Independence number of random trees

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École Polytechnique

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Définition

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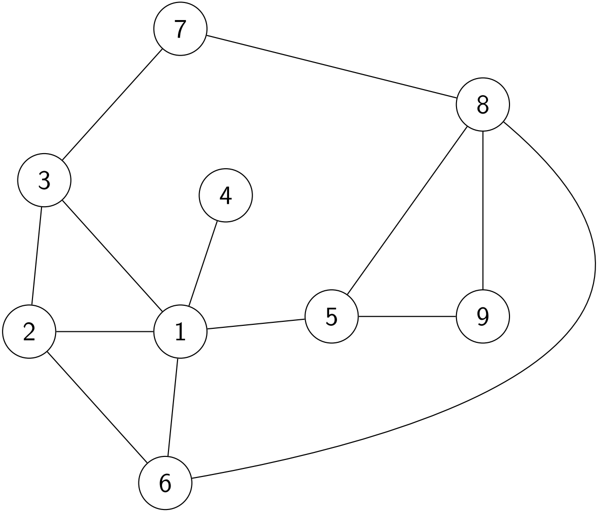
- A maximal independent set if it is an independent set with maximal cardinality.

Exemple



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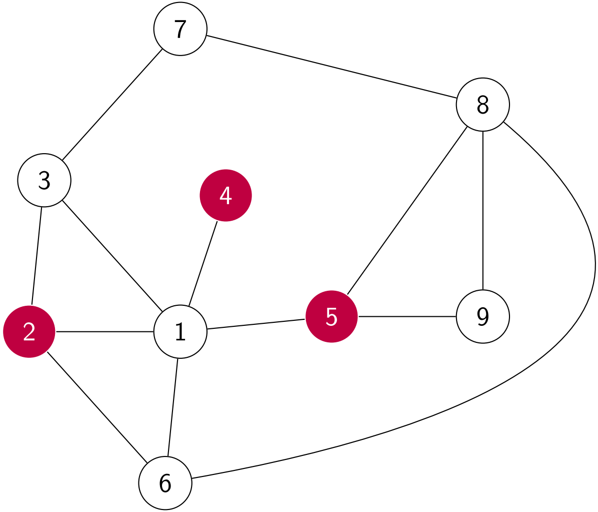

$S = \{2, 4, 5\}$ is an independent set .

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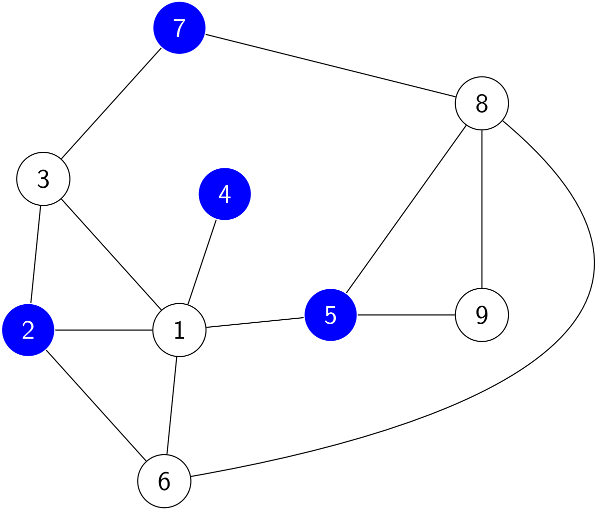
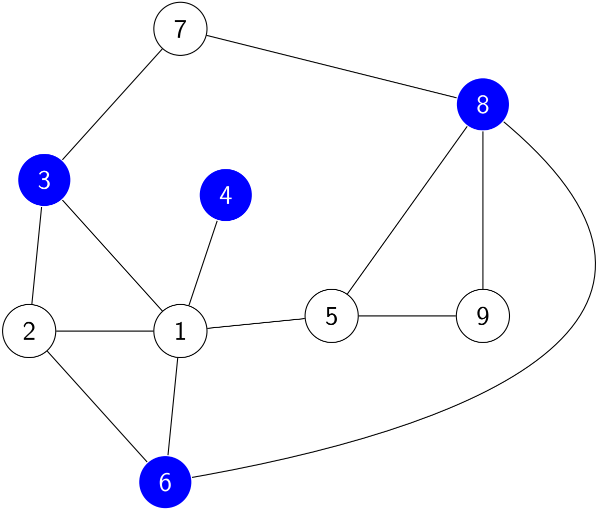


$S = \{2, 4, 5, 7\}$ is a maximal independent set .

