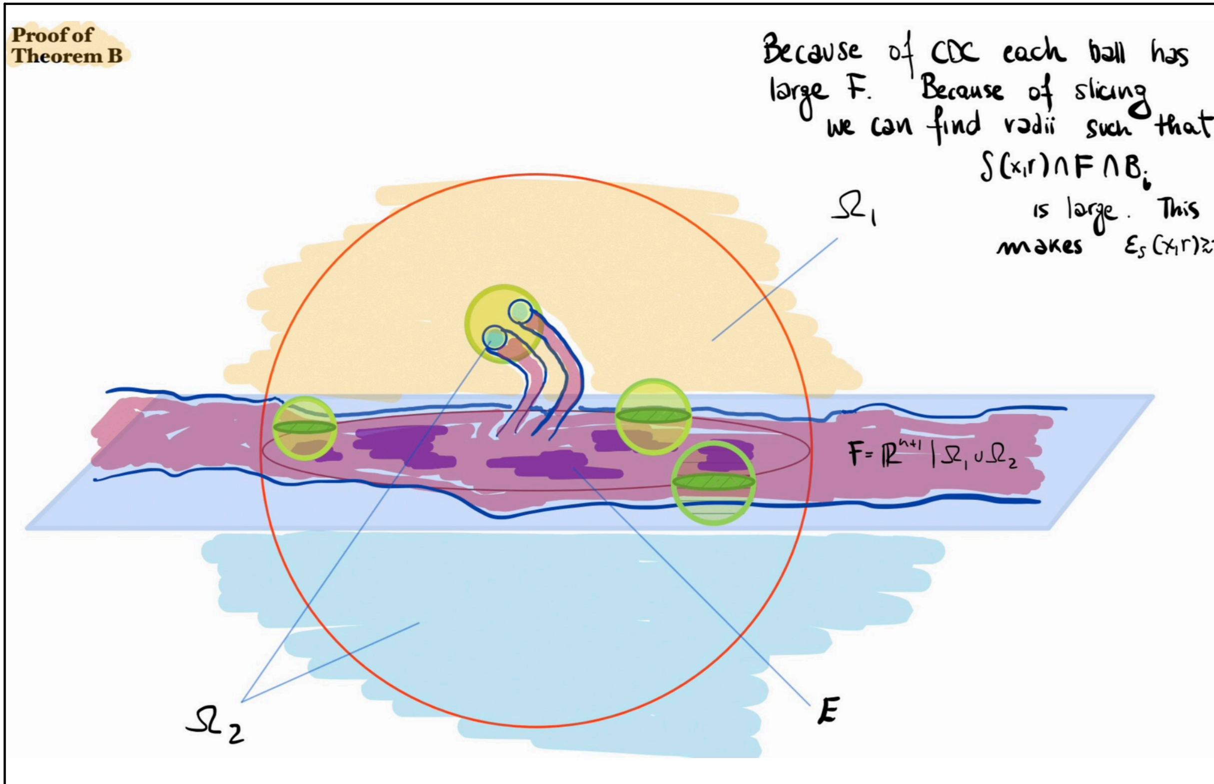
A Brownian particle almost never lands on a Cantor set

Les journées de rentrée FMJH, September 2-3d 2024

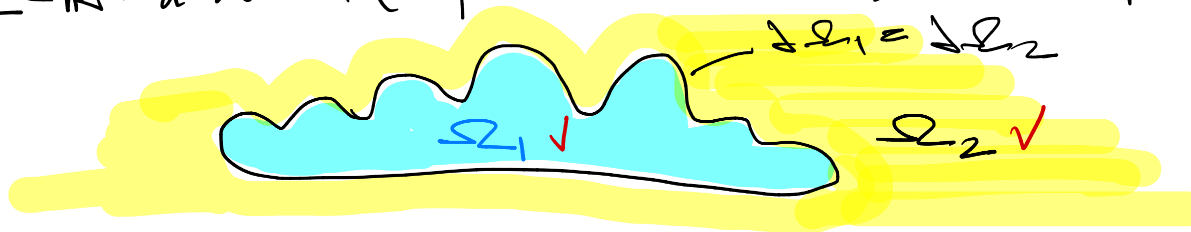


PDEs, Harmonic Analysis, Geometric Measure Theory are fun — you are allowed to draw pictures and it helps!

picture credit: Michele Villa

② So, what Brownian particles got to do with (Harmonic) Analysis?

