad \forall i \in I, \qquad \forall j \in J.$$

Hence the game has a value, w and x^*, y^* are optimal strategies.

The proof shows more:
- any accumulation point $\bar{x}$ of $\bar{x}_T$ belongs to $X(A)$, set of optimal strategies of player 1 in the game A,
- the average payoff along the trajectory $\frac{1}{T}\int_0^T x_s A y_s ds$ converges to the value.

Minmax theorem via unilateral process

Lehrer and Sorin, 2001 [25]

A) **Preliminary result**

Le C be a non empty closed subset of $\mathbb{R}^k$ (endowed with the Euclidean scalar product $\langle \, , \, \rangle$).

For $z \in \mathbb{R}^k$, $\mathscr{P}_C(z)$ stands for a closest point to z in C. Let $\{z_n\}$ be a bounded sequence in $\mathbb{R}^k$: $\|z_n\| \leq M$.

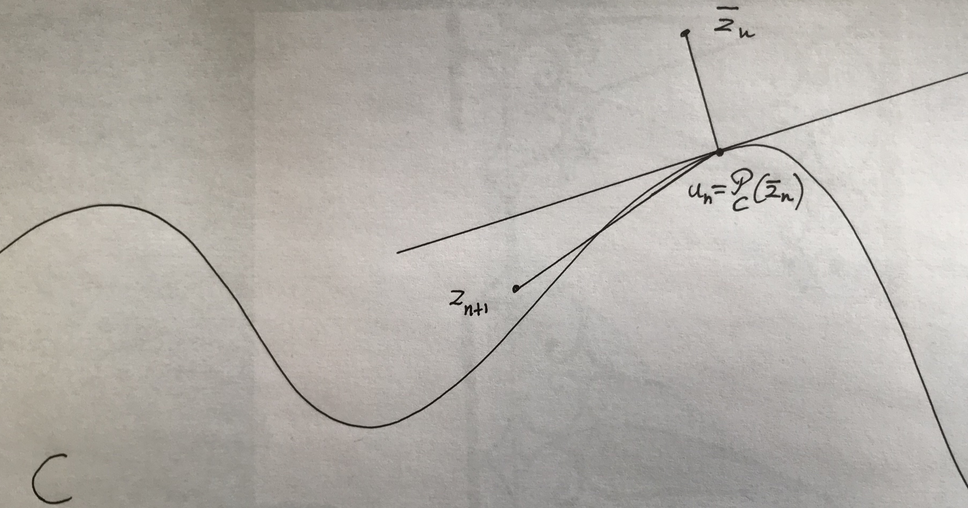
$\bar{z}_n$ denotes the Cesaro mean up to stage n of the sequence $\{z_m\}$:

$$\bar{z}_n = \frac{1}{n} \sum_{m=1}^{n} z_m.$$

Definition 2.1

$\{z_n\}$ is a Blackwell C-sequence, Blackwell, 1956 [4], if it satisfies :

$$\langle z_{n+1} - \mathscr{P}_C(\bar{z}_n), \bar{z}_n - \mathscr{P}_C(\bar{z}_n) \rangle \leq 0, \quad \forall n. \tag{4}$$


$\bar{z}_n$
$u_n = P_C(\bar{z}_n)$
z_{n+1}
C

Lemma 2.2

If $\{z_n\}$ *is a Blackwell* C*-sequence,* $d_n = d(\bar{z}_n, C)$ *converges to* 0*.*

Proof : Let $u_n = \mathscr{P}_C(\bar{z}_n)$then :

$$d_{n+1}^2 \leq \|\bar{z}_{n+1} - u_n\|^2 = \|\bar{z}_n - u_n\|^2 + \|\bar{z}_{n+1} - \bar{z}_n\|^2 + 2\langle \bar{z}_{n+1} - \bar{z}_n, \bar{z}_n - u_n\rangle$$

Decompose:

$$\langle \bar{z}_{n+1} - \bar{z}_n, \bar{z}_n - u_n\rangle = (\frac{1}{n+1})\langle z_{n+1} - \bar{z}_n, \bar{z}_n - u_n\rangle$$
$$= (\frac{1}{n+1})(\langle z_{n+1} - u_n, \bar{z}_n - u_n\rangle - \|\bar{z}_n - u_n\|^2).$$

Using the hypothesis $\langle z_{n+1} - u_n, \bar{z}_n - u_n\rangle \leq 0$, we obtain:

$$d_{n+1}^2 \leq (1 - \frac{2}{n+1})\, d_n^2 + (\frac{1}{n+1})^2 \|z_{n+1} - \bar{z}_n\|^2.$$

From: $\|z_{n+1} - \bar{z}_n\|^2 \leq 2\|z_{n+1}\|^2 + 2\|\bar{z}_n\|^2 \leq 4M^2$, one deduces:

$$d_{n+1}^2 \leq (\frac{n-1}{n+1})\, d_n^2 + (\frac{1}{n+1})^2\, 4M^2$$

and by induction :

$$d_n \leq \frac{2M}{\sqrt{n}}.$$

B) Consequence: minmax theorem.

Let A be a $I \times J$ matrix and assume that the minmax is 0 :

$$\bar{v} = \min_{y \in \Delta(J)} \max_{x \in \Delta(I)} xAy = \min_{y \in \Delta(J)} \max_{i \in I} e^i Ay = 0.$$

Proposition 2.1

Player 1 can guarantee 0, i.e. $\underline{v} \geq 0$.

Proof : Let us construct by induction a sequence $z_n \in \mathbb{R}^J$.
The first term z_1 is any row of the matrix A.
Given $z_1, z_2, ..., z_n$, define z_{n+1} as follows :
Let $\bar{z}_n^+$ be the vector with j^{th} coordinate equals to $\max(\bar{z}_n^j, 0)$.
If $\bar{z}_n = \bar{z}_n^+$, take z_{n+1} as any row of A.
Otherwise let $a > 0$ such that :

$$y_n = \frac{\bar{z}_n^+ - \bar{z}_n}{a} \in \Delta(J).$$

Since $\bar{v} = 0$, there exists $i_{n+1} \in I$ such that $e^{i_{n+1}} Ay_n \geq 0$. Define z_{n+1} as such a line i_{n+1} of the matrix A.