Exemple



$S = \{3, 4, 6, 8\}$ is also a maximal independent set .

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Remark : The maximal independent set is not unique.

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The size of a maximal independent set of G is called the independence number of G and is denoted by $\alpha(G)$.

Complexité

Théorème

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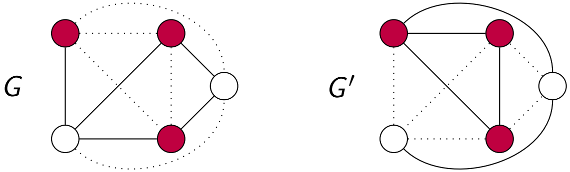
Proof :

- NP : OK
- NP-hard : Take $G' = (V, E')$ where E' is the complementary of E. Then G has an independent set of size $\geq k$ is equivalent to say that G' has a clique of size $\geq k$. And the clique problem is well known to be NP-hard. □



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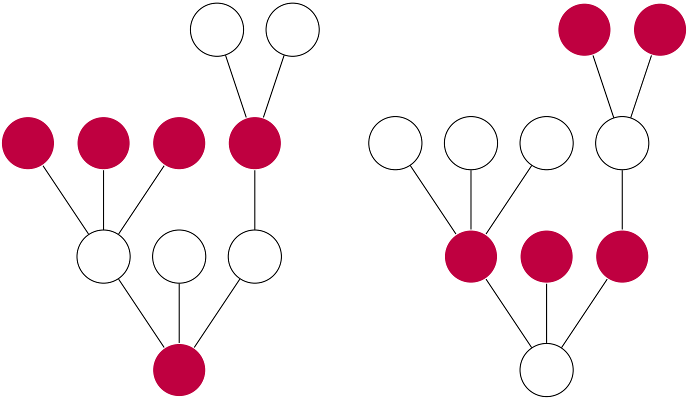

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- More generally, if G is a bipartite graph with n vertices, then $\alpha(G) \geq n/2$.

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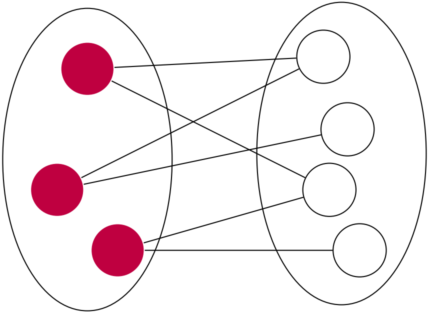

Tricoloration

Définition

Let T be a tree and v a vertex of T, we colour v in :

- green if it belongs to **no** maximal independent set
- red if it belongs to **all** maximal independent set
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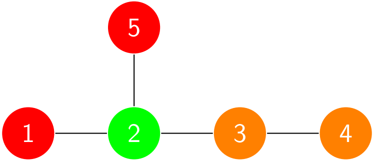

Tricoloration

$n_g(T)$:= number of green vertices.
$n_r(T)$:= number of red vertices.
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Proposition

- $\alpha(T) = n_r(T) + \frac{n_o(T)}{2}$.

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Proposition

- $\alpha(T) = n_r(T) + \frac{n_o(T)}{2}$.
- $\beta(T) = n_g(T) + \frac{n_o(T)}{2} = n - \alpha(T)$.

Arbres de BGW

Définition

Let μ be a distribution on $\mathbb{N}$. A Bienaymé-Galton-Watson (BGW) tree with reproduction law μ is a tree, starting from a single root, where each vertex breeds in a i.i.d manner with law μ.

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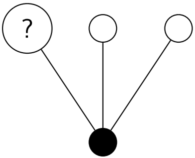

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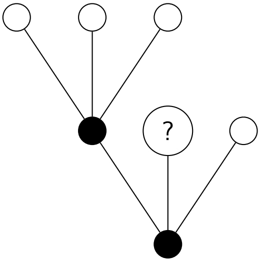

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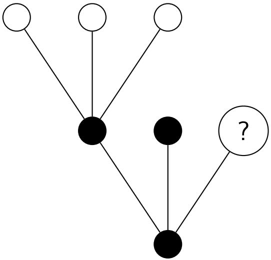

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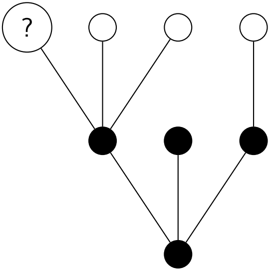

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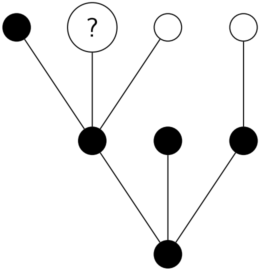

