Mean-Field Games

First lecture: Formulation, Solvability

François Delarue (Nice – J.-A. Dieudonné)

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Based on joint works with R. Carmona, P. Cardaliaguet, A. Cecchin, D. Crisan, J.F. Chassagneux, R. Foguen, D. Lacker, J.M. Lasry, P.L. Lions, K. Ramaman

Part I. Motivation

Part I. Motivation

a. General philosophy

Basic purpose

- Interacting particles / players
  - controlled players in mean-field interaction
  - particles have dynamical states $\rightsquigarrow$ stochastic diff. equation
  - mean-field $\rightsquigarrow$ symmetric interaction with whole population
no privileged interaction based on the labels
- Associate cost functional with each player
  - find $\boxed{\text{equilibria}}$ w.r.t. cost functionals
  - shape of the equilibria for a large population?

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- Associate cost functional with each player
  - ∘ find $\boxed{\text{equilibria}}$ w.r.t. cost functionals
  - ∘ shape of the equilibria for a large population?
- Different notions of equilibria

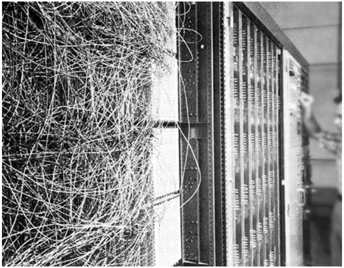
∘ players decide on their own $\rightsquigarrow$ find a compromise inside the population $\Rightarrow$ notion of $\boxed{\text{Nash equilibrium}}$

∘ players obey a common center of decision $\rightsquigarrow$ minimize the global cost to the collectivity $\Rightarrow$ notion of $\boxed{\text{Social optimizer}}$

- Both cases $\rightsquigarrow$ asymptotic equilibria as the number of players $\uparrow \infty$?

Typical examples...



Asymptotic formulation

- Paradigm

  ◦ mean-field / symmetry $\rightsquigarrow$ $\boxed{\text{propagation of chaos / LLN}}$

  ◦ reduce the asymptotic analysis to one typical player with interaction with a theoretical distribution of the population?

  ◦ decrease the complexity to solve asymptotic formulation first

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- Program
  - Existence of $\boxed{\text{asymptotic equilibria}}$? Uniqueness? Shape?
  - Use asymptotic equilibria as quasi-equilibria in finite-game
  - Prove convergence of equilibria in finite-player-systems

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- Asymptotic formulation of Nash equilibria $\rightsquigarrow$ $\boxed{\text{Mean-field games!}}$ [Lasry-Lions (06), Huang-Caines-Malhamé (06), Bertucci, Cardaliaguet, Achdou, Gangbo, Gomes, Porreta (PDE), Bensoussan, Carmona, Cecchin, D., Djete, Lacker (Probability)]
- Common center of decision $\rightsquigarrow$ optimal control of McKean-Vlasov SDEs or Fokker-Planck PDEs $\rightsquigarrow$ $\boxed{\text{Mean-field control!}}$

Part I. Motivation

b. Nash equilibria within a finite system

General formulation

- Controlled system of N interacting particles with mean-field interaction through the global state of the population

  - dynamics of particle number $i \in \{1, \dots, N\}$

$$\underbrace{dX_t^i}_{\in \mathbb{R}^d} = b(X_t^i, \text{global state of the collectivity}, \alpha_t^i)dt$$

$$+ \sigma(X_t^i, \text{global state}) \quad \underbrace{dW_t^i}_{\text{idiosyncratic noises}}$$

$$+ \sigma^0(X_t^i, \text{global state}) \quad \underbrace{dB_t}_{\text{common/systemic noise}}$$

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- Rough description of the probabilistic set-up

  - $(B_t, W_t^1, \dots, W_t^N)_{0 \le t \le T}$ independent B.M. with values in $\mathbb{R}^d$

  - $(\alpha_t^i)_{0 \le t \le T}$ progressively-measurable processes with values in A (closed convex $\subset \mathbb{R}^k$)

  - i.i.d. initial conditions $\perp\!\!\!\perp$ noises

Empirical measure

- Code the state of the population at time t through $\boxed{\bar{\mu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}}$

$\rightsquigarrow$ probability measure on $\mathbb{R}^d$

  - $\mathcal{P}_2(\mathbb{R}^d) \rightsquigarrow$ set of probabilities on $\mathbb{R}^d$ with finite 2nd moments

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- Express the coefficients as $\begin{array}{l} b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \to \mathbb{R}^d, \\ \sigma, \sigma^0 : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\times d}, \end{array}$

  - examples: $b(x,\mu,\alpha) = b(x, \int_{\mathbb{R}^d} \varphi d\mu, \alpha), \quad \int_{\mathbb{R}^d} b(x,v,\alpha) d\mu(v)$
  - rewrite the dynamics of the particles

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- $\boxed{\text{Cost functional}}$ to player $i \in \{1, \dots, N\}$

$$J^i(\alpha^1, \alpha^2, \dots, \alpha^N) = \mathbb{E}\Big[g(X_T^i, \bar{\mu}_T^N) + \int_0^T f(X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt\Big]$$

  - same (f, g) for all i but J^i depends on the others through $\bar{\mu}^N$

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- $\boxed{\text{Cost functional}}$ to player $i \in \{1, \dots, N\}$ (an example)

$$J^i = \mathbb{E}\Big[\frac{1}{N}\sum_{j=1}^N g(X_T^i - X_T^j) + \int_0^T \Big(\frac{1}{N}\sum_{j=1}^N f(X_t^i - X_t^j) + |\alpha_t^i|^2\Big)dt\Big]$$