By construction:

$$0 \leq a\, e^{i_{n+1}} A y_n = \left\langle z_{n+1}, \bar{z}_n^+ - \bar{z}_n \right\rangle.$$

Since $\langle \bar{z}_n^+, \bar{z}_n^+ - \bar{z}_n \rangle = 0$ one gets :

$$\langle z_{n+1} - \bar{z}_n^+, \bar{z}_n - \bar{z}_n^+ \rangle \leq 0. \tag{5}$$

Let $C = \{z \in \mathbb{R}^J;\ \ z \geq 0\}$.
Note that : $\bar{z}_n^+ = \Pi_C(\bar{z}_n) = \mathscr{P}_C(\bar{z}_n)$ (where Π_C denotes the orthogonal projection on the convex closed set C) so that (5) gives (4):
$\{z_n\}$ is a Blackwell C-sequence.
Finally write $\bar{z}_n = \bar{x}_n A$.
Any accumulation point $\hat{x}$ of the sequence $\{\bar{x}_n\} \in \Delta(I)$ satisfies $\hat{x}A \in C$.
Hence $\hat{x}Ay \geq 0$, $\forall y \in \Delta(J)$, thus $\underline{v} \geq 0$. ∎

Fictitious play

Let A be a $I \times J$ real matrix.
The following process, called fictitious play, has been introduced by Brown, 1951 [8].
Consider two players playing in a repeated way the matrix game A.
At each stage $t = 1, ..., n, ...$, each player is aware of the previous action (move) of her opponent and compute the empirical distribution of the actions used in the past. Player 1 (resp. 2) plays then an action i_t (resp. j_t) which is a best reply to this average.
Explicitly, starting with any (i_1, j_1) in $I \times J$, consider at each stage n,
$x_n = \frac{1}{n} \sum_{t=1}^{n} e^{i_t}$, viewed as an element of $\Delta(I)$, and similarly
$y_n = \frac{1}{n} \sum_{t=1}^{n} e^{j_t} \in \Delta(J)$.

Definition 2.2

A sequence $(i_n, j_n)_{n\geq 1}$ with values in $I \times J$ is the realization of a fictitious play process for the matrix A if, for each $n \geq 1$, i_{n+1} is a best reply of player 1 to y_n for A:

$$i_{n+1} \in BR^1(y_n) = \{i \in I : e^i A y_n \geq e^k A y_n, \forall k \in I\}$$

and j_{n+1} is a best reply of player 2 to x_n for A ($j_{n+1} \in BR^2(x_n)$, defined in a dual way).

The main properties of this procedure are given by the next result.

Theorem 2.1 (Robinson, 1951 [36])

Let $(i_n, j_n)_{n\geq 1}$ *be the realization of a fictitious play process for the matrix* A.

1) The distance from (x_n, y_n) *to the set of optimal strategies* $X(A) \times Y(A)$ *goes to 0, as* $n \to \infty$.

Explicitely : $\forall \varepsilon > 0, \exists N, \forall n \geq N, \forall x \in \Delta(I), \forall y \in \Delta(J)$:

$$x_n A y \geq \mathtt{val}(A) - \varepsilon \text{ and } x A y_n \leq \mathtt{val}(A) + \varepsilon.$$

2) The average payoff on the trajectory, $\frac{1}{n}\sum_{t=1}^{n} A_{i_t,j_t}$, *converges to* $\mathtt{val}(A)$.

Proof :

We will prove the theorem by considering the continuous time analog.

Take as variables the empirical frequencies x_n and y_n, so that the discrete dynamics for player 1 writes :

$$x_{n+1} = \frac{1}{n+1}[i_{n+1} + nx_n] \qquad \text{with} \qquad i_{n+1} \in BR^1(y_n)$$

hence satisfies :

$$x_{n+1} - x_n \in \frac{1}{n+1}[BR^1(y_n) - x_n].$$

The corresponding system in continuous time is now :

$$\dot{x}_t \in \frac{1}{t}\left[BR^1(y_t) - x_t\right]. \tag{6}$$

This is a differential inclusion which defines, with a similar condition for player 2, the process (CFP): continuous fictitious play.

Write the payoff as $f(x,y) = xAy$ and for $(x,y) \in \Delta(I) \times \Delta(J)$, let :

$$L(y) = \max_{x' \in \Delta(I)} f(x', y) \qquad M(x) = \min_{y' \in \Delta(J)} f(x, y').$$

The duality gap at (x,y) is defined as :

$$W(x,y) = L(y) - M(x) \geq 0$$

and the pair (x,y) are optimal strategies in A if and only if $W(x,y) = 0$.

Proposition 2.2 (Harris, 1998 [17]; Hofbauer and Sorin, 2006 [20])

For the (CFP) process, the duality gap converges to 0 at a speed $O(1/t)$.

Proof : Make the time change $z_t = x_{e^t}$ in (6) which leads to the autonomous differential inclusion:

$$\dot{x}_t \in \left[BR^1(y_t) - x_t\right], \quad \dot{y}_t \in \left[BR^2(x_t) - y_t\right]. \tag{7}$$

known as the best reply dynamics (BR), Gilboa and Matsui, 1991 [15]).
Let now $(x_t, y_t)_{t \geq 0}$ be a solution of (BR), see Aubin and Cellina,1984 [1].
Denote by $w_t = W(x_t, y_t)$ the evaluation of the duality gap on the trajectory, and write $\alpha_t = x_t + \dot{x}_t \in BR^1(y_t)$ and $\beta_t = y_t + \dot{y}_t \in BR^2(x_t)$.
One has $L(y_t) = f(\alpha_t, y_t)$, thus:

$$\frac{d}{dt} L(y_t) = \dot{\alpha}_t D_1 f(\alpha_t, y_t) + \dot{y}_t D_2 f(\alpha_t, y_t).$$

