Postdocs in Paris

Special Day

Artin groups & (Δ-ed) categories

09 - Nov - 2022

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Artin-Tits groups (Generalised braid groups, ~1960s)

↳ groups defined by (labelled) graphs

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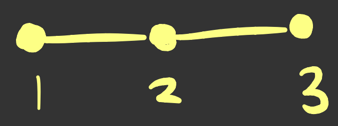
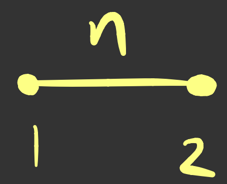
E.g.: ① $A_3 :=$ 

$$\mathbb{B}(A_3) = \left\langle s_1, s_2, s_3 \;\middle|\; \begin{array}{l} s_1 s_3 = s_3 s_1, \\ s_1 s_2 s_1 = s_2 s_1 s_2, \quad s_2 s_3 s_2 = s_3 s_2 s_3 \end{array} \right\rangle$$

② $I_2(n) :=$ 

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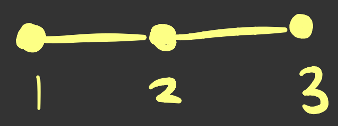
E.g.: ① $A_3 :=$ 

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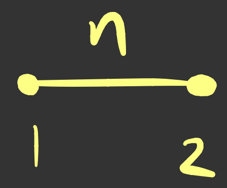
② $I_2(n) :=$ 

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Q. ① Word problem?

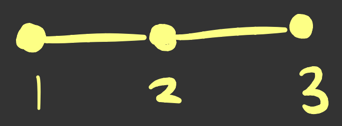
② Torsion, centre?

③ Dynamics?

④ Cohomology?

Coxeter groups

↳ groups defined by (labelled) graphs

E.g.: ① $A_3 :=$ 

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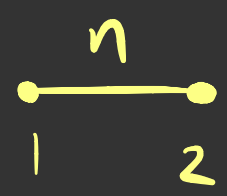
② $I_2(n) :=$ 

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S_4

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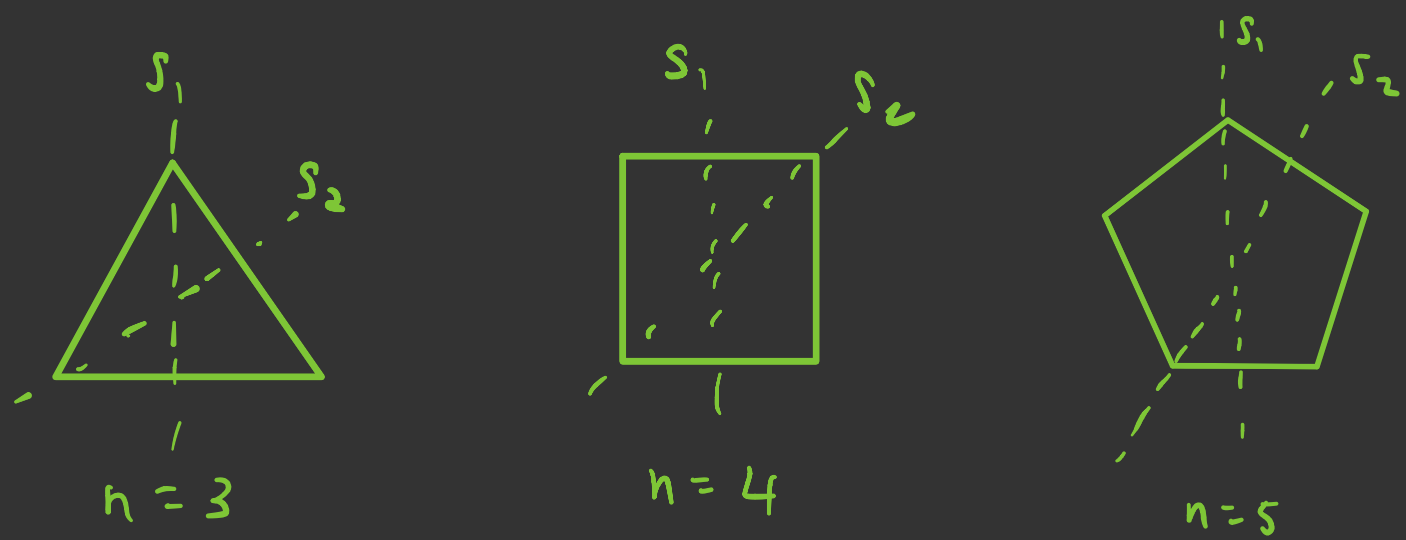