  - cost reads potential energy plus kinetic energy

Nash equilibrium

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- Say that a N-tuple of strategies $(\alpha^{1,\star}, \dots, \alpha^{N,\star})$ is a compromise if
  - no interest for any player to leave the compromise
  - change $\alpha^{i,\star} \rightsquigarrow \alpha^i \Rightarrow J^i \nearrow$

$$J^i(\alpha^{1,\star}, \dots, \alpha^{i,\star}, \dots, \alpha^{N,\star}) \le J^i(\alpha^{1,\star}, \dots, \alpha^{i}, \dots \alpha^{N,\star})$$

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- Meaning of $\boxed{\text{freezing}}$ $\alpha^{1,\star}, \ldots, \alpha^{i-1,\star}, \alpha^{i+1,\star}, \alpha^{N,\star}$
  - freezing the processes $\rightsquigarrow$ Nash equilibrium in open loop
  - $\alpha^i_t = \alpha^i(t, X^1_t, \ldots, X^N_t)$ $\rightsquigarrow$ each function α^i is a Markov feedback

$\rightsquigarrow$ Nash over of Markov loop

  - leads to different equilibria! but expect that there is no difference in the asymptotic setting

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- $\boxed{\text{Central planner}}$ $\rightsquigarrow$ everybody uses same feedback! and minimize the common cost!

Part I. Motivation

c. Example

Exhaustible resources [Guéant Lasry Lions]

- N producers of oil $\rightsquigarrow X_t^i$ (estimated reserve) at time t

$$dX_t^i = -\alpha_t^i dt + \sigma X_t^i dW_t^i$$

  - $\alpha_t^i \rightsquigarrow$ instantaneous production rate
  - σ common volatility for the perception of the reserve
  - should be a constraint $X_t^i \geq 0$

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- Optimize the profit of a producer

$$J^i(\alpha^1, \ldots, \alpha^N) = \mathbb{E} \int_0^\infty \exp(-rt)(\alpha^i_t P_t - c(\alpha^i_t))dt$$

  - P_t is selling price, c cost production
  - mean-field constraint $\rightsquigarrow$ selling price is a function of the mean-production

$$P_t = P(\frac{1}{N} \sum_{i=1}^N \alpha^i_t)$$

  - slightly different! $\rightsquigarrow$ interaction through the law of the control

$\rightsquigarrow$ $\boxed{\text{extended MFG}}$ [Gomes al., Carmona D., Cardaliaguet Lehalle]

Part I. Motivation

d. Central planner

Optimization problem over the whole population

- Same dynamics as before! rewrite the dynamics of the particles

$$dX_t^i = b(X_t^i, \bar{\mu}_t^N, \alpha_t^i)dt + \sigma(X_t^i, \bar{\mu}_t^N)dW_t^i + \sigma^0(X_t^i, \bar{\mu}_t^N)dB_t$$

- $\boxed{\text{Same cost functional!}}$ to player $i \in \{1, \ldots, N\}$

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- Reduce to Markov feedback policies $\alpha_t^i = \alpha^i(t, X_t^1, \ldots, X_t^N)$
- $\boxed{\text{Central planner!}}$ $\Rightarrow$ Forces all the players to use the same $\alpha^i = \alpha$!

  ◦ exchangeability (symmetry in law) $\Rightarrow J^1 = \cdots = J^N$ is the cost to the society

  ◦ minimize any J^i with respect to α!

Part II. From propagation of chaos to MFG

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a. McKean-Vlasov SDEs

NO COMMON NOISE

General uncontrolled particle system

- $\boxed{\text{Remove the control}}$ and the common noise!

$$dX_t^i = b(X_t^i, \bar{\mu}_t^N)dt + \sigma(X_t^i, \bar{\mu}_t^N)dW_t^i$$

  - $X_0^1, \ldots, X_N^i$ i.i.d. (and $\perp\!\!\!\perp$ of noises), $\quad \bar{\mu}_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}$

- $\exists!$ if the coefficients are Lipschitz in all the variables $\rightsquigarrow$ need a suitable distance on space of measures

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- Use the Wasserstein distance on $\mathcal{P}_2(\mathbb{R}^d)$

$$\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d), \quad W_2(\mu, \nu) = \left(\inf_\pi \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\pi(x,y)\right)^{1/2},$$

where π has μ and ν as marginals on $\mathbb{R}^d \times \mathbb{R}^d$ (●)

  ◦ X and X' two r.v.'s $\Rightarrow W_2(\mathcal{L}(X), \mathcal{L}(X')) \le \mathbb{E}[|X - X'|^2]^{1/2}$

- Example $W_2\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}, \frac{1}{N}\sum_{i=1}^N \delta_{x_i'}\Big) \le \Big(\frac{1}{N}\sum_{i=1}^N |x_i - x_i'|^2\Big)^{1/2}$ (●)

McKean-Vlasov SDE

- Expect some $\boxed{\text{decorrelation / averaging}}$ in the system as $N \uparrow \infty$ (●)
  - replace the empirical measure by the theoretical law

$$dX_t = b(X_t, \mathcal{L}(X_t))dt + \sigma(X_t, \mathcal{L}(X_t))dW_t$$

- Cauchy-Lipschitz theory
  - assume b and σ Lipschitz continuous on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \Rightarrow$ unique solution for any given initial condition in L^2
  - proof works as in the standard case taking advantage of

$$\mathbb{E}\Big[\big|(b,\sigma)(X_t, \mathcal{L}(X_t)) - (b,\sigma)(X'_t, \mathcal{L}(X'_t))\big|^2\Big] \le C\mathbb{E}[|X_t - X'_t|^2]$$