The envelope's theorem (see e.g., Bonnans and Shapiro (2000) [5]) shows that the first term collapses and the second term is $f(\alpha_t, \dot{y}_t)$ (since f is linear w.r.t. the second variable). Then we obtain :

$$\begin{aligned}
\dot{w}(t) &= \frac{d}{dt}L(y_t) - \frac{d}{dt}M(x_t) = f(\alpha_t, \dot{y}_t) - f(\dot{x}_t, \beta_t) \\
&= f(x_t, \dot{y}_t) - f(\dot{x}_t, y_t) = f(x_t, \beta_t) - f(\alpha_t, y_t) \\
&= M(x_t) - L(y_t) = -w(t)
\end{aligned}$$

thus :

$$w_t = w(0)\, e^{-t}.$$

There is convergence of w_t to 0 at exponential speed, hence convergence to 0 at a speed $O(1/t)$ in the original problem before the time change.The convergence to 0 of the duality gap implies by uniform continuity the convergence of (x_t, y_t) to the set of optimal strategies $X(A) \times Y(A)$.

■

The result is actually stronger: the set $X(A) \times Y(A)$ is a global attractor for the best reply dynamics, Hofbauer and Sorin, 2006 [20], which implies the convergence of the discrete time version, hence of the fictitious play process, i.e. part $1)$ of proposition 2.1.

For an alternative proof in the same spirit, see Lehrer and Sorin, 2007 [26].

We finally prove part $2)$ of proposition 2.1, Rivière, 1994 [35], Monderer and Sela, 1996 [30].

Proof :

Let us consider the sum of the realized payoffs : $R_n = \sum_{p=1}^{n} f(i_p, j_p)$.
Writing : $U_m^i = \sum_{k=1}^{m} f(i, j_k)$, one obtains :

$$R_n = \sum_{p=1}^{n} (U_p^{i_p} - U_{p-1}^{i_p}) = \sum_{p=1}^{n} U_p^{i_p} - \sum_{p=1}^{n-1} U_p^{i_{p+1}} = U_n^{i_n} + \sum_{p=1}^{n-1} (U_p^{i_p} - U_p^{i_{p+1}})$$

but the fictitious play property implies that :

$$U_p^{i_p} - U_p^{i_{p+1}} \leq 0.$$

Hence $\limsup_{n\to\infty} \frac{R_n}{n} \leq \limsup_{n\to\infty} \max_i \frac{U_n^i}{n} \leq \mathtt{val}(A)$, since $\frac{U_n^i}{n} = f(i, y_n) \leq \mathtt{val}(A) + \varepsilon$ for n large enough by part 1) of proposition 2.1.

The dual inequality thus implies the result. ∎

Note that Part 1 and Part 2 of the theorem are independent.

In general convergence of the average marginal trajectories on moves does not imply any property of the average payoff on the trajectory. For example in the "matching pennies" game :

1	-1
-1	1

convergence in average of both strategies to $(1/2, 1/2)$ is compatible with a sequence of payoffs 1 or -1.

Proposition 2.3 (Loomis, 1946 [27])

Let A and B be two $I \times J$ real matrices, with $B \gg 0$. Then there exist (x, y, v) in $\Delta(I) \times \Delta(J) \times \mathbb{R}$ such that :

$$xA \geq v\, xB \text{ and } Ay \leq v\, By.$$

With $B_{ij} = 1$ for all $(i,j) \in I \times J$, one recovers von Neumann's theorem.

Proof : The proof is obtained by induction on the dimension $|I| + |J| = m + n$. The result is clear for $m = n = 1$.
1) Assume the result is true for $m + n - 1$.
Let $\lambda_0 = \max\{\lambda \in \mathbb{R}, \exists s \in \Delta(I), sA \geq \lambda\, sB\}$ and $\mu_0 = \min\{\mu \in \mathbb{R}, \exists t \in \Delta(J), At \leq \mu Bt\}$ so that $\lambda_0 \leq \mu_0$.
If $\lambda_0 = \mu_0$, the result holds, hence assume that $\lambda_0 < \mu_0$.
Let s_0 and t_0 be optimal, and note that $s_0 A = \lambda_0\, s_0 B$ and $At_0 = \mu_0\, Bt_0$ cannot both hold.
Assume then that $\bar{j} \in J$ is such that $s_0 A e^{\bar{j}} > \lambda_0\, s_0 B e^{\bar{j}}$ and let $J' = J \setminus \{\bar{j}\}$.

Using the induction hypothesis, introduce $v' \in \mathbb{R}$ and $s' \in \Delta(I)$ associated to the $I \times J'$ submatrices A' of A and B' of B, with $s'A' \geq v' \, s'B'$.
Now $v' \geq \mu_0 > \lambda_0$ since there are less constraints for player 1 in the reduced game, then consider $s = \alpha s' + (1-\alpha)s_0$ with $\alpha \in (0,1)$ and note that for α small enough:

$$(\alpha s' + (1-\alpha)s_0)(A - \lambda_0 B)e^j > 0,$$

for $\bar{j}$, thanks to s_0 and for $j \in J'$, because of s' : a contradiction to the definition of λ_0. ∎

Application: Let B be a square matrix with positive entries. Take $A = Id$, then there exists an eigenvector associated to a positive eigenvalue with positive components (Perron–Frobenius).

Equilibria and variational inequalities

Equilibrium: existence

The main result is the following:

Theorem 3.1 (Nash, 1951 [34], Glicksberg, 1952 [16], Fan, 1952 [13])

1) If S^i is a compact convex subset of a topological vector space, g^i is continuous, quasi concave w.r.t. s^i, for all $i \in I$, the set of equilibria is compact and non empty.
2) If S^i is compact , g^i is continuous, for all $i \in I$, the mixed extension of the game has an equilibrium.

Proof : 1) By continuity and compactness, for each profile $t \in S$, the set $E(t)$ of profiles not eliminated by t, (recall that a profile t eliminates a profile s if there exists a player $i \in I$ with $g^i(t^i, s^{-i}) > g^i(s)$), is a compact subset of S.
By the intersection property, to prove the existence of an equilibrium, it is enough to show that for any family $\{t(k) \in S; k \in K \text{ finite}\}$ the intersection $\cap_{k \in K} E(t(k))$ is not empty.