<u>Def</u> $\Omega \subset \mathbb{R}^n$ – a domain (= open connected subset) with a compact boundary $\partial\Omega$



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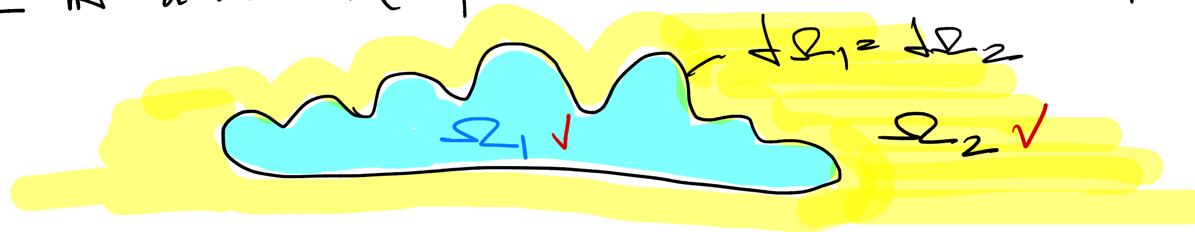
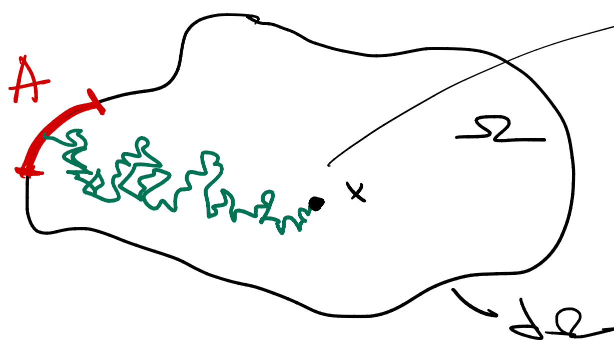
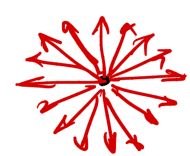


Harmonic measure ω^x with a pole $x \in \Omega$ of a subset A of the boundary $\partial\Omega$ is the probability that a Brownian particle, starting from x, will hit the boundary $\partial\Omega$ for the first time inside A.



at each moment, the particle decides in which direction to go



all directions are <u>equiprobable</u>

③ How to understand what does this measure look like?

harmonic measure

→ we can try to compare it with a measure we know well

"the natural measure on the boundary"

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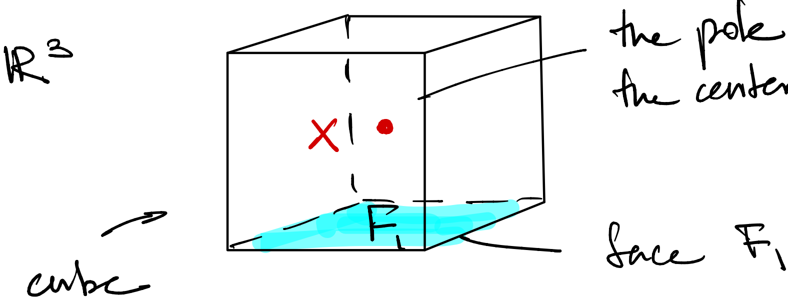
Let's see a couple of examples of how it compares:

- $\mathbb{R}^3$



? $w^x(F_1)$

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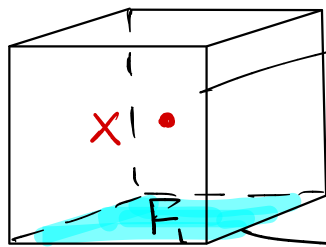

cube

the pole is the center

face F_1

$$w^{x}(F_1) \sim S(F_1)$$

proportional $\left(\text{or} = \dfrac{S(F_1)}{S(\text{cube})}\right)$

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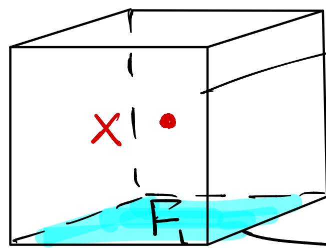