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- Propagation of chaos
  - each $(X^i_t)_{0\leq t\leq T}$ converges in law to the solution of MKV SDE
  - particles get $\boxed{\text{independent}}$ in the limit $\rightsquigarrow$ for k fixed:

$$(X^1_t, \ldots, X^k_t)_{0\leq t\leq T} \underset{\mathcal{L}}{\longrightarrow} \mathcal{L}(\text{MKV})^{\otimes k} = \mathcal{L}((X_t)_{0\leq t\leq T})^{\otimes k} \quad \text{as } N \nearrow \infty$$

  - $\lim_{N\nearrow\infty} \sup_{0\leq t\leq T} \mathbb{E}[(W_2(\bar{\mu}^N_t, \mathcal{L}(X_t))^2] = 0$

Part II. From propagation of chaos to MFG

b. Formulation of the asymptotic problems

Ansatz

- Go back to the finite game
- Ansatz $\rightsquigarrow$ at equilibrium

$$\boxed{\alpha_t^{i,\star} = \alpha^N(t, X_t^i, \bar{\mu}_t^N) \approx \alpha(t, X_t^i, \bar{\mu}_t^N)}$$

  - particle system at equilibrium

$$dX_t^i \approx b\Big(X_t^i, \bar{\mu}_t^N, \alpha(t, X_t^i, \bar{\mu}_t^N)\Big)dt + \sigma\Big(X_t^i, \alpha(t, X_t^i, \bar{\mu}_t^N)\Big)dW_t^i$$

  - particles should decorrelate as $N \nearrow \infty$
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- What about an $\boxed{\text{intrinsic interpretation of } \mu_t}$?
  - should describe the global state of the population in equilibrium
  - in the limit setting, any particle that leaves the equilibrium

should not modify $\mu_t \rightsquigarrow$ leaving the equilibrium means that the cost increases $\rightsquigarrow$ any particle in the limit should solve an optimal control problem in the environment $(\mu_t)_{0 \le t \le T}$

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(2) solve the stochastic optimal control problem in the environment $(\mu_t)_{0\le t\le T}$

$$dX_t = b(X_t, \mu_t, \alpha_t)dt + \sigma(X_t, \mu_t)dW_t$$

∘ with $X_0 = \xi$ being fixed on some set-up $(\Omega, \mathbb{F}, \mathbb{P})$ with a d-dimensional B.M.

∘ with $\boxed{\text{cost}}$ $J(\alpha) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T f(X_t, \mu_t, \alpha_t)dt\Big]$

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- with $\boxed{\text{cost}}$ $J(\alpha) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T f(X_t, \mu_t, \alpha_t)dt\Big]$

(3) let $(X_t^{\star,\mu})_{0\le t\le T}$ be the unique optimizer (under nice assumptions)
$\rightsquigarrow$ find $(\mu_t)_{0\le t\le T}$ such that

$$\mu_t = \mathcal{L}(X_t^{\star,\mu}), \quad t \in [0, T]$$

- Not a proof of convergence!

Part II. From propagation of chaos to MFG

c. Forward-backward systems

PDE point of view: HJB

- $\boxed{\text{PDE characterization of the optimal control problem}}$ when σ is the identity

- Value function in environment $(\mu_t)_{0 \le t \le T}$

$$u(t,x) = \inf_{\alpha \text{ processes}} \mathbb{E}\Big[g(X_T, \mu_T) + \int_t^T f(X_s, \mu_s, \alpha_s) ds | X_t = x \Big]$$

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- u solution $\boxed{\text{Backward}}$ HJB (●)

$$\Big(\partial_t u + \frac{\Delta_x u}{2}\Big)(t,x) + \underbrace{\inf_{\alpha \text{ deterministic}} \big[b(x,\mu_t,\alpha)\partial_x u(t,x) + f(x,\mu_t,\alpha)\big]}_{\text{standard Hamiltonian in HJB}} = 0$$

- $H(x,\mu,\alpha,z) = b(x,\mu,\alpha)\cdot z + f(x,\mu,\alpha)$ (●)
  - $\alpha^\star(x,\mu,z) = \text{argmin}_{\alpha\in A} H(x,\mu,\alpha,z) \rightsquigarrow \alpha^\star = \alpha^\star(x,\mu_t,\partial_x u(t,x))$
- Terminal boundary condition: $u(T,\cdot) = g(\cdot,\mu_T)$
- Pay attention that u depends on $(\mu_t)_t$!

Fokker-Planck

- Need for a $\boxed{\text{PDE characterization of } (\mathcal{L}(X_t^{\star,\boldsymbol{\mu}}))_t}$
- Dynamics of $X^{\star,\boldsymbol{\mu}}$ at equilibrium

$$dX_t^{\star,\boldsymbol{\mu}} = b(X_t^{\star,\boldsymbol{\mu}}, \mu_t, \alpha^\star(X_t^{\star,\boldsymbol{\mu}}, \mu_t, \partial_x u^{\boldsymbol{\mu}}(t, X_t^{\star,\boldsymbol{\mu}})))dt + dW_t$$

- Law $(X_t^{\star,\boldsymbol{\mu}})_{0\le t\le T}$ satisfies Fokker-Planck (FP) equation ( )

$$\partial_t \mu_t = -\mathrm{div}_x(\underbrace{b(x, \mu_t, \alpha^\star(x, \mu_t, \partial_x u^{\boldsymbol{\mu}}(t,x))}_{b^\star(t,x)}\mu_t) + \frac{1}{2}\Delta_x \mu_t$$