We are then in a finite dimensional framework (replace each S^i by $\text{co}(\{t^i(k), k \in K\})$) and an equilibrium of the reduced game will be in $\cap_{k\in K} E(t(k))$.
Now g^i quasi-concave w.r.t. s^i implies that for all s, $BR(s)$ is convex. By continuity and compactness, $BR(s)$ is compact and non-empty for each s. The joint continuity hypothesis implies that the graph of the correspondence BR is closed.
Then use a fixed point theorem, Fan, 1952 [13], for the correspondence BR on S.
The corresponding fixed point is an equilibrium.
2) If S^i is compact, $\Sigma^i = \Delta(S^i)$ is convex and compact (for the weak$*$ topology).
Similarly if g^i is continuous on S, its extension to $\Sigma = \prod_{j\in I} \Sigma^j$ is continuous (again for the weak$*$ topology), using for example the Stone-Weierstrass theorem to get the joint continuity, and multilinear. Then use Part 1. ∎

For more results, see e.g. MSZ I.4, LRS Ch. 4.

Equilibrium: finite case

We consider here the case of a finite game: finitely many players, each player $i \in I$ having finitely many strategies in S^i.
The finiteness assumption allows for a more precise analysis of equilibria.

Lemma 3.1

σ is a mixed equilibrium iff for all i and all $s^i \in S^i$:

$$g^i(s^i, \sigma^{-i}) < \max_{t^i \in S^i} g^i(t^i, \sigma^{-i}) \Rightarrow \sigma^i(s^i) = 0.$$

Proof :
Follows from $g^i(\sigma^i, \sigma^{-i}) = \max_{t^i \in S^i} g^i(t^i, \sigma^{-i})$ and $g^i(\sigma^i, \sigma^{-i}) = \sum_{t^i \in S^i} \sigma^i(t^i) g^i(t^i, \sigma^{-i})$.

∎

In other words, the support of σ^i is included in $BR^i(\sigma^{-i})$, for all $i \in I$.

Theorem 3.2 (Nash, 1950 [33])

Every finite game G has a mixed equilibrium.

Proof :

Define the Nash map f from Σ to Σ by:

$$f(\sigma)^i(s^i) = \frac{\sigma^i(s^i) + (g^i(s^i, \sigma^{-i}) - g^i(\sigma))^+}{1 + \sum_{t^i \in S^i} (g^i(t^i, \sigma^{-i}) - g^i(\sigma))^+}$$

where $a^+ = max(a, 0)$.
f is well defined and with values in Σ: $f(\sigma)^i(s^i) \geq 0$ and $\sum_{s^i \in S^i} f(\sigma)^i(s^i) = 1$.
Since f is continuous and Σ convex, compact, Brouwer's fixed point theorem, Brouwer, 1910 [6], implies the existence of $\sigma \in \Sigma$ with $f(\sigma) = \sigma$.
Let us prove that such σ is an equilibrium.

Otherwise there exists $i \in I$ and $u^i \in S^i$ with $g^i(u^i, \sigma^{-i}) - g^i(\sigma) > 0$ hence $\sum_{t^i \in S^i} (g^i(t^i, \sigma^{-i}) - g^i(\sigma))^+ > 0$.
Since there exists s^i with $\sigma^i(s^i) > 0$ and $g^i(s^i, \sigma^{-i}) \leq g^i(\sigma)$ one obtains:

$$\sigma^i(s^i) = f(\sigma)^i(s^i) = \frac{\sigma^i(s^i)}{1 + \sum_{t^i \in S^i} (g^i(t^i, \sigma^{-i}) - g^i(\sigma))^+} < \sigma^i(s^i)$$

a contradiction.
Note that reciprocally any equilibrium is a fixed point of f since all quantities $(g^i(t^i, \sigma^{-i}) - g^i(\sigma))^+$ vanish. ∎

Algebraic approach

Recall that each S^i is finite, with cardinal m^i. Let $m = \Pi_i m^i$.
A game can thus be identified with a point in $\mathbb{R}^{Nm}$. For example, with 2 players having each 2 strategies one obtains $g \in \mathbb{R}^8$ specified by:

	L	R
T	(a_1, a_2)	(a_3, a_4)
B	(a_5, a_6)	(a_7, a_8)

Proposition 3.1

The set of equilibria is defined by a finite family of large polynomial inequalities.

Proof : σ is an equilibrium iff:

$$\sum_{s^i \in S^i} \sigma^i(s^i) - 1 = 0, \quad \sigma^i(s^i) \geq 0, \quad \forall s^i \in S^i, \forall i \in I,$$

$$g^i(\sigma) = \sum_{s=(s^1,\ldots,s^N)\in S} [\prod_i \sigma^i(s^i)] g^i(s) \geq g^i(t^i, \sigma^{-i}), \forall t^i \in S^i, \forall i \in I,$$

the unknown being the family $\{\sigma^i(s^i)\}$.
We used the linearity to make the comparison only to extreme points.

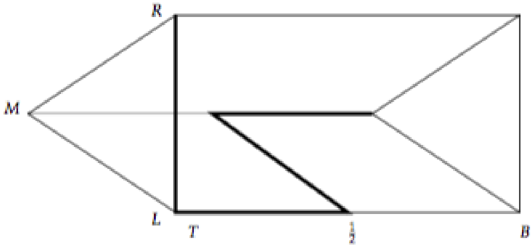
Corollary 3.1

The set of equilibria is semi-algebraic.
It is a finite union of closed connected components.

Example 1:

	L	M	R
T	$(2,1)$	$(1,0)$	$(1,1)$
B	$(2,0)$	$(1,1)$	$(0,0)$

The set of equilibria is described by the thick line below.

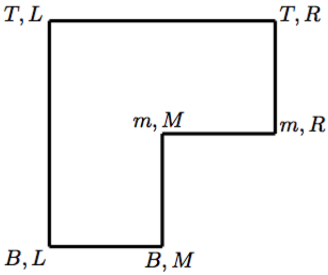


There is one connected component, homeomorphic to a segment.

Example 2 : Kohlberg and Mertens, 1986 [23].