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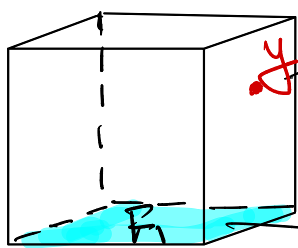
- $\mathbb{R}^3$



same cube

the pole is somewhere else

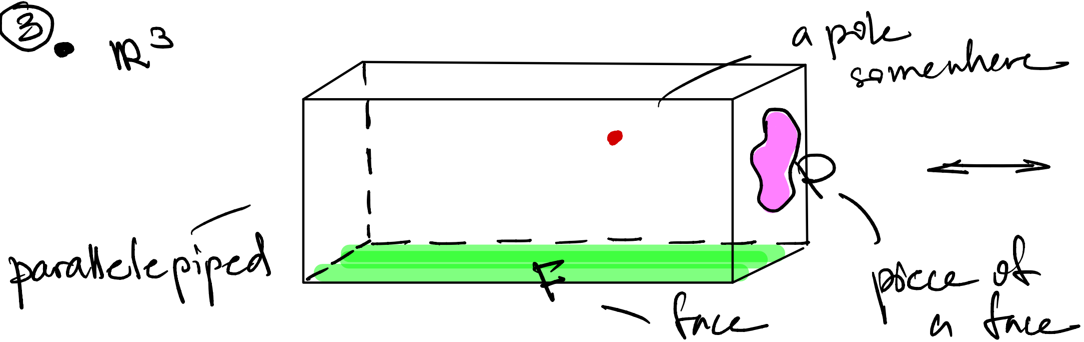
same face

$\longleftrightarrow$ still, $w^y(F_1) \sim S(F_1)$

and $w^x \simeq w^y$

$\boxed{\text{so write } w \text{ not } w^x}$

③ • $\mathbb{R}^3$



$\longleftrightarrow$ $w(F) \sim S(F)$ intuitively clear,

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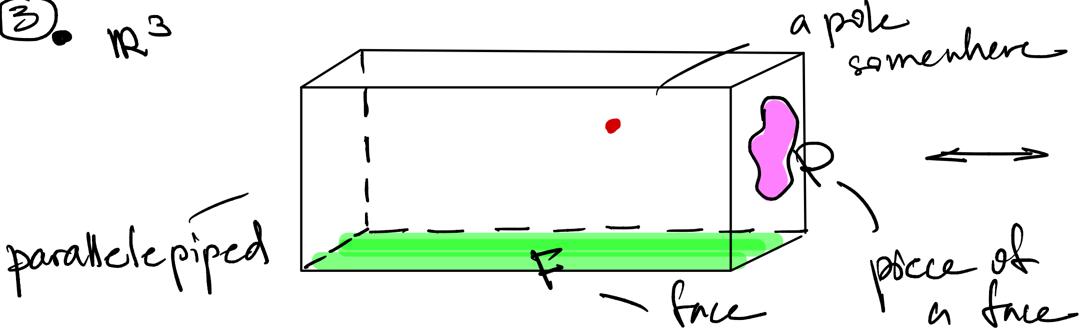

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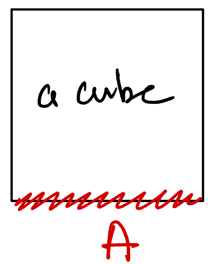

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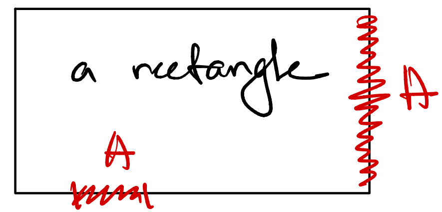
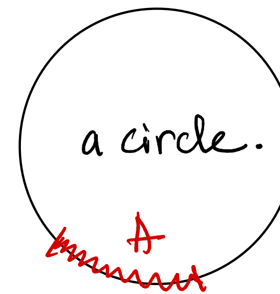