Fokker-Planck

- Need for a $\boxed{\text{PDE characterization of } (\mathcal{L}(X_t^{\star,\mu}))_t}$
- Dynamics of $X^{\star,\mu}$ at equilibrium

$$dX_t^{\star,\mu} = b(X_t^{\star,\mu}, \mu_t, \alpha^\star(X_t^{\star,\mu}, \mu_t, \partial_x u^\mu(t, X_t^{\star,\mu})))dt + dW_t$$

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- MFG equilibrium described by $\boxed{\text{forward-backward}}$ in ∞ dimension

Fokker-Planck (forward)
HJB (backward)

  - ∞ dimensional analogue of

$$\dot{x}_t = b(x_t, y_t)dt, \quad x_0 = x^0$$
$$\dot{y}_t = -f(x_t, y_t)dt, \quad y_T = g(x_T)$$

Part II. From propagation of chaos to MFG

d. FBSDE formulation

Optimal control and FBSDEs

- Environment $(\mu_t)_{0\le t\le T}$ is fixed and cost functional of the type

$$J(\alpha) = \mathbb{E}\Big[g(X_T,\mu_T) + \int_0^T f(X_t,\mu_t,\alpha_t)dt\Big]$$

  - assume f and g continuous and at most of quadratic growth

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- Interpret optimal paths as the forward component of an FBSDE $\rightsquigarrow$

On $(\Omega,\mathbb{F},\mathbb{P})$ with $\mathbb{F}$ generated by $(\xi,(W_t)_{0\le t\le T})$

$$X_t = \xi + \int_0^t b\left(X_s,\mu_s,Y_s,Z_s\right) ds + \int_0^t \sigma(X_s,\mu_s)dW_s$$

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  - σ invertible, H strict convex in α and coeff. bounded in $x \Rightarrow$

$((G,F)=(g,f)) \Rightarrow$ $\boxed{\text{represent value function!}}$

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  - choose $(\mu_t)_{0\le t\le T}$ as the law of optimal path! $\Rightarrow$ characterize by FBSDE of McKean-Vlasov type

MKV FBSDE for the value function

- Consider, on $(\Omega, \mathbb{F}, \mathbb{P})$, the MKV FBSDE

$$
\begin{aligned}
X_t &= \xi + \int_0^t b\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Z_s\sigma^{-1}(X_s, \mathcal{L}(X_s)))\Big)\, ds \\
&\quad + \int_0^t \sigma(X_s, \mathcal{L}(X_s)) dW_s \\
Y_t &= g(X_T, \mathcal{L}(X_T)) \\
&\quad + \int_t^T f\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Z_s\sigma^{-1}(X_s, \mathcal{L}(X_s)))\Big)\, ds - \int_t^T Z_s dW_s
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- Connection with PDE formulation

$$\boxed{Y_s = u(s, X_s), \qquad Z_s = \partial_x u(s, X_s)\sigma(X_s, \mu_s)}$$

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$$\boxed{Y_s = u(s, X_s), \qquad Z_s = \partial_x u(s, X_s)\sigma(X_s, \mu_s)}$$

- Unique minimizer for each $(\mu_t)_{0 \le t \le T}$ if
  - $b, f, g, \sigma, \sigma^{-1}$ bounded in (x, μ), Lipschitz in x
  - b linear in α and f strictly convex and loc. Lip in α, with $\mathrm{Lip}(f)$ at most of linear growth in α

MKV FBSDE for the Pontryagin principle

- Consider, on $(\Omega, \mathbb{F}, \mathbb{P})$, the MKV FBSDE

$$X_t = \xi + \int_0^t b\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Y_s)\Big)\, ds + \int_0^t \sigma(X_s, \mathcal{L}(X_s)) dW_s$$

$$\begin{aligned} Y_t &= \partial_x g(X_T, \mathcal{L}(X_T)) \\ &\quad + \int_t^T \partial_x H\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Y_s), Y_s\Big)\, ds - \int_t^T Z_s dW_s \end{aligned}$$

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  - σ indep. of x and $b(x, \mu, \alpha) = b_0(\mu) + b_1 x + b_2 \alpha$
  - $\partial_x f, \partial_\alpha f, \partial_x g$ L-Lipschitz in (x, α)
  - g and f convex in (x, α) with f strict convex in α

Part III. Solving MFG

a. Picture

Seeking a solution

- Any way $\rightsquigarrow$ two-point-boundary-problem $\Rightarrow$
  - Cauchy-Lipschitz theory in small time only
  - if Lipschitz coefficients (including the direction of the measure)

$\rightsquigarrow$ existence and uniqueness in short time (see later on)

$\rightsquigarrow$ existence and uniqueness of MFG equilibria in small time

Seeking a solution

- Any way $\rightsquigarrow$ two-point-boundary-problem $\Rightarrow$

  ◦ Cauchy-Lipschitz theory in small time only

  ◦ if Lipschitz coefficients (including the direction of the measure) $\rightsquigarrow$ existence and uniqueness in short time (see later on)