	L	M	R
T	$(1,1)$	$(0,-1)$	$(-1,1)$
m	$(-1,0)$	$(0,0)$	$(-1,0)$
B	$(1,-1)$	$(0,-1)$	$(-2,-2)$

There is only one connected component of equilibria which is of the form:



hence homeomorphic to a circle in Σ.

In addition each point of the circle is the limit of a sequence of equilibria of close-by games, like the next one, with $\varepsilon > 0$:

	L	M	R
T	$(1, 1-\varepsilon)$	$(\varepsilon, -1)$	$(-1-\varepsilon, 1)$
m	$(-1, -\varepsilon)$	$(-\varepsilon, \varepsilon)$	$(-1+\varepsilon, -\varepsilon)$
B	$(1-\varepsilon, -1)$	$(0, -1)$	$(-2, -2)$

with equilibrium $[(\varepsilon/(1+\varepsilon), 1/(1+\varepsilon), 0); (0, 1/2, 1/2)]$ close to $[(0, 1, 0); (0, 1/2, 1/2)]$.

Equilibria and variational inequalities

For various classes of games, equilibria can be represented as solutions of variational inequalities, see e.g. Sorin and Wan, 2016 [42].

Finite games

Define the vector payoff function $Vg^i : \Sigma^{-i} \longrightarrow \mathbb{R}^{S^i}$ by $Vg^i(\sigma^{-i})^u = g^i(u, \sigma^{-i}), u \in S^i$. Hence $g^i(\sigma) = \langle Vg^i(\sigma^{-i}), \sigma^i \rangle$ and $\sigma \in \Sigma$ is a Nash equilibrium iff:

$$\langle Vg^i(\sigma^{-i}), \sigma^i - \tau^i \rangle \geq 0, \qquad \forall \tau^i \in \Sigma^i, \qquad \forall i \in I.$$

Concave games

I is a finite set of players, for each $i \in I$, $X^i \subset H^i$(Hilbert) is the convex set of actions of player i and $G^i : X = \prod_j X^j \longrightarrow \mathbb{R}$ his payoff function.
Assume G^i concave and $\mathscr{C}^1$ w.r.t. x^i.
Then $x \in X$ is a Nash equilibrium iff:

$$\langle \nabla_i G^i(x), x^i - y^i \rangle_{H^i} \geq 0, \qquad \forall y^i \in X^i, \qquad \forall i \in I.$$

where ∇_i stands for the gradient of G^i w.r.t. x^i.

Population games

I is a finite set of populations of non atomic players; for each $i \in I$, S^i is the finite set of actions of population i and $X^i = \Delta(S^i)$ is the simplex over S^i. x^{iu} is the proportion of players in population i that play $u \in S^i$. Given $K^i : X \longrightarrow \mathbb{R}^{S^i}$, $K^{iu}(x)$ is the payoff of a member of population i using the action $u \in S^i$ given the configuration x.

Then $x \in X$ is an equilibrium, Wardrop (1952) [48], iff:

$$x^{iu} > 0 \Rightarrow K^{iu}(x) \geq K^{iv}(x), \quad \forall u, v \in S^i, \forall i \in I. \tag{8}$$

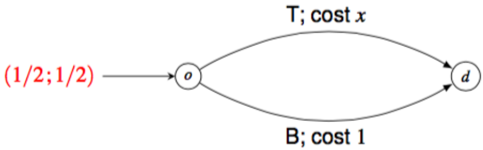
which is:

$$\boxed{\langle K^i(x), x^i - y^i \rangle = \sum_{u \in S^i} K^{iu}(x)(x^{iu} - y^{iu}) \geq 0, \qquad \forall y^i \in X^i, \qquad \forall i \in I.}$$

or

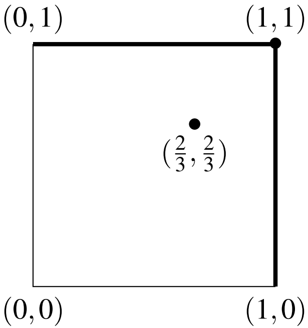
$$\langle K(x), x - y \rangle = \sum_i \langle K^i(x), x^i - y^i \rangle \geq 0, \qquad \forall y \in X$$

A typical example is congestion games: i corresponds to the type of the agent and u to the link in a network.
Consider the following Pigou's example.
Two roads, T and B link the origin o to the destination d. On T the cost is x if the congestion is x. On B there is a constant cost of 1.
Consider two populations of size $1/2$ each.



Since the agents are nonatomic they will all choose T if $x < 1$, hence the only equilibrium is $(s^1, s^2) = (1, 1)$, where s^i is the proportion of T for i, thus inducing a cost of 1.

Note that in the case of two players controlling each a mass $1/2$ the only equlibrium is $(2/3,2/3)$.
Finally if the mass is not splittable, the players use mixed strategies and the set of equilibria is of the form: one player uses T and the other is indifferent between T and B.



Remark that the social optimum is obtained for $s^i = 1/2, i = 1,2$.

General case

Consider a finite collection of convex compact sets $X^i \subset H^i$ (Hilbert), and evaluation mappings $\phi^i : X = \prod_j X^j \to H^i$, $i \in I$.

Definition 3.1

$NE(\phi)$ is the set of $x \in X$ satisfying:

$$\langle \phi(x), x - y \rangle \geq 0, \qquad \forall y \in X \tag{9}$$

where $\langle \phi(x), x - y \rangle = \sum_i \langle \phi^i(x), x^i - y^i \rangle_{H^i}$.

Remark that all the previous sets of equilibria can be written this way. We denote by $\Gamma(\phi)$ a game with evaluation ϕ.

Let Π_C denotes the projection from H to the closed convex set C and $\mathbf{T}$ the map from X to itself defined by:

$$\mathbf{T}(x) = \Pi_X[x + \phi(x)]$$

Proposition 3.2

$NE(\phi)$ is the set of fixed points of $\mathbf{T}$.