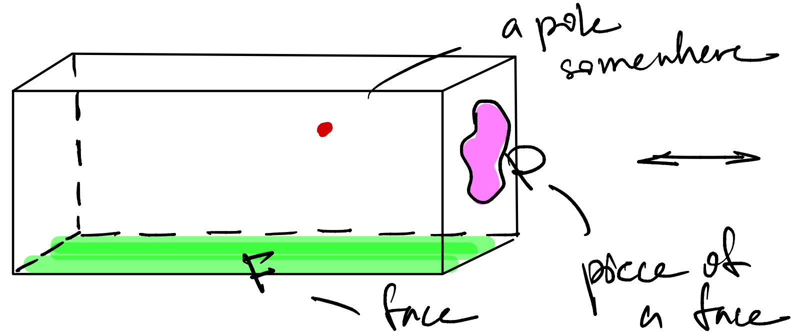
or even



③

• $\mathbb{R}^3$

parallelepiped



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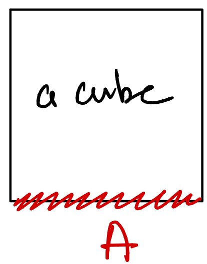

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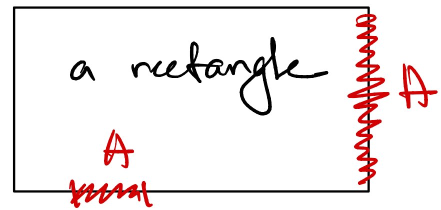

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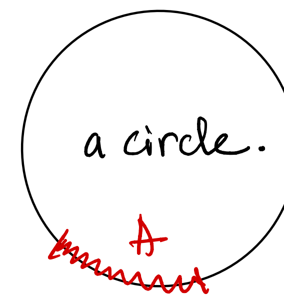


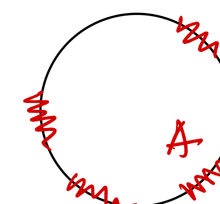
here, if you don't trust your intuition, you can

– „prove" that of $A = \frac{1}{2}$ of the circle $w \sim \frac{1}{2}$

– same for any interval of length $\frac{1}{2^k}$

– apply the philosophy

„if true for dyadic intervals $\Rightarrow$ true for any reasonable sets", e.g.



③ • same story for a non-compact $d\Omega$ and a pole far away

• — pole at ∞

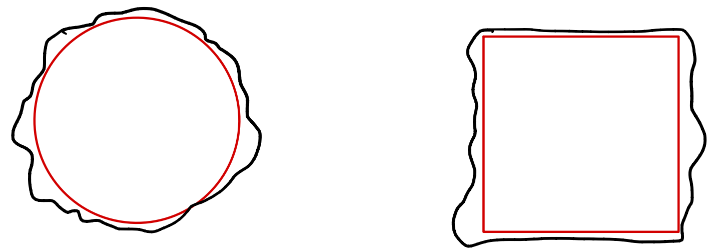
— a line

③

- same story for a non-compact $\partial\Omega$ and a pole far away
  - — pole at ∞

— a line

- same for small perturbations of pictures we saw before:



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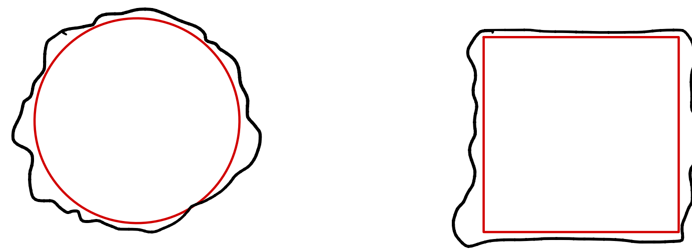

- same if your boundary is a Lipschitz curve
  ~ graph of a Lipschitz function f

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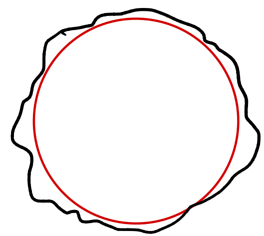
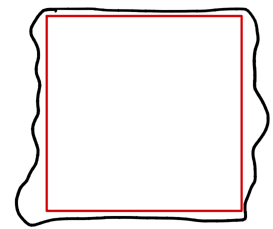