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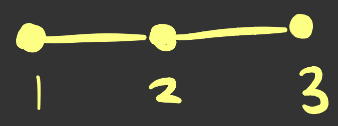
E.g.: ① $A_3 :=$ 

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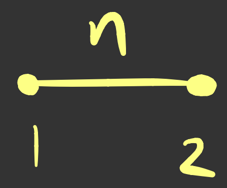
② $I_2(n) :=$ 

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③ Finite groups

A: 

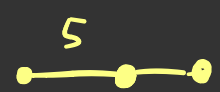
B = C: 

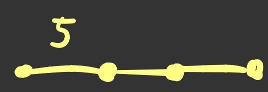
D: 

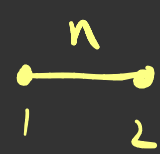
F: 

E: exceptionals

G_2: 

H_3: 

H_4: 

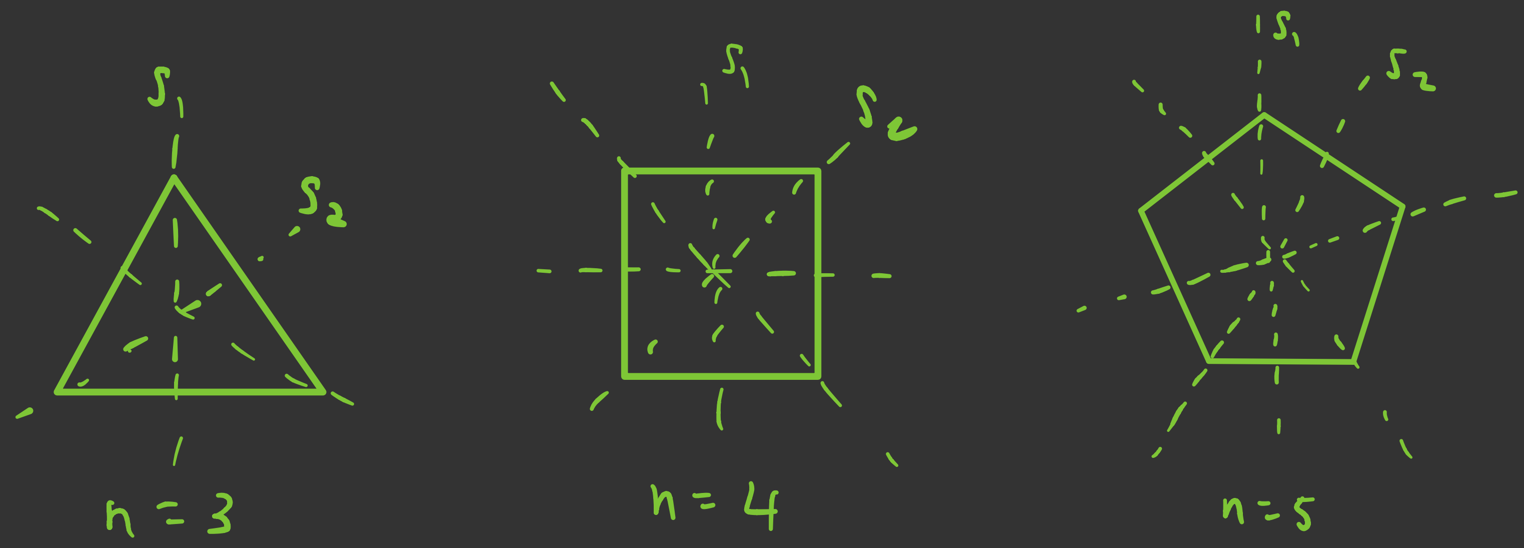
$I_2(n)$: 

Coxeter groups

Thm. [Tits]

All Coxeter groups are linear.