Proof : The characterization of the projection gives:

$$\langle x + \phi(x) - \Pi_X[x + \phi(x)], y - \Pi_X[x + \phi(x)]\rangle \leq 0, \qquad \forall y \in X,$$

hence $\Pi_C[x + \phi(x)] = x$ is the solution iff $x \in NE(\phi)$. ∎

Corollary 3.2

Assume ϕ continuous on X. Then $NE(\phi) \neq \emptyset$.

Proof : The map $x \mapsto \Pi_C[x + \phi(x)]$ is continuous from the convex compact set X to itself, hence a fixed point exists. ∎

Lemma 3.2

An alternative characterization of $NE(\phi)$ *is the set of solutions of :*

$$\Pi_{TX(\hat{x})}(\phi(\hat{x})) = 0 \qquad (10)$$

where $TC(x)$ *is the tangent cône to* C*, closed and convex, at* $x \in C$*.*

Proof :
Recall that:

$$\Pi_{TX(x)}(y) = \lim_{h \to 0} \frac{\Pi_X(x + h\,y) - x}{h}.$$

■

These results are standard in the theory of Variational Inequalities, Kinderlehrer and Stampacchia,1980 [22], Facchinei and Pang, 2007 [12], and used in Operations Research areas, see e.g. Dafermos, 1980 [10], Dupuis and Nagurney, 1993 [11], Nagurney and Zhang,1996 [32].

Specific classes

Supermodular games

Endow the euclidean space $\mathbb{R}^n$, with the product (partial) order $x \geq y$ iff $x_i \geq y_i$ for all i.
$S \subset \mathbb{R}^n$ is a lattice if for all $x, y \in S$: $\sup\{x, y\} \in S$ and $\inf\{x, y\} \in S$.
Recall the version of the fixed point theorem in this framework.

Theorem 3.3 (Tarski, 1955 [43])

Let $S \subset \mathbb{R}^n$ be a non empty compact lattice and f an increasing function from S to itself.
Then f has a fixed point.

Consider a strategic game G, where for each $i \in I$, S^i is a non-empty compact subset of $\mathbb{R}^{m_i}$ and g^i is upper semi continuous in s^i for each fixed s^{-i}.

Assume moreover that the game is supermodular, i.e. :

(i) For all i, S^i is a lattice in $\mathbb{R}^{m_i}$.

(ii) g^i has increasing differences in (s^i, s^{-i}):

$$g^i(s^i, s^{-i}) - g^i(s'^i, s^{-i}) \geq g^i(s^i, s'^{-i}) - g^i(s'^i, s'^{-i})$$

as soon as $s^i \geq s'^i$ and $s^{-i} \geq s'^{-i}$.

(iii) g^i is supermodular w.r.t. s^i: $\forall s^{-i} \in S^{-i}$,

$$g^i(s^i, s^{-i}) + g^i(s'^i, s^{-i}) \leq g^i(s^i \vee s'^i, s^{-i}) + g^i(s^i \wedge s'^i, s^{-i}).$$

Proposition 3.3 (Topkis, 1979 [45])

Under the previous hypotheses, the game G has an equilibrium.

Proof :
For each i and s^{-i}, $BR^i(s^{-i})$ is a non-empty compact lattice of $\mathbb{R}^{m_i}$.
If $s^{-i} \geq s'^{-i}$, $\forall t'^i \in BR^i(s'^{-i})$, $\exists t^i \in BR^i(s^{-i})$ such that $t^i \geq t'^i$.
Apply Tarski's theorem to the maximal element of the best reply map.

■

Potential games

A) *Finite case*

A real function P defined on S is a potential, Monderer and Shapley, 1996 [31], for the finite game (g,S) if:

$$g^i(s^i,u^{-i}) - g^i(t^i,u^{-i}) = P(s^i,u^{-i}) - P(t^i,u^{-i}), \forall s^i,t^i \in S^i, u^{-i} \in S^{-i}, \forall i \in I. \tag{11}$$

This means that the impact due to a change of action of player i is the same on g^i and on P, for all $i \in I$. In particular one can use P to check the equilibrium condition.

B) *Evaluation functions*

A real function W, of class $\mathscr{C}^1$ on a neighborhood Ω of X, is a potential for the game with evaluation ϕ if for each $i \in I$, there is a strictly positive function $\mu^i(x)$ defined on X such that:

$$\left\langle \nabla_i W(x) - \mu^i(x)\phi^i(x), y^i - x^i \right\rangle = 0, \quad \forall x,y \in X, \forall i \in I, \tag{12}$$

where ∇_i is the gradient w.r.t. x^i.

Theorem 3.4

Let $\Gamma(\phi)$ be a game with potential W.
1. Every local maximum of W is an equilibrium of $\Gamma(\phi)$.
2. If W is concave on X, then any equilibrium of $\Gamma(\phi)$ is a global maximum of W on X.

Proof : The condition implied by a local maximum is :

$$\langle \nabla W(x), x - y \rangle \geq 0, \quad \forall y \in X$$

hence in particular :

$$\langle \nabla_i W(x), x^i - y^i \rangle \geq 0, \quad \forall y^i \in X^i$$

so that :

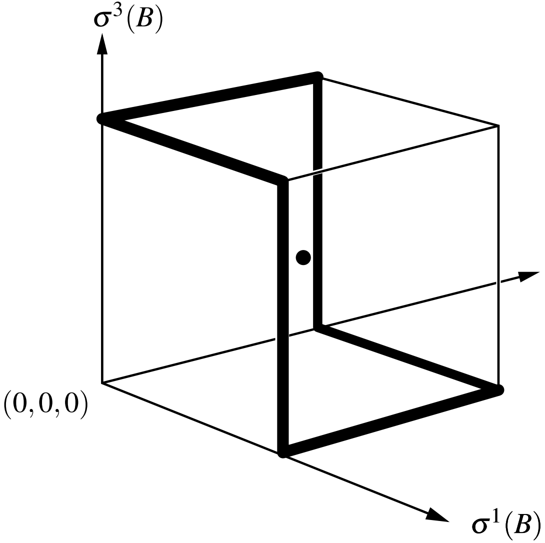
$$\langle \nabla_i \phi^i(x), x^i - y^i \rangle \geq 0, \quad \forall y^i \in X^i.$$

The inverse statement is clear. ∎

Minority game

Consider a three player game where each of the players has to choose one of two rooms A,B. The payoff is one for a player being alone (if any) and 0 otherwise.