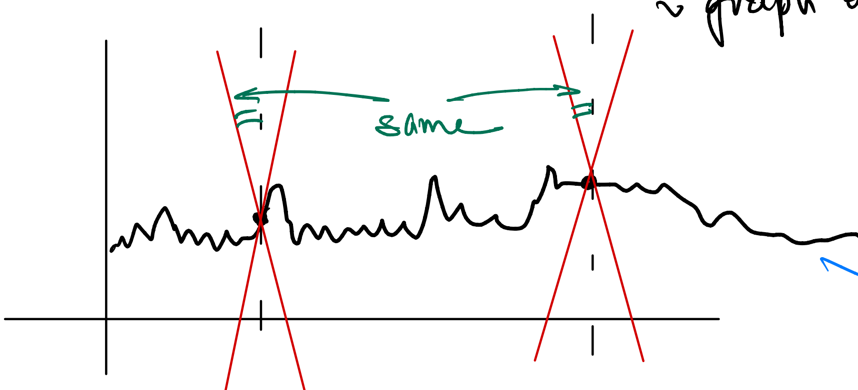


- same if your boundary is a Lipschitz curve

  ~ graph of a Lipschitz function $f: \mathbb{R} \to \mathbb{R}$ ($\mathbb{R}^{n-1} \to \mathbb{R}$)

Def if $\exists\, C$:

$$|f(x) - f(y)| \leq C|x-y|$$



can look quite bad!

4 So, since even for quite ugly one-dimensional boundaries

$$\boxed{w(A) \sim \lambda(A) \sim \ell(A) \text{ in } \mathbb{R}^2}$$

is it true for all one dimensional boundaries?

④ So, since even for quite ugly one-dimensional boundaries

$$\boxed{\omega(A) \sim \lambda(A) \sim \rho(A) \text{ in } \mathbb{R}^2}$$

is it true for <u>all</u> one dimensional boundaries?

– well, <u>no</u> : the (standard) four-corner Cantor set is a one counterexample

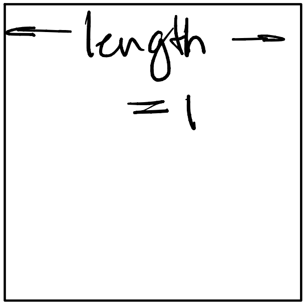
Some philosophy!

– counterexamples help to uncover the correct general pattern

– by now it is known (in any dimension!) that the behaviour of the harmonic measure $\omega(A) \sim \rho(A)$ actually characterizes the geometry of the boundary too

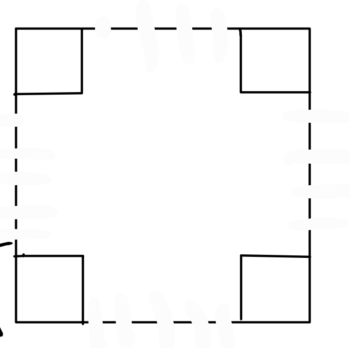
– thanks to counterexamples like this

4 **Def** the (standard four-corner) Cantor set



set K_0

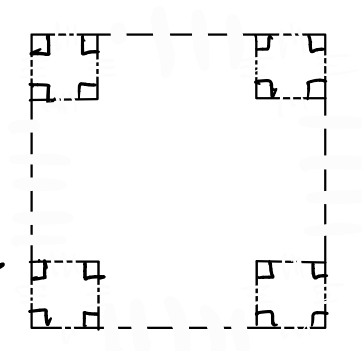
step 0



set K_1

step 1

- in the corners of the square from step 0, build squares of sidelength = 1/4
- erase the rest



set K_2

step 2

repeat step 1 for the 4 squares we got at step 1 $\longrightarrow$ ...

$$K = \bigcap_{n=0}^{\infty} K_n$$

⑤ But in what sense this object is one dimensional ?