$W(\Gamma) \curvearrowright V_{\mathbb{R}}$



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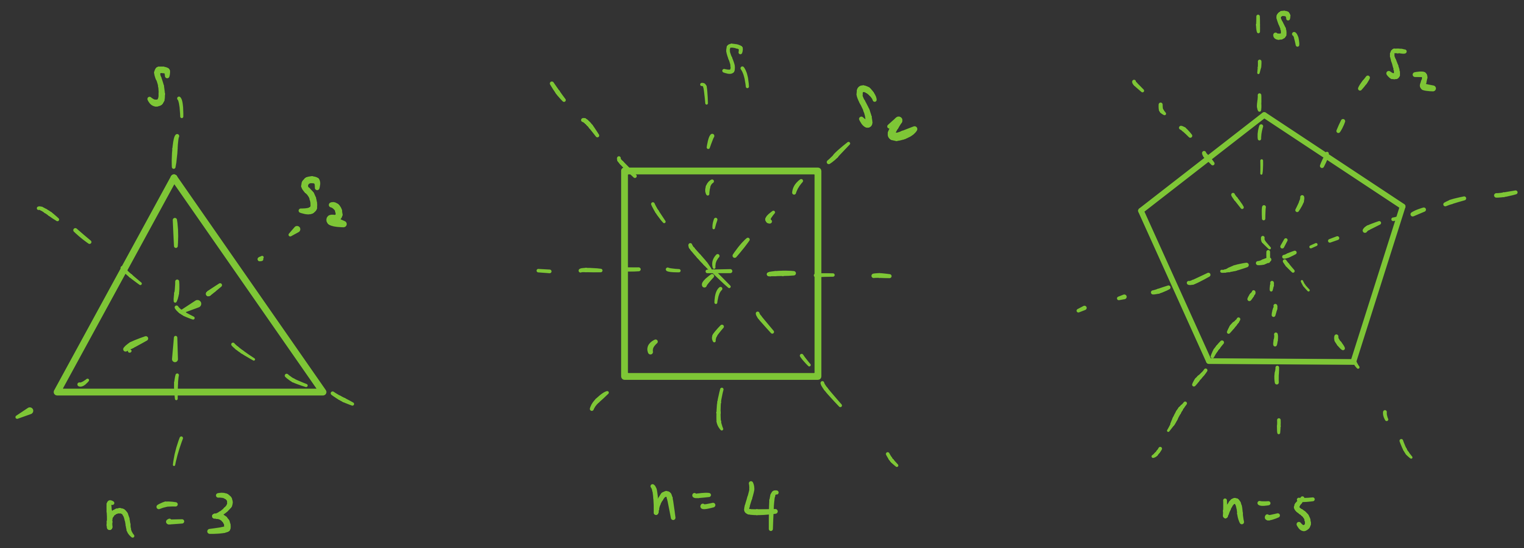

Thm. [van der Lek 1983]

$$W(\Gamma) \curvearrowright X := (V_{\mathbb{R}} + iV_{\mathbb{R}}) \setminus \bigcup_{\alpha} H_\alpha + iH_\alpha$$

* Technically $V_{\mathbb{R}}$ should be the Tits cone.

is free and properly discontinuous, and

$$\boxed{\pi_1(X / W(\Gamma)) \cong \mathbb{B}(\Gamma)}$$

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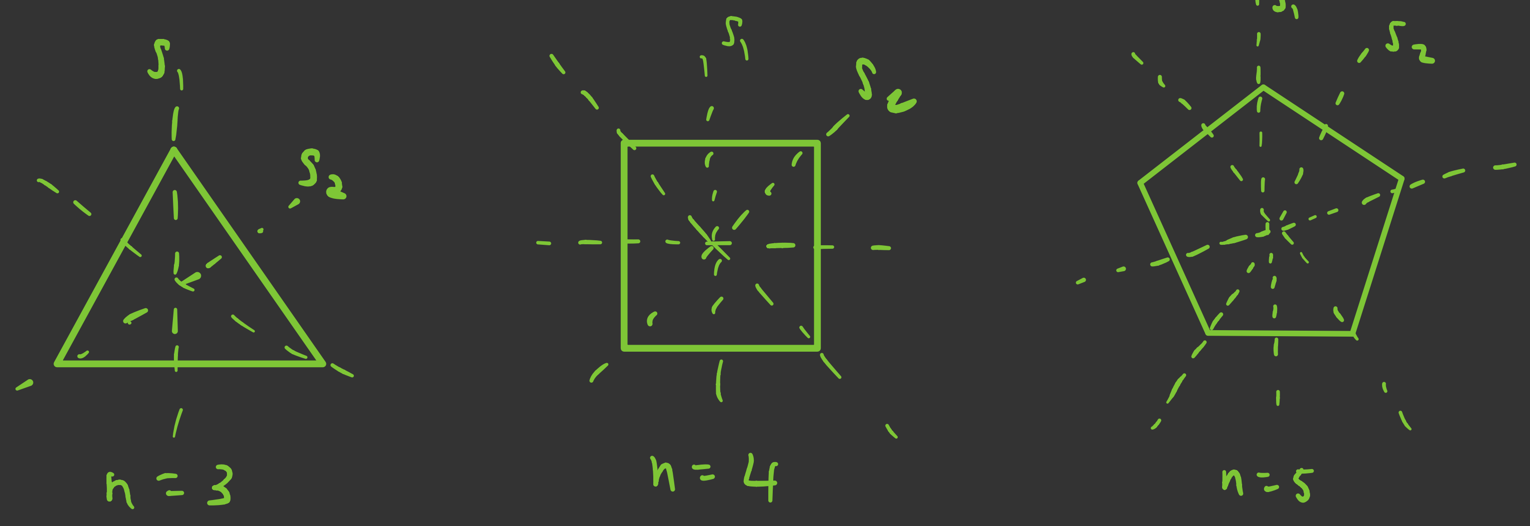