Pure equilibria are of the form (A,B,B) and mixed equilibria of the form $(A,B,?)$. In addition there is a symmetric one where each player uses $(1/2,1/2)$. The set of equilibria is homeomorphic to a circle plus an isolated point.



This is a potential game.

Dissipative games

A game $\Gamma(\phi)$ is dissipative, if ϕ is dissipative i.e. satisfies:

$$\langle \phi(x) - \phi(y), x - y \rangle \leq 0, \qquad \forall\, (x,y) \in X \times X.$$

Alternatively, $-\phi$ is "monotone"; recall that the gradient of a $\mathscr{C}^1$ convex function is monotone.

This notion appears for strategic games in Rosen, 1965 [38]. Hofbauer and Sandholm, 2009 [19] use the terminology “stable games” in the framework of population games.

A basic example corresponds to two-person zero-sum games.

Proposition 3.4 (Rockafellar, 1970 [37])

Assume $f : X = X^1 \times X^2 \to \mathbb{R}$, $\mathscr{C}^1$ *and concave/convex.*
Then $\phi = (\nabla_1 f, -\nabla_2 f)$ *is dissipative.*

Proof :

$$f(y^1,x^2) - f(x^1,x^2) \leq \langle \nabla_1 f(x^1,x^2), y^1 - x^1 \rangle$$
$$f(x^1,y^2) - f(y^1,y^2) \leq \langle \nabla_1 f(y^1,y^2), x^1 - y^1 \rangle$$
$$f(x^1,y^2) - f(x^1,x^2) \geq \langle \nabla_2 f(x^1,x^2), y^2 - x^2 \rangle$$
$$f(y^1,x^2) - f(y^1,y^2) \geq \langle \nabla_2 f(y^1,y^2), x^2 - y^2 \rangle$$

so that :

$$\langle \nabla_1 f(x^1,x^2) - \nabla_1 f(y^1,y^2), x^1 - y^1 \rangle + \langle -\nabla_2 f(x^1,x^2) + \nabla_2 f(y^1,y^2), x^2 - y^2 \rangle \leq 0$$

which is

$$\langle \phi(x) - \phi(y), x - y \rangle \leq 0.$$

■

Definition 3.2

Let $SE(\phi)$ be the set of $x \in X$ satisfying:

$$\langle \phi(y), x - y \rangle \geq 0, \qquad \forall y \in X.$$

Note that $SE(\phi)$ is convex but may be empty.

Proposition 3.5 (Minty, 1967 [29])

Assume ϕ dissipative and X convex, compact.
Then $SE(\phi)$ is non-empty.

Proof : Let :

$$S_y = \{x \in X; \langle \phi(y), x - y \rangle \geq 0, \}$$

so that $SE(\phi) = \cap_{y \in X} S_y$ hence by compactness it is enough to establish the following:

Claim

For any finite collection $y_i \in X, i \in I$, there exists $x \in \mathrm{co}\{y_i, i \in I\}$ such that:

$$\langle \phi(y_i), x - y_i \rangle \geq 0, \qquad \forall i \in I. \tag{13}$$

Consider the finite two-person zero-sum game defined by the following $I \times I$ matrix A :

$$A_{ij} = \langle \phi(y_j), y_i - y_j \rangle.$$

Introduce the decomposition : $A = B + C$ with $B = \frac{1}{2}[A +^t A]$ and $C = \frac{1}{2}[A -^t A]$.
The crucial point is that B has non negative coefficients since:

$$2B_{ij} = \langle \phi(y_j), y_i - y_j \rangle + \langle \phi(y_i), y_j - y_i \rangle = \langle \phi(y_j) - \phi(y_i), y_i - y_j \rangle \geq 0.$$

Hence an optimal strategy $u \in \Delta(I)$ in the game C (antisymmetric hence with value 0) gives $uAe^j = uBe^j + uCe^j \geq 0, \forall j \in I$, i.e. :

$$\sum_{i \in I} u_i \langle \phi(y_j), y_i - y_j \rangle \geq 0, \quad \forall j \in J.$$

Letting $x = \sum_i u_i y_i$ this writes as (13).

■

Proposition 3.6 (Minty, 1967 [29])

If $\Gamma(\phi)$ is dissipative and ϕ is continuous:

$$SE(\phi) = NE(\phi).$$

Proof :

One direction is clear and does not use continuity.
If ϕ is dissipative and x is an equilibrium, then:

$$\langle \phi(y), x - y \rangle \geq \langle \phi(x), x - y \rangle \geq 0, \quad \forall y \in X.$$

On the other hand, given $z \in X$, $x \in SE(\phi)$ and $t \in (0, 1]$, let $y = x + t(z - x)$, hence:

$$\langle \phi(x + t(z - x)), t(x - z) \rangle \geq 0.$$

Dividing by t and then letting t go to 0 gives, by continuity of ϕ, the result. ∎

- Aubin J.-P. and A. Cellina (1984) *Differential inclusions*, Springer.
- Aumann R.J. and M. Maschler (1972) Some thoughts on the minmax principle, *Management Science*, **18**, 53-63.
- Berge C. (1966) *Espaces Topologiques, Fonctions Multivoques*, Dunod.
- Blackwell D. (1956) An analog of the minmax theorem for vector payoffs, *Pacific Journal of Mathematics*, **6**, 1-8.
- Bonnans J.-F. and A. Shapiro (2000) *Perturbation Analysis of Optimization Problems*, Springer.
- Brouwer L. E. J. (1910) Uber Abbildung von Mannigfaltikeiten, *Mathematische Annalen*, **71**, 97-115.
- Brown G.W. (1949) Some notes on computation of games solutions, *RAND Report*, P-78, The RAND Corporation.
- Brown G.W. (1951) Iterative solutions of games by fictitious play, in *Activity Analysis of Production and Allocation*, T.C. Koopmans (ed.), Wiley, 374-376.