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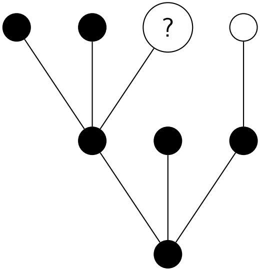

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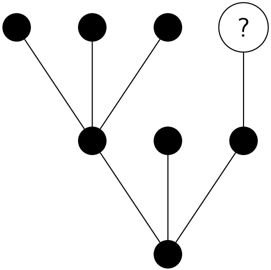

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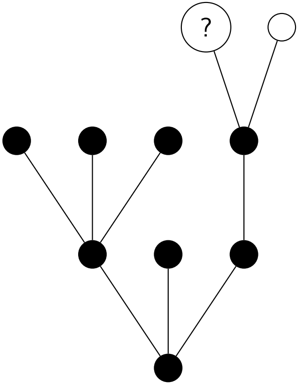

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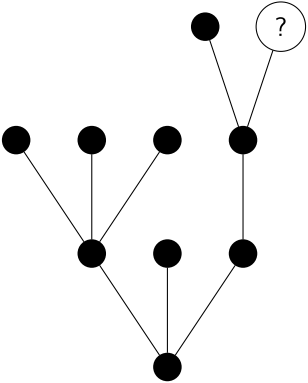

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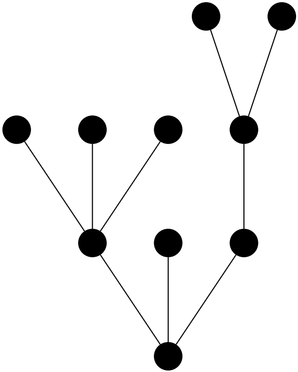


Un dessin

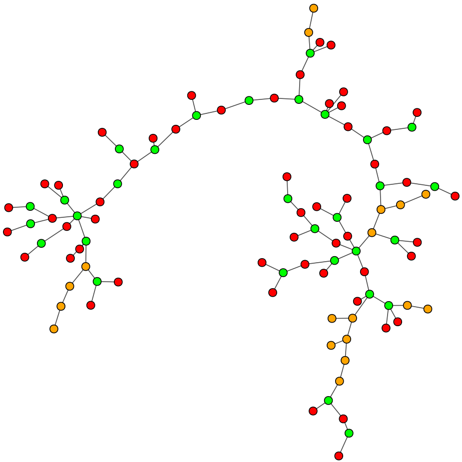


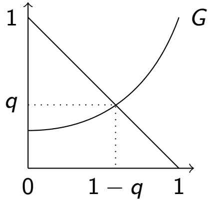
Figure: Tricolouration of a BGW tree with 100 vertices and a Poisson offspring distribution of parameter 1.

Résultat

Let T_n be a BGW tree with reproduction law μ, conditionned on having n vertices.Let

$$G(t) := \sum_{k=0}^{\infty} \mu(k) t^k.$$

Let q be the unique solution of $G(1-q) = q$ in $[0,1]$. Suppose that μ has mean 1.



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Théorème (B.)

The following convergences hold in L^p for every $p > 0$:

$$\frac{n_r(T_n)}{n} \xrightarrow[n\to\infty]{L^p} \frac{q}{1+G'(1-q)}, \qquad \frac{n_o(T_n)}{n} \xrightarrow[n\to\infty]{L^p} \frac{2\,q\,G'(1-q)}{1+G'(1-q)},$$

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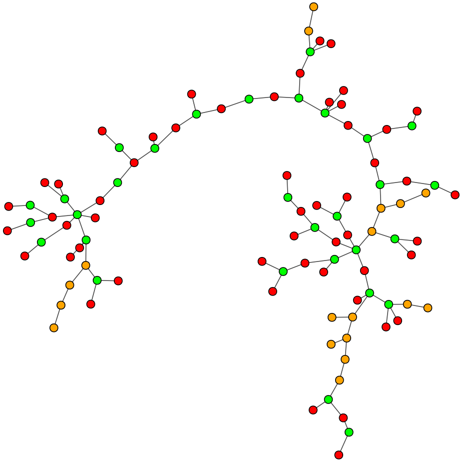
Fin

Merci de votre attention



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Théorème (Caro & Wei, Alon & Spencer)

For any graph $G = (V, E)$ we have the lower bound :