  $\rightsquigarrow$ existence and uniqueness of MFG equilibria in small time

- $\boxed{\text{What about arbitrary time?}}$

  ◦ existence $\rightsquigarrow$ fixed point over the measure argument by means of compactness arguments

$$\boxed{\text{Schauder's theorem}}$$

  ◦ uniqueness $\rightsquigarrow$ require additional assumption

Part III. Solving MFG

b. Schauder fixed point theorem without common noise

Statement of the Schauder fixed point theorem

- Generalisation of Brouwer's theorem from finite to infinite dimension
- Let $(V, \|\cdot\|)$ be a normed vector space
  - $\emptyset \neq E \subset V$ with E closed and convex
  - $\phi : E \to E$ continuous such that $\phi(E)$ is relatively compact
  - $\Rightarrow$ existence of a fixed point to ϕ

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  - $\phi : E \to E$ continuous such that $\phi(E)$ is relatively compact
  - $\Rightarrow$ existence of a fixed point to ϕ
- In MFG $\rightsquigarrow$ $\boxed{\text{what is } V\text{, what is } E\text{, what is } \phi\text{?}}$
  - recall that MFG equilibrium is a flow of measures $(\mu_t)_{0 \leq t \leq T}$

$$E \subset \mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d))$$

  - need to embed into a linear structure

$$\mathcal{C}([0,T], \mathcal{P}_2(\mathbb{R}^d)) \subset \mathcal{C}([0,T], \mathcal{M}_1(\mathbb{R}^d))$$

  - $\mathcal{M}_1(\mathbb{R}^d)$ set of signed measures ν with $\int_{\mathbb{R}^d} |x| d|\nu|(x) < \infty$

Compactness on the space of probability measures

- Equip $\mathcal{M}_1(\mathbb{R}^d)$ with a norm $\|\cdot\|$ ( ) and restrict to $\mathcal{P}_1(\mathbb{R}^d)$ such that
  - convergence of $(\nu_n)_{n\geq 1}$ in $\mathcal{P}_1(\mathbb{R}^d)$ implies weak convergence

$$\forall h \in C_b(\mathbb{R}^d, \mathbb{R}), \quad \lim_{n\to\infty} \int_{\mathbb{R}^d} h d\nu_n = \int_{\mathbb{R}^d} h d\nu$$

  - if $(\nu_n)_{n\geq 1}$ has uniformly bounded moments of order $p > 2$

$$\text{Unif. square integrability} \Rightarrow W_2(\nu_n, \nu) \to 0$$

  - says that the input in the coefficients varies continuously!

$$b(x, \nu_n, \alpha),\ f(x, \nu_n, \alpha),\ g(x, \nu_n)$$

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- Compactness $\rightsquigarrow$ if $(\nu_n)_{n\geq 1}$ has bounded moments of order $p > 2$
  - $(\nu_n)_{n\geq 1}$ admits a weakly convergent subsequence
  - then convergence for W_2 by unif. integrability and for $\|\cdot\|$ also

Application to MKV FBSDE

- Choose E as continuous $(\mu_t)_{0 \le t \le T}$ from $[0,T]$ to $\mathcal{P}_2(\mathbb{R}^d)$

$$\sup_{0 \le t \le T} \int_{\mathbb{R}^d} |x|^4 d\mu_t(x) \le K \qquad \text{for some } K$$

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- Construct $\phi \rightsquigarrow$ fix $(\mu_t)_{0\le t\le T}$ in E and solve HJB

$$\Big(\partial_t u^{\boldsymbol{\mu}} + \frac{\Delta_x u^{\boldsymbol{\mu}}}{2}\Big)(t,x) + \inf_{\alpha \ \text{deterministic}}\big[b(x,\mu_t,\alpha)\partial_x u^{\boldsymbol{\mu}}(t,x) + f(x,\mu_t,\alpha)\big] = 0$$

  - optimal trajectory given by

$$dX_t^{\star,\boldsymbol{\mu}} = b(X_t^{\star,\boldsymbol{\mu}},\mu_t,\alpha^\star(X_t^{\star,\boldsymbol{\mu}},\mu_t,\partial_x u(t,X_t^{\star,\boldsymbol{\mu}})))dt + dW_t$$

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  ◦ let $\boxed{\phi\Big(\mu = (\mu_t)_{0\le t\le T}\Big) = (\mathcal{L}(X_t^\mu))_{0\le t\le T}}$

- Assume bounded coefficients and $\mathbb{E}[|X_0|^4] < \infty$

  ◦ choose K such that $\mathbb{E}[|X_t^\mu|^4] \le K$

$$\Rightarrow \boxed{E \text{ stable by } \phi}$$

  ◦ $W_2(\mathcal{L}(X_t^{\star,\mu}), \mathcal{L}(X_s^{\star,\mu})) \le C\mathbb{E}[|X_t^{\star,\mu} - X_s^{\star,\mu}|^2]^{1/2} \le C|t-s|^{1/2}$

Conclusion

- Consider continuous $\boldsymbol{\mu} = (\mu_t)_{0\le t\le T}$ from $[0,T]$ to $\mathcal{P}_2(\mathbb{R}^d)$
  - for any $t \rightsquigarrow (\phi(\boldsymbol{\mu}))_t$ in a compact subset of $\mathcal{P}_2(\mathbb{R}^d)$
  - $[0,T] \ni t \mapsto (\phi(\boldsymbol{\mu}))_t$ is uniformly continuous in $\boldsymbol{\mu}$
  - by Arzelà-Ascoli $\Rightarrow$ output lives in a compact subset of

$E \subset \mathcal{C}([0,T],\mathcal{P}_2(\mathbb{R}^d))$ (and thus of $\mathcal{C}([0,T],\mathcal{M}_1(\mathbb{R}^d))$)

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- Continuity of ϕ on $E \rightsquigarrow$ stability of the solution of non-degenerate HJB with respect to a continuous perturbation of the environment