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The $K(\pi,1)$ conjecture:

$X/W(\Gamma)$ is a $K(\pi,1)$ space for $\mathbb{B}(\Gamma)$.

↳ ① finite type [Deligne, Brieskorn-Saito 1972]

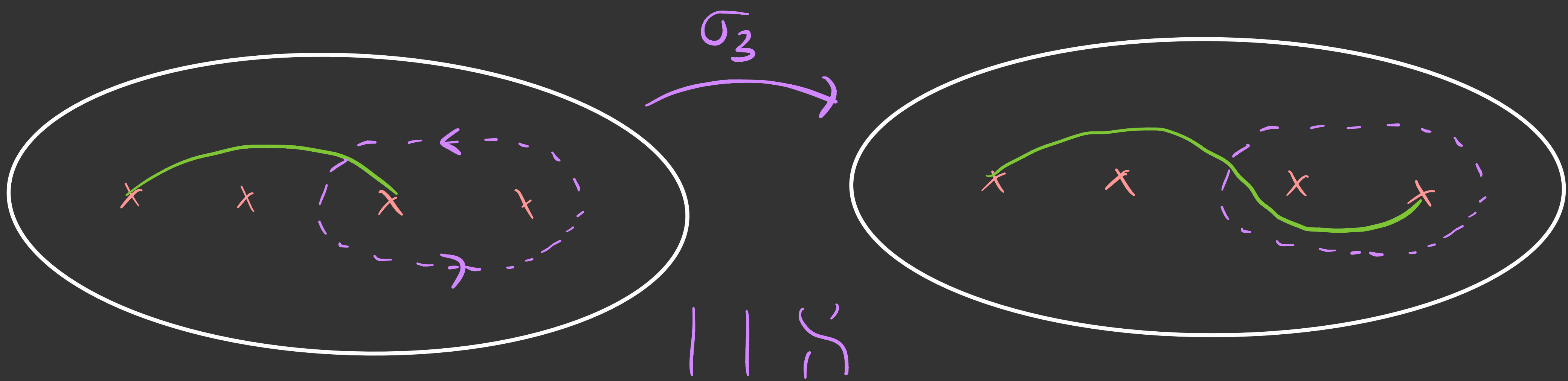
② FC type [Charney-Davis 1995]

③ Affine type [Paolini-Salvetti 2021]

☆ Many remain unknown!

In type A (arguably the most well-understood type)

$$\mathbb{B}(A_{n-1}) \cong Br_n \cong MCG(D_n)$$



TEICHMULLER THEORY

Thm. (Nielsen - Thurston classification)

Every $\varphi \in MCG(D_n)$ is either:

(i) periodic

(ii) reducible

(iii) pseudo-Anosov

Thm. [Kaliman '75]

$Teich(\mathbb{D}_{n+1}) \times \mathbb{C}^2 \twoheadrightarrow Conf_n(\mathbb{C})$

is a universal cover

Cor. [Fox - Neuwirth '62]