(Hausdorff) dimension of K =

$$= \sup \left\{ d > 0 : \lim_{\delta \to 0} \inf \left\{ \sum_i \operatorname{diam}(B_i)^d, K \subseteq \bigcup B_i, \operatorname{diam} B_i < \delta \right\} = \infty \right\}$$

$= d$- Hausdorff measure $H^d(K)$

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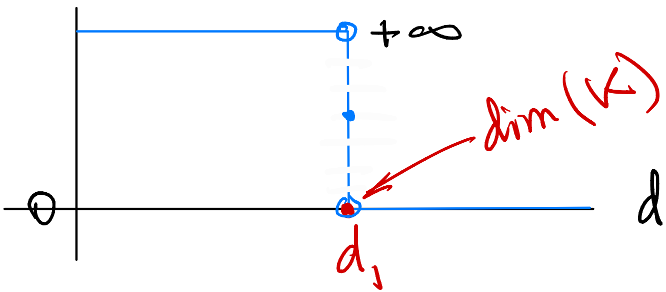

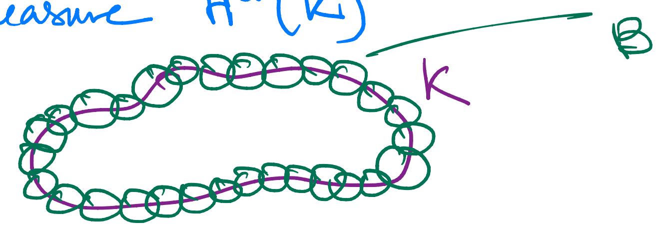

a set K fixed –
looks like this for any set

- $\exists$ for d_1 $H^{d_1}(K) < +\infty$

$\xrightarrow[\delta \to 0]{} 0$

- for $d > d_1$

$$\sum \operatorname{diam} B_i^d = \sum \operatorname{diam} B_i^{d_1} \cdot \underline{\operatorname{diam} B_i^{d-d_1}}$$

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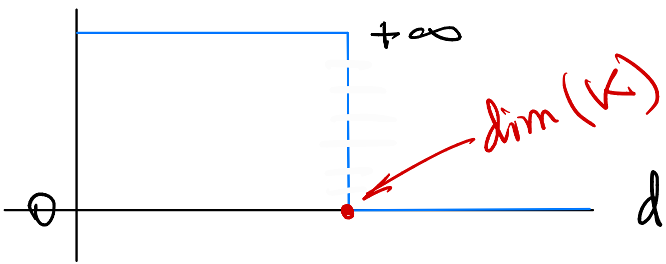

a set K fixed -
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for the Cantor set K,

$$H^d(K) = 4^n \cdot \left(\frac{1}{4^n}\right)^d, \ \forall n \geq 0$$

$< \infty$ and > 0 as soon as $d = 1$

H^1 measure = probability measure that equidistributes mass over 4^n squares at the construction step n

⑥ So we want to prove that, on K, w is not comparable to H^1

→ the proof is due to Athanassios Batakis, mid 90s + works in $\mathbb{R}^n$!
(PhD Université Paris-Sud - Orsay 1997)

→ on $\mathbb{R}^2$, earlier proofs use Complex Analysis + Dynamics

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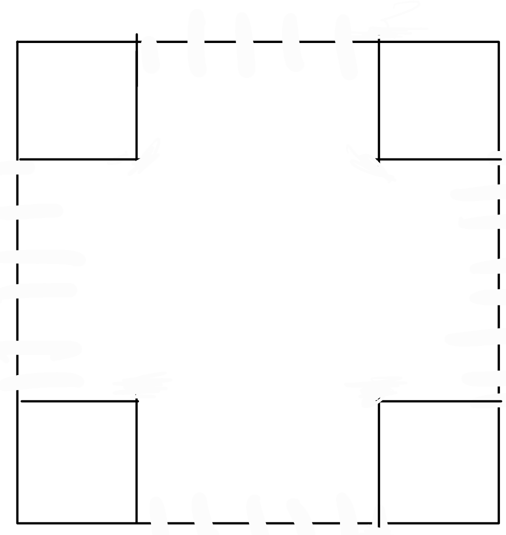
The main idea of the proof:

Imagine the particle starts moving from far away from K

(remember $w^x \sim w^y$)



look at different stages of the constructions of K / different levels of approximation

step 1:

all 4 squares potentially look the same

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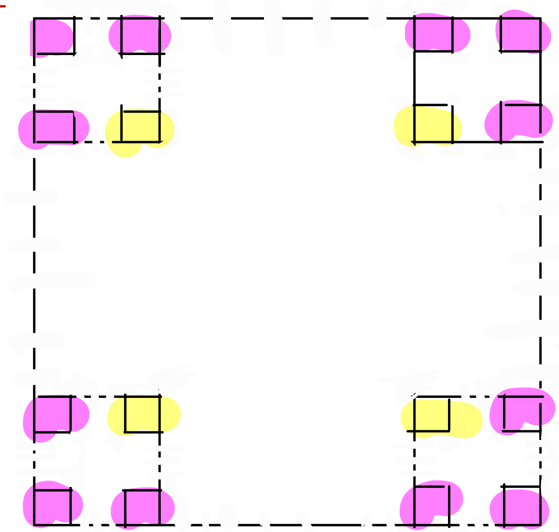

step 2:
from far away the "boundary" cubes can be reached easier, than the "inner" ones

// though you can argue that side holes are large

⑥ So we want to prove that, on K, ω is not comparable to $\mathcal{H}^1$

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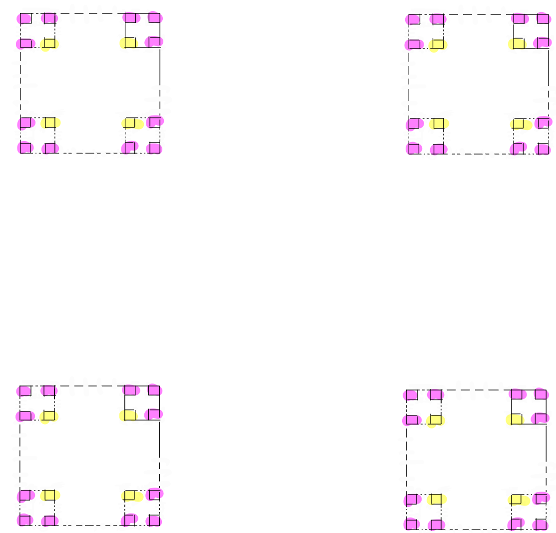

step 3 certainly convinces you more that pink cubes are more „attended" than the yellow ones

Cor: the measure w does not distribute itself uniformly

⑦ The proof „dans les grandes lignes"

$\mathcal{E}_k :=$ all the little cubes at step k of the construction $\cap K$

$L_k := \{ J \in \mathcal{E}_k \mid w(J) \geq \ell(J)^{1-\varepsilon} \}$, for some $\varepsilon > 0$

„sidelength" of $J = 4^{-k}$

- Observe: $\sum_{J \in L_k} \ell(J)^{1-\varepsilon} < \sum_{J \in L_k} w(J) \leq 1$
- $\operatorname{supp} w \subset \mathcal{E}_k$, $H^{1-\varepsilon}(\operatorname{supp} w) \sim \sum_{J \in \mathcal{E}_k \cap \operatorname{supp} w} \ell(J)^{1-\varepsilon}$
- Imagine we prove: $\boxed{\exists\, \varepsilon > 0 \text{ and } \{n_i\} \text{ such that } \lim_{i \to \infty} \sum_{J \notin L_{n_i}} w(J) = 0}$
- This implies: $H^{1-\varepsilon}(\operatorname{supp} w) \sim \sum_{J \in \mathcal{E}_k \cap \operatorname{supp} w} \ell(J)^{1-\varepsilon} \sim \sum_{J \in L_{n_i}} \ell(J)^{1-\varepsilon} < 1$
- So, $H^1(\operatorname{supp} w) = 0$ and w cannot be comparable to H^1

8 **Lm 1** (the main geometric Lemma)

$\exists\ N_1 \in \mathbb{N}$ independent of n (the step in the construction) and

a place in the construction / a cube $I \in \mathcal{E}_n$ such that

$\forall\ I \in \mathcal{E}_n$ $\exists\ J \in \mathcal{E}_{n+N_1} \cap I$, a child of I of generation N_1,

with the property $\boxed{w(J) < \frac{1}{4} \frac{w(I)}{4^{N_1}}}$.