- $\boxed{\text{Refinements}}$ to allow for unbounded coefficients

  ◦ example $\rightsquigarrow$ b linear in α, f strictly convex in α, with derivatives in α at most of linear growth in α

  ◦ other refinements to treat other types of growth

  $\rightsquigarrow$ typical instance is linear-quadratic mean field games $\Rightarrow$ HJB has a quadratic solution that may be found (almost) explicitly

Part III. Solving MFG

c. Uniqueness criterion

Need for monotonicity

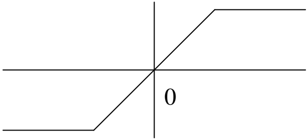
- $\boxed{\text{Simple } 1d \text{ forward backward system}}$ $\begin{cases} \dot{x}_t = -y_t \\ \dot{y}_t = 0 \end{cases}$ $\qquad y_T = g(x_T)$
  - finite-dimensional master PDE $\rightsquigarrow$ inviscid $y_t = v(t, x_t)$ Burgers equation

$$-\partial_t v = -v \partial_x v, \quad v(T, x) = g$$

  - well-posed if $g \nearrow \Rightarrow$! of characteristics
  - if not $\Rightarrow$ $\boxed{\text{shocks may emerge in finite time}}$ $\Rightarrow$ may loose !

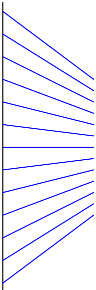
- Plots of the characteristics

if $g(x) = (-1 \vee x \wedge 1)$



$T = 1$

0



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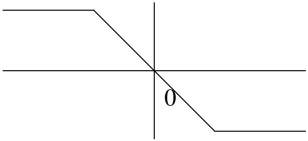

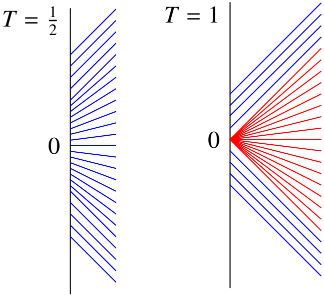

Lasry Lions monotonicity condition

- Recall following FBSDE result
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- Lasry Lions monotonicity condition
  - b, σ do not depend on μ
  - $f(x,\mu,\alpha) = f_0(x,\mu) + f_1(x,\alpha)$ (μ and α are separated)
  - monotonicity property for f_0 and g w.r.t. μ

$$\int_{\mathbb{R}^d} (f_0(x,\mu) - f_0(x,\mu'))d(\mu - \mu')(x) \geq 0$$

$$\int_{\mathbb{R}^d} (g(x,\mu) - g(x,\mu'))d(\mu - \mu')(x) \geq 0$$

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- Recall following FBSDE result
  - what is monotonicity condition in the direction of the measure?
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$$\int_{\mathbb{R}^d} (g(x,\mu) - g(x,\mu'))d(\mu-\mu')(x) \geq 0$$

- $\boxed{\text{Example}}$:

$$h(x,\mu) = \int_{\mathbb{R}^d} L(z, \rho \star \mu(z))\rho(x-z)dz$$

  - where L is $\nearrow$ in second variable and ρ is even

Monotonicity restores uniqueness

- Assume that for any input $\boldsymbol{\mu} = (\mu_t)_{0 \le t \le T}$ unique optimal control $\boldsymbol{\alpha}^{\star,\boldsymbol{\mu}}$
  - + existence of an MFG for a given initial condition

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$$\mathbb{E}\Big[\underbrace{g(X_T^{\star,\boldsymbol{\mu}}, \mu_T') - g(X_T^{\star,\boldsymbol{\mu}}, \mu_T)}_{\int_{\mathbb{R}^d} (g(x,\mu_T') - g(x,\mu_T)) d\mu_T(x)} - \underbrace{\Big(g(X_T^{\star,\boldsymbol{\mu}'}, \mu_T') - g(X_T^{\star,\boldsymbol{\mu}'}, \mu_T)\Big)}_{\int_{\mathbb{R}^d} (g(x,\mu_T') - g(x,\mu_T)) d\mu_T'(x)} + \dots \Big] > 0$$

  - same for $f_0 \Rightarrow$ LHS must be ≤ 0

□ Recall the form of an Ordinary DE (with values in $\mathbb{R}^d$)

$$\dot{X}_t = b(t, X_t), \quad \text{or} \quad dX_t = b(t, X_t)dt,$$

for a velocity field b with values in $\mathbb{R}^d$, called drift

□ call Stochastic DE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t,$$

for a volatility σ with values in $\mathbb{R}^{d\times d}$, and where

$\rightsquigarrow$ $dW_t = (dW_t^1, \cdots, dW_t^d)$

$\rightsquigarrow$ $dW_t^1, \cdots, dW_t^d$ are $\perp\!\!\!\perp$ and $\mathcal{N}(0, dt)$

$\rightsquigarrow$ dW_t is independent of the past before t

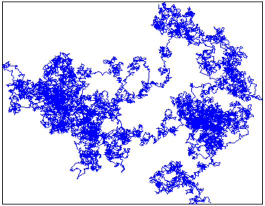
□ $(W_t)_{t\geq 0}$ B.M. with values in $\mathbb{R}^d$ is

$$W_t = (W_t^1, \cdots, W_t^d)$$

$\rightsquigarrow$ $(W_t^1)_{t\geq 0}, \cdots, (W_t^d)_{t\geq 0}$ independent

$\rightsquigarrow$ $W_{t+dt}^i - W_t^i$ of the past before t and $\mathcal{N}(0, dt)$ distributed