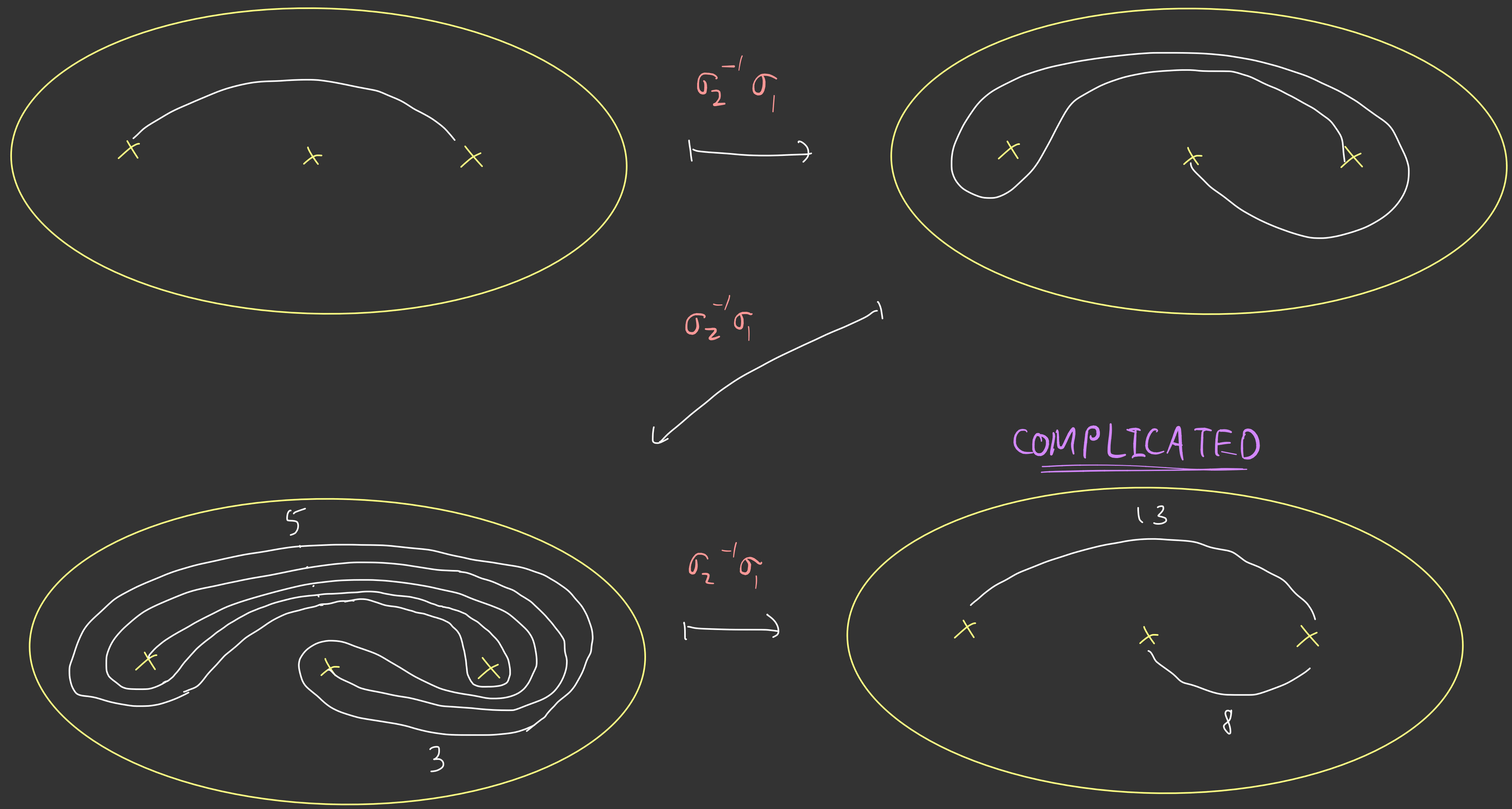
$Conf_n(\mathbb{C})$ is a $K(\pi,1)$-space for Br_n.

Dynamics

Thm. (Nielsen-Thurston Classification)

Let S be a compact orientable surface. Every mapping class element $\varphi \in MCG(S)$ is either:

(1) periodic (2) reducible (3) pseudo-Anosov. large "complexity"



Action of Artin groups on surfaces

PLAN: search for "nice" action of $\mathbb{B}(T)$ on some surface.

Thm. (Tits conjecture) [Crisp - Paris '01]
↳ proved using actions on surfaces.

However!!!

Thm. [Wajnryb '99]
The type E Artin group has no faithful action on surfaces.

General Analogy

[Bridgeland-Smith '14, Haiden-Katzarkov-Kontsevich '17]

"Theory of stability conditions is Teichmuller theory for triangulated categories."