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The intuition:

choose any place in the construction

after a couple of iterations, we see that there are cubes with are harder to reach

(9) We need to find $\{n_i\}$. $n_i := 3N_1 \cdot i$ (probs $n_i = 2N_1 \cdot i$ is ok)

Lm 2 (Cauchy-Schwartz inequality quantified)

$\exists\, \beta < 1 : \forall\, \varepsilon > 0$ and $I \in \mathcal{E}_{n_i}$ we have

$$\boxed{\sum_{J \in \mathcal{E}_{n_{i+1}} \cap I} w(J)^{1/2} \ell(J)^{\frac{1+\varepsilon}{2}} \le \beta^{n_{i+1} - n_i}\, w(I)^{1/2} \ell(I)^{\frac{1+\varepsilon}{2}}.}$$

$\ell(J) = -4^{n_{i+1}}$ $\qquad\qquad$ $\ell(I) = -4^{n_i}$

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$\ell(J) = 4^{-n_{i+1}}$ $\qquad$ $\ell(I) = 4^{-n_i}$

Why believable: • for $\varepsilon = 0$ this is almost Cauchy-Schwartz!

$$\boxed{\sum x_i y_i \le \left(\sum x_i^2\right)^{1/2} \left(\sum y_i^2\right)^{1/2}}$$

take $x_i = w(J)^{1/2}$, $y_i = \ell(J)^{1/2}$

• plus we know that $=$ in C-S iff $x_i = \text{const}$, $y_i = \text{const}$

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• plus we know that $=$ in C-S iff $x_i = \text{const}$, $y_i = \text{const}$

• for $\varepsilon > 0$, C-S gets us $\sum w(J)^{1/2} \ell(J)^{\frac{1+\varepsilon}{2}} \le w(I)^{1/2} \left(\sum \ell(J)^{1+\varepsilon}\right)^{1/2}$

$$\left(\sum \ell(J)^{1+\varepsilon}\right)^{1/2} = \left(4^{-n_{i+1}(1+\varepsilon)} \cdot 4^{n_{i+1}-n_i}\right)^{1/2} = 4^{\frac{-n_{i+1}\cdot\varepsilon - n_i}{2}} < 4^{\frac{-n_i - n_i\cdot\varepsilon}{2}},$$

⑩ Finishing the proof

✓ defined already

Recall that we need

$$\boxed{\begin{gathered}\exists\, \varepsilon>0,\ \{n_i\}: \quad \lim_{i\to+\infty} \sum_{\delta\notin L_{n_i}} w(\delta) = 0, \\ L_{n_i} = \{\delta\in\mathcal{E}_{n_i} \mid w(\delta) > \ell(\delta)^{1-\varepsilon}\}\end{gathered}}$$

$$\sum_{\delta\notin L_{n_i}} w(\delta) = \sum_{\delta\notin L_{n_i}} w(\delta)^{1/2} w(\delta)^{1/2} \le \sum_{\delta\in\mathcal{E}_{n_i}} w(\delta)^{1/2}\, \ell(\delta)^{\frac{1+\varepsilon}{2}-\varepsilon} =$$

$$= 4^{n_i\varepsilon} \sum_{\delta\in\mathcal{E}_{n_i}} w(\delta)^{1/2}\, \ell(\delta)^{\frac{1+\varepsilon}{2}} \underset{\text{Lm 2}}{\le} 4^{n_i\varepsilon}\, \beta^{n_i - n_{i-1}} \sum_{\delta\in\mathcal{E}_{n_{i-1}}} w(\delta)^{1/2}\, \ell(\delta)^{\frac{1+\varepsilon}{2}}$$

after iterating get $\displaystyle\sum_{\delta\notin L_{n_i}} w(\delta) \le 4^{n_i\varepsilon}\beta^{n_i} = (4^{\varepsilon}\beta)^{n_i}$

$4^{\varepsilon}\cdot\beta<1$

Choose $\varepsilon>0$ small: $\cancel{4^{\varepsilon}<1}$, we have $\beta<1 \Rightarrow (4^{\varepsilon}\cdot\beta)^{n_i} \longrightarrow 0$

$\blacksquare$

Thank you for your attention!