$\rightsquigarrow$ plot in $d = 2$



□ Recall Law of Large Numbers

$\rightsquigarrow$ take sequence $(X_n)_{n\geq 1}$ of independent r.v. with values in $\mathbb{R}^d$
$\rightsquigarrow$ take $\varphi : \mathbb{R}^d \to \mathbb{R}$ bounded continuous

$$\mathbb{P}\Big(\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \varphi(X_n) = \mathbb{E}[\varphi(X_1)]\Big) = 1$$

□ Define the empirical measure $\bar{\mu}^N = \frac{1}{N}\sum_{k=1}^{N} \delta_{X_k}$ and call $\mu = \mathcal{L}(X_1)$

$$\mathbb{P}\Big(\lim_{N\to\infty} \int_{\mathbb{R}^d} \varphi d\bar{\mu}^N = \int_{\mathbb{R}^d} \varphi d\mu\Big) = 1$$

$\rightsquigarrow$ take φ in countable subset of $C_0(\mathbb{R}^d)$, set of bounded continuous functions that tend to 0 at infinity,

$$\mathbb{P}\Big(\forall \varphi \in C_0(\mathbb{R}^d), \quad \lim_{N\to\infty} \int_{\mathbb{R}^d} \varphi d\bar{\mu}^N = \int_{\mathbb{R}^d} \varphi d\mu\Big) = 1$$

and then $\boxed{\mathbb{P}(\mu^N \Rightarrow \mu) = 1!}$

□ u (classical) solution

$$\Big(\partial_t u + \frac{\Delta_x u}{2}\Big)(t,x) + \inf_{\beta \in A}\big[b(x,\mu_t,\beta)\partial_x u(t,x) + f(x,\mu_t,\beta)\big] = 0$$

$\rightsquigarrow H(x,\mu,\alpha,z) = b(x,\mu,\alpha)\cdot z + f(x,\mu,\alpha)$

$\rightsquigarrow \alpha^\star(x,\mu,z) = \mathrm{argmin}_{\alpha\in A} H(x,\mu,\alpha,z) \rightsquigarrow \alpha^\star = \alpha^\star(x,\mu_t,\partial_x u(t,x))$

□ (Exercise) take $dX_t = b(X_t,\mu_t,\alpha_t)dt + dW_t$

$\rightsquigarrow$ Apply Itô' s formula to prove that

$$d\Big(u(t,X_t) + \int_0^t f(X_s,\mu_s,\alpha_s)ds\Big) \geq \Big[H(X_t,\mu_t,\alpha_t,\partial_x u(t,X_t)) - \inf_\beta H(X_t,\mu_t,\beta,\partial_x u(t,X_t))\Big]dt + dm_t$$

with $(m_t)_{0\leq t\leq T}$ zero expectation

$\rightsquigarrow$ take expectation and $\int_0^T \dots dt$ and get $J(\boldsymbol{\alpha}) \geq \mathbb{E}[u(0,X_0)]$

optimal feedback is $\alpha^\star(x,\mu_t,\partial_x u(t,x))$

- Recall $H(x, \mu, \alpha, z) = b(x, \mu, \alpha) \cdot z + f(x, \mu, \alpha)$
  - $\rightsquigarrow$ take $b(x, \mu, \alpha) = \alpha$
  - $\rightsquigarrow$ take $f(x, \mu, \alpha) = f(x, \mu) + \frac{1}{2}|\alpha|^2$
  - $\rightsquigarrow$ take $A = \mathbb{R}^d$
- Then, $\alpha^\star(x, \mu, z) = -z$
  - $\rightsquigarrow$ HJB becomes

$$\Big(\partial_t u + \frac{1}{2}\Delta_x u\Big)(t, x) - \frac{1}{2}|\partial_x u(t, x)|^2 + f(x, \mu_t) = 0$$

□ Take

$$dX_t = b(t, X_t)dt + dW_t$$

⇝ find an equation for $(\mu_t = \mathcal{L}(X_t))_{0 \le t \le T}$?

□ Fokker-Planck

$$\partial_t \mu_t = -\mathrm{div}_x(b(t,x)\mu_t) + \frac{1}{2}\Delta_x \mu_t$$

⇝ take φ smooth test function (with compact support) and expand

$$\mathbb{E}[\varphi(X_t)] = \int_{\mathbb{R}^d} \varphi(x) d\mu_t(x)$$

⇝ Itô's formula yields

$$\frac{d}{dt}\mathbb{E}[\varphi(X_t)] = \mathbb{E}\Big[b(t, X_t) \cdot \partial_x \varphi(X_t) + \frac{1}{2}\Delta_x \varphi(X_t)\Big]$$

that is

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi(x) d\mu_t(x) = \int_{\mathbb{R}^d} b(t,x) \cdot \partial_x \varphi(x) d\mu_t(x) + \frac{1}{2}\int_{\mathbb{R}^d} \Delta_x \varphi(x) d\mu_t(x)$$

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that is

$$\frac{d}{dt}\langle \varphi, \mu_t \rangle = \Big\langle \varphi, -\mathrm{div}_x(b(t,\cdot)\mu_t + \frac{1}{2}\Delta_x \mu_t \Big\rangle$$

□ Take θ random variable with uniform distribution on $\{1, \cdots, N\}$ and, for $\boldsymbol{x}, \boldsymbol{x}'$ in $(\mathbb{R}^d)^N$, consider

$$X = \theta_{\boldsymbol{x}}, \quad X' = \theta_{\boldsymbol{x}'}$$

$$\rightsquigarrow \mathcal{L}(X) = \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}, \mathcal{L}(X') = \frac{1}{N} \sum_{i=1}^{N} \delta_{x'_i}$$

□ Backward SDE

$$Y_t = G(X_T) + \int_t^T F\big(X_s, Y_s, Z_s\big)\,ds - \int_t^T Z_s dW_s$$

$\rightsquigarrow$ strange to have two unknowns and one equation!