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{\deg(v) + 1}.$$

Proof : Let $\preceq$ be a total ordering of V chosen uniformly at random. Set

$$I := \{v \in V \ : \forall w \in V, \{v, w\} \in E \implies v \preceq w\}.$$

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Thus we can find a total order $\preceq$ such that $\#I \geq \sum_{v \in V} \frac{1}{\deg(v)+1}$. To conclude, notice that I is an independent set □

Bornes supérieures

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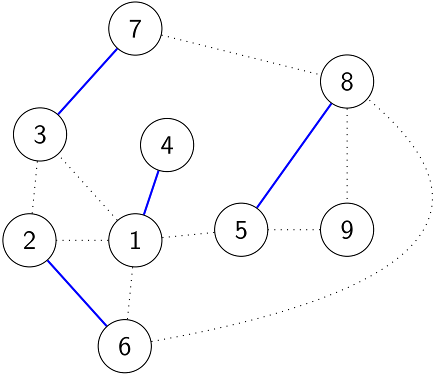

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Denote by $\beta(G)$ the size of a maximal matching of G.

Théorème (Konig, Egervary)

For any graph $G = (V, E)$ with n vertices, we have the upper bound :

$$\alpha(G) \leq n - \beta(G)$$

with equality if G is bipartite.

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Remark : $\beta(G)$ can be computed in $O(n^3)$ (Edmond, Gabow).

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Proof that $\mathbb{E}\left[\frac{n_r(T_n)}{n}\right] \to \frac{q}{1+G'(1-q)}$:

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We say that (τ_n, u_n) converges locally towards (τ, u) if for every $h \in \mathbb{N}$ there is a n_0 such that for all $n \geq n_0$,

$$[(\tau_n, u_n)]_h = [(\tau, u)]_h.$$

This convergence defines a topology called the local topology.

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Théorème (Stufler)

(T_n, v_n) *converges in distribution for the local topology towards a random pointed tree* (T^*, u_0).

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$$\mathbb{P}\left(u_0 \text{ is red in } T^*\right) = \ ?$$

Arbre limite

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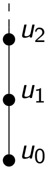

Start with an infinite spine of vertices $u_0, u_1, u_2, \ldots$

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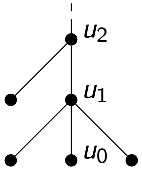

Each vertex u_i with $i > 0$, gets offspring according to the law

$$\hat{\mu}(k) := k\mu(k).$$

And u_{i-1} is identified to one of u_i's children uniformly at random.

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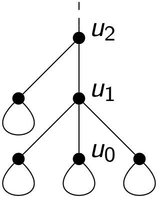

Finally, each vertex of the current tree gives birth to a independent copy of a BGW tree with law μ.

For $T_1, \ldots, T_k$ rooted trees we denote by $T_1 * \cdots * T_k$ the tree obtained by linking all the roots of $T_1, \ldots, T_k$ to a new vertex.

Un lemme

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Lemme

The root of $T_1 * \cdots * T_k$ is red iff all the roots of $T_1, \ldots, T_k$ are non-red.

Fin de la preuve

Denote by q the probability that the root of a BGW tree with law μ is red. From the previous lemma we deduce that

$$q = \sum_{k \geq 0} \mu(k)(1-q)^k = G(1-q).$$

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Let $\widetilde{T}$ be the tree obtained from T^* by cutting the edge between u_0 and u_1 and keeping the component containing u_1. Let $\widetilde{q}$ be the probability that u_1 is red in $\widetilde{T}$. Then, from the lemma again,

$$\widetilde{q} = \sum_{k \geq 1} k\mu(k)(1-q)^{k-1}(1-\widetilde{q}) = (1-\widetilde{q})G'(1-q).$$

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Denote by q the probability that the root of a BGW tree with law μ is red. From the previous lemma we deduce that

$$q = \sum_{k \geq 0} \mu(k)(1-q)^k = G(1-q).$$

Let $\widetilde{T}$ be the tree obtained from T^* by cutting the edge between u_0 and u_1 and keeping the component containing u_1. Let $\widetilde{q}$ be the probability that u_1 is red in $\widetilde{T}$. Then, from the lemma again,

$$\widetilde{q} = \sum_{k \geq 1} k\mu(k)(1-q)^{k-1}(1-\widetilde{q}) = (1-\widetilde{q})G'(1-q).$$

Finally,

$$\mathbb{P}\left(u_0 \text{ is red in } T^*\right) = \sum_{k \geq 0} \mu(k)(1-q)^k(1-\widetilde{q}) = \frac{q}{1+G'(1-q)}.$$

Fin

Merci de votre attention