Brown, G.W. and J. von Neumann (1950) Solutions of games by differential equations, in *Contributions to the Theory of Games, I*, H.W. Kuhn and A.W. Tucker (eds.), Annals of Mathematical Studies, 24, Princeton University Press, 73-79.

Dafermos S. C. (1980) Traffic equilibrium and variational inequalities, *Transportation Sci.*, **14**, 42–54.

Dupuis P. and A. Nagurney (1993) Dynamical systems and variational inequalities, *Ann. Oper. Res.*, **44**, 9–42.

Facchinei F. and J. Pang (2007) *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Springer.

Fan K. (1952) Fixed-points and minmax theorems in locally convex topological linear spaces, *Proceedings of the National Academy of Sciences of the U.S.A*, **38**, 121-126.

Gale D., H. Kuhn and A.W. Tucker (1950) On symmetric games, in *Contributions to the Theory of Games*, I, Annals of Mathematical Studies, **24**, H.W. Kuhn and A.W. Tucker (eds.), Princeton University Press, 81-87.

- Gilboa I. and A. Matsui (1991) Social stability and equilibrium, *Econometrica*, **58**, 859-67.
- Glicksberg I. (1952) A further generalization of the Kakutani fixed point theorem, with applications to Nash equilibrium points, *Proceedings of the American Mathematical Society*, **3**, 170–174.
- Harris C. (1998) On the rate of convergence of continuous time fictitious play, *Games and Economic Behavior*, **22**, 238-259.
- Hofbauer J. (2018) Minmax via replicator dynamics, *Dynamic Games and Applications*, **8**, 637-640.
- Hofbauer J. and W.H. Sandholm (2009) Stable games and their dynamics, *J. Econom. Theory*, **144**, 1665–1693.
- Hofbauer J. and S. Sorin (2006) Best response dynamics for continuous zero-sum games, *Discrete and Continuous Dynamical Systems-series B*, **6**, 215-224.
- Kakutani S. (1941) A generalization of Brouwer's fixed point theorem, *Duke Mathematical Journal*, **8**, 416–427.

- Kinderlehrer D. and G. Stampacchia (1980) *An Introduction to Variational Inequalities and their Applications*, Academic Press.
- Kohlberg E. and J.-F. Mertens (1986) On the strategic stability of equilibria, *Econometrica*, **54**, 1003-37.
- Laraki R., J. Renault and S. Sorin (2019) *Mathematical Foundations of Game Theory*, Springer.
- Lehrer E. and S. Sorin (2001) Approachability and applications, preprint.
- Lehrer E. and S. Sorin (2007) Minmax via differential inclusion, *Journal of Convex Analysis*, **14**, 271-274.
- Loomis L. H. (1946) On a theorem of von Neumann, *Proceeding of the National Academy of Sciences of the U.S.A*, **32**, 213-215.
- Mertens J.-F., S. Sorin and S. Zamir (2015) *Repeated Games*, Cambridge University Press.
- Minty G. J. (1967) On the generalization of a direct method of the calculus of variations, *Bulletin A.M.S.*, **73**, 315-321.

Monderer D. and A. Sela (1996) A 2x2 game without the fictitious play, *Games and Economic Behavior*, **14**, 144-148.

Monderer D. and L.S. Shapley (1996) Potential games, *Games and Economic Behavior*, **14**, 124-143.

Nagurney A. and D. Zhang (1996) *Projected Dynamical Systems and Variational Inequalities with Applications* , Kluwer.

Nash J. (1950) Equilibrium points in n-person games, *Proceedings of the National Academy of Sciences*, **36**, 48–49.

Nash J. (1951) Non-cooperative games, *Annals of Mathematics*, **54**, 286-295.

Rivière P. (1994) *Quelques Modèles de Jeux d' Evolution*, Thèse, Université Paris 6, 1997.

Robinson J. (1951) An iterative method of solving a game, *Annals of Mathematics*, **54**, 296-301.

Rockafellar R.T. (1970) Monotone operators associated with saddle-functions and minmax problems, *Nonlinear Functional*

Analysis, F. Browder, ed., Proceedings of Symposia in Pure Math, **18**, AMS, 241-250.

- Rosen J.B. (1965) Existence and uniqueness of equilibrium points for concave N-person games, *Econometrica*, **33**, 520-534.
- Sandholm W.H. (2001) Potential games with continuous player sets, *J. Econom. Theory*, **97**, 81–108.
- Shapley L.S. (1964) Some topics in two-person games, in *Advances in Game Theory*, M. Dresher, L.S. Shapley and A. W. Tucker (eds.), Annals of Mathematical Studies, **52**, Princeton University Press, 1-28.
- Sion M. (1958) On general minimax theorems, *Pacific Journal of Mathematics*, **8**, 171–176.
- Sorin S. and C. Wan (2016) Finite composite games: equilibria and dynamics, *Journal of Dynamics and Games*, **3**, 101-120.
- Tarski A.(1955) A lattice theoretical fixed point theorem and its applications, *Pacific Journal of Mathematics*, **5**, 285-308.

- Taylor P.D. and L.B. Jonker (1978) Evolutionary stable strategies and game dynamics, *Math. Biosci.*, **40**, 145–156.
- Topkis D. (1979) Equilibrium points in non zero-sum n person submodular games, *SIAM Journal of Control and Optimization*, **17**, 773-787.
- Ville J. (1938) Sur la théorie générale des jeux où intervient l'habileté des joueurs, in E. Borel, *Traité du Calcul des Probabilités et de ses Applications*, Tome IV, Gauthier-Villars, 105-113.
- Von Neumann J. (1928) Zur Theorie der Gesellschaftsspiele, *Mathematische Annalen*, **100**, 295–320.
- Wardrop G. (1952) Some theoretical aspects of road traffic research, *Proc. Inst. Civ. Eng.*, **1**, 325-362.
- Weyl H. (1950) Elementary proof of a minimax theorem due to von Neumann, in *Contributions to the Theory of Games, I*, H. W. Kuhn and A. W. Tucker (eds.), Annals of Mathematical Studies, **24**, Princeton University Press, 19-25.