□ make it clear with $F \equiv 0$

$\rightsquigarrow$ always want Y_t not to anticipate on the future

$\rightsquigarrow$ cannot be $Y_t = G(X_T)$

$\rightsquigarrow$ only way is $Y_t = \mathbb{E}[G(X_T)\,|\,\mathcal{F}_t]$

$\rightsquigarrow$ Theorem from probability that there exists a unique Z!

□ Choose $\|\cdot\|$ as Kantorovich Rubinstein norm (see Bogachev)

$$\|\mu\| = |\mu(\mathbb{R}^d)| + \sup\Big\{ \int_{\mathbb{R}^d} \ell(x) d\mu(x), \quad \ell \text{ 1-Lip}, \ \ell(0) = 0 \Big\}$$

□ Then Kantorovich formula for $\mathcal{W}_1$

$$\mathcal{W}_1(\mu, \mu') = \|\mu - \mu'\|$$

□ If $p \geq 1$ and $\mathcal{P}_p(\mathbb{R}^d)$ probability measures with finite p-moments

$$\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d), \qquad W_p(\mu, \nu) = \left(\inf_\pi \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\pi(x, y)\right)^{1/p},$$

where π has μ and ν as marginals on $\mathbb{R}^d \times \mathbb{R}^d$

□ $\boxed{\text{First order condition of optimality with noise}}$

$$dX_t = b(X_t, \mu_t, \alpha_t)dt + \sigma dW_t$$

⇝ Pontryagin system (Peng)

$$X_t = X_0 + \int_0^t b\Big(X_s, \mu_s, \alpha^\star(X_s, \mu_s, Y_s)\Big)\, ds$$
$$+ \sigma W_t$$
$$Y_t = \partial_x g(X_T, \mu_T) + \int_t^T \partial_x H\Big(X_s, \mu_s, \alpha^\star(X_s, \mu_s, Y_s), Y_s\Big)\, ds$$
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☐ $\boxed{\text{First order condition of optimality with noise}}$

$$dX_t = b(X_t, \mu_t, \alpha_t)dt + \sigma dW_t$$

$\rightsquigarrow$ Pontryagin system (Peng)

$$X_t = X_0 + \int_0^t b\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Y_s)\Big)\, ds$$
$$+ \sigma W_t$$
$$Y_t = \partial_x g(X_T, \mathcal{L}(X_T)) + \int_t^T \partial_x H\Big(X_s, \mathcal{L}(X_s), \alpha^\star(X_s, \mathcal{L}(X_s), Y_s), Y_s\Big)\, ds$$
$$- \int_t^T Z_s dW_s$$

□ Summary: Forward-Backward systems may be ill-posed! But:

$\rightsquigarrow$ Noise restores uniqueness!

$\rightsquigarrow$ Monotonicity ($\leftrightarrow$ convexity)*restoresuniqueness*!

□ Hint: Either use monotonicity or interpret the FB system as the Pontryagin system of a standard optimal control problem with linear–convex coefficients

$$\rightsquigarrow b(t,x,\alpha) = (a_t x + a'_t)x + b_t \alpha_t$$

$$\rightsquigarrow g(x) = \frac{1}{2} q(q + q')x^2$$

$$\rightsquigarrow f(t,x,\alpha) = \frac{1}{2}[\alpha^2 + m_t(m_t + m'_t)x^2]$$

□ $\boxed{\text{Exercise}}$: What does monotonicity for the MFG mean for the control problem?

□ $\boxed{\text{Hint}}$: Write monotonicity as

$$\int_{\mathbb{R}^d}\Big[\int_{\mathbb{R}^d} F(x-y)dm(y) - \int_{\mathbb{R}^d} F(x-y)dm'(y)\Big]d(m-m')(x) \geq 0$$

$$\Leftrightarrow \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x-y)d(m-m')(y)d(m-m')(x) \geq 0$$

$\rightsquigarrow$ second-order term is positive in linearization $\Leftrightarrow$ convexity!

□ $\boxed{\text{Examples}}$:

$\rightsquigarrow F(z) = -|z|^2$

$\rightsquigarrow F(z) = \displaystyle\int_{\mathbb{R}^d} \exp(iz \cdot s)d\lambda(s)$, where λ is symmetric positive finite measure

(take λ a Gaussian, take λ a Cauchy, take λ a combination of two Dirac masses...)

□ Make a convex perturbation of $\mu \in \mathcal{P}(\mathbb{R}^d)$

$\rightsquigarrow$ take $\nu \in \mathcal{P}(\mathbb{R}^d)$ and expand

$$
\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x-y)d((1-\varepsilon)\mu(x)+\varepsilon\nu(x))d((1-\varepsilon)\mu(x)+\varepsilon\nu(x)) \\
&= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x-y)d\mu(x)d\mu(y) \\
&\quad + \varepsilon\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x-y)d\mu(x)d(\nu-\mu)(y) \\
&\quad + \varepsilon^2\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x-y)d(\nu-\mu)(x)d(\nu-\mu)(y)
\end{aligned}
$$

$\rightsquigarrow$ regard $\nu-\mu$ as direction of linearization