Δ-ed categories, D	Surfaces, S
$Aut(D)$	$MCG(S)$
objects, C	curves, c
$Hom(C_1, C_2)$	intersection numbers
stabilty conditions	(flat) metric
mass	length

Triangulated category & surface correspondence

[Khovanov-Seidel '01 (A), Gadbled-Thiel-Wagner '15 (Ext. $\hat{A}$), H.-Nge '19 (B)]

$\mathbb{B}(\Gamma) \curvearrowright \mathcal{D} = \mathrm{Kom}^b(\mathcal{A}\text{-mod})$ $\qquad$ $\mathbb{B}(\Gamma) \cong \mathrm{MCG}(\mathbb{D}) \curvearrowright \mathbb{D}$

- objects, $\mathcal{L}(c)$ $\xleftarrow{\ \mathcal{L}\ }$ - simple (closed) curves, c
- $\mathrm{Hom}(\mathcal{L}(c_1), \mathcal{L}(c_2))$ $\xrightarrow[\text{dimension}]{}$ - intersection numbers

"decategorify"

$\mathbb{B}(A_n) \curvearrowright K_0(\mathcal{D})$

(first) homology

Burau representation

$$\begin{array}{ccc} \mathbb{B}(\Gamma) & \dashrightarrow & GL_m(\mathbb{Z}[q, q^{-1}]) \\ \downarrow & & \downarrow {\scriptstyle q=-1} \\ W(\Gamma) & \longrightarrow & GL_m(\mathbb{Z}) \end{array}$$

Other types

$\mathbb{B}(\Gamma) \curvearrowright \mathcal{D} = \mathrm{Kom}^b(\mathcal{T}\text{-mod})$

- objects, C $\xleftarrow{?}$
- $\mathrm{Hom}(C_1, C_2)$ $\xrightarrow{?}$

BLACK BOX

?

"decategorify"

$\mathbb{B}(\Gamma) \curvearrowright K_0(\mathcal{D})$

(?)

Burau representation

- $$\begin{tikzcd} \mathbb{B}(\Gamma) \arrow[r, dashed] \arrow[d] & GL_m(\mathbb{R}[q, q^{-1}]) \arrow[d, "q=-1"] \\ W(\Gamma) \arrow[r] & GL_m(\mathbb{R}) \end{tikzcd}$$

Other types

Thm. [Huerfano - Khovanov '02]

For each simply-laced Γ, there exist a (f.d. quotient) quiver algebra A_Γ such that $\mathbb{B}(\Gamma)$ acts on $\mathrm{Kom}^b(A\text{-prgrmod})$, categorifying Burau rep.

Thm. [H. '22]

Extend the above to inc. non-simply laced Γ by constructing "quiver" algebra (monoid) in fusion categories : $\overline{\mathrm{Rep}}(U_q sl_2)$, q = root of unity

Thm. [Brav - Thomas '10, Licata - Queffelec '17 (ADE)] [H. '22 (other finite)]

The action above is faithful.

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☆ Faithful even for type E !!

$\mathbb{B}(T) \curvearrowright$

Δ-category $\mathcal{D}_T$

[Dmitrov-Haiden-Katzarkov-Kontsevich '17]

Categorical Dynamics

Thm. [H. '22]

Every element $\sigma \in \mathbb{B}(I_2(n))$ is either

① periodic ② reducible ③ pseudo-Anosov

defined using categorical entropy (function)

Q. Type E ?

$\mathbb{B}(\Gamma) \curvearrowright$

Δ-category $\mathcal{D}_\Gamma$

[Dmitrov-Haiden-Katzarkov-Kontsevich '17]

Categorical Dynamics

$K(\pi, 1)$

(simply-laced)

[Bridgeland '09 + Ikeda '14]

$\mathrm{Stab}(\mathcal{D}_{\tilde{\Gamma}})$

covering

$X_{\tilde{\Gamma}}$

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[Dimitrov-Haiden-Katzarkov-Kontsevich '17]

Categorical Dynamics

(non-simply laced) $\rightsquigarrow$ $\mathrm{Stab}_e(\mathcal{D}_\Gamma)$

$K(\pi,1)$

(simply-laced) [Bridgeland '09 + Ikeda '14] $\rightsquigarrow$ $\mathrm{Stab}(\mathcal{D}_{\tilde{\Gamma}})$

$\mathrm{Stab}_e(\mathcal{D}_\Gamma) \subseteq_{\text{submfld.}} \mathrm{Stab}(\mathcal{D}_{\tilde{\Gamma}})$

[H.-Licata ~] $\mathrm{Stab}_e(\mathcal{D}_\Gamma) \xrightarrow{\text{covering}} X_\Gamma$

$\mathrm{Stab}(\mathcal{D}_{\tilde{\Gamma}}) \xrightarrow{\text{covering}} X_{\tilde{\Gamma}}$

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Note: Contractible $\Rightarrow K(\pi,1)$!

Q. Type E?

THANK YOU !

$\mathcal{T} = I_2(5)$

The golden ratio δ