Une introduction à la Théorie des Modèles
An introduction to Model Theory

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Introduction

Model Theory is a relatively recent (1930's) branch of Mathematical Logic. It is much closer to abstract algebra and algebraic geometry than to questions about the Foundations of Mathematics. It is still not very well known in spite of its numerous applications to algebra, geometry and, more recently, to combinatorics.
Informal introduction to the subject, building from a classical example, with some very informal definitions and some more precise ...
Try to nearly give a proof , or give the strategy of a proof of :

Theorem

(James Ax 1969) *Let f be a polynomial map from $\mathbb{C}^n$ into $\mathbb{C}^n$ ($n \geq 1$). If f is injective, f is surjective.*

– $\mathbb{C}$: the field of complex numbers
– $f : \mathbb{C}^n \mapsto \mathbb{C}^n$ is polynomial means
$f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots f_n(x_1, \ldots, x_n))$ where, for each i,
$f_i \in \mathbb{C}[X_1, \ldots, X_n]$ (f_i is a polynomial over $\mathbb{C}$ in n variables)

Theorem

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First remarks: of course one needs an assumption about the map f.
1. the fact that f goes from $\mathbb{C}^n$ to $\mathbb{C}^n$ is essential.
2. the fact it is $\mathbb{C}$ and not any field is essential (true also for $\mathbb{R}$ in fact).
3. f polynomial is essential. There are examples of holomorphic maps which are injective and not surjective.

is it easy to prove? For $n = 1$ yes. Do not need injective
Take $f(X) \in \mathbb{C}[X]$ of degree at least one, and any $a \in \mathbb{C}$ want $b \in \mathbb{C}$ with $f(b) = a$. The equation $f(x) - a = 0$ has a solution in $\mathbb{C}$ because every polynomial of degree ≥ 1 has a root in $\mathbb{C}$.
But for $n \geq 2$...

This is a simple version,
The real theorem is about f from V^n to V^n where V is an algebraic variety or more generally a scheme of finite type.
But the idea of the proof will be exactly the same for V.

The proofs?
- was proved for $\mathbb{R}$ in 1962, first proof also for $\mathbb{C}^n$ in there.
- then for varieties and schemes,
– James Ax (69) using model theory
– Borel (69) using algebraic geometry.
More different proofs later...

How can we prove Ax's theorem?

Model-theoretic proof : We want to use a TRANSFER principle from the cases where it is very easy to prove to the case of $\mathbb{C}$.
For which fields is it obviously true?
1. Finite fields Well if $K$is a finite field, or just a finite set!!! ANY $f : K^n \mapsto K^n$which is injective is also surjective!.
2. Now easy to deduce : For locally finite fields: K is a *locally finite field* if every finitely generated subfield is finite.

Proof of Ax's theorem for K locally finite

Let $f = (f_1, \ldots, f_n) : K^n \mapsto K^n$ be injective
and $a = (a_1, \ldots, a_n) \in K^n$.
We want : there exists $b = (b_1, \ldots, b_n) \in K^n$ such that
$f(b_1, \ldots, b_n) = (a_1, \ldots, a_n)$, that is $f_i(b_1, \ldots, b_n) = a_i$ for $1 \leq i \leq n$.
Let $K_0 < K$ subfield finitely generated by $\{a_1, \ldots a_n\}$ and all the coefficients of the polynomials $f_i \in K[X_1, \ldots, X_n]$.
$f_0 := f|K_0$ (the restriction of f to K_0.)
$f_0 = (f_1|K_0, \ldots, f_n|K_0)$, where each $f_i \in K_0[X_1, \ldots, K_n]$. so f_0 polynomial map over K_0 and for each $(x_1, \ldots, x_n) \in {K_0}^n$,
$f_0(x_1, \ldots x_n) \in {K_0}^n$ So $f_0 : {K_0}^n \mapsto {K_0}^n$, is polynomial and is also injective.
Now K_0 is finite, so f_0 is surjective by case 1, and as
$a = (a_1 \ldots a_n) \in {K_0}^n$, there is some $b \in {K_0}^n$ such that $a = f_0(b)$,
$b \in {K_0}^n$.

Why is this useful?

$\mathbb{C}$ has characteristic zero , no field of characteristic zero can be locally finite !!

But $\mathbb{C}$ is algebraically closed. Some other important algebraically closed fields are locally finite .

Now need to do a little algebra

About algebraically closed fields

1. A field K is algebraically closed if every polynomial of $K[X]$ of degree ≥ 1 has a root (= racine) in K .
$\mathbb{R}$, the field of reals is not algebraically closed : $x^2 = -1$ has no solution.
$\mathbb{C}$ is algebraically closed.
2. Let k be a field, k has an extension K which is algebraically closed.
3. Let k be a field, there is an extension of k, $\bar{k}$ which is both algebraically closed and algebraic over k. We call $\bar{k}$ an algebraic closure of k. Two algebraic closures of k are isomorphic over k.

About algebraically closed fields, characteristic 0

We will use only one fact about $\mathbb{C}$:

4. In characteristic 0

$\mathbb{C}$ is algebraically closed and every algebraically closed field of characteristic 0 with same cardinality as $\mathbb{C}$ is isomorphic to $\mathbb{C}$ (the cardinality of $\mathbb{C}$ is the cardinality of the continuum = cardinality of $\mathcal{P}(\mathbb{N})$ = cardinality of $\mathbb{R} = 2^{\aleph_0}$)

About algebraically closed fields, characteristic p

5. Fix an algebraic closure of $\mathbb{F}_p$, $\overline{\mathbb{F}_p}$:
So $\mathbb{F}_p < \overline{\mathbb{F}_p}$
In $\overline{\mathbb{F}_p}$, $\mathbb{F}_p$ has one and only one extension of degree n $\mathbb{F}_p < \mathbb{F}_{p^n}$
$\mathbb{F}_{p^n} < \mathbb{F}_{p^m}$ iff n divides m
Then $\overline{\mathbb{F}_p}$ is the inductive limit of the F_{p^n} for $1 \leq n$: the union of the F_{p^n}, with $\mathbb{F}_{p^n} < \mathbb{F}_{p^m}$ iff n divides m
$\longrightarrow$ every finite subset of $\overline{\mathbb{F}_p}$ is contained in one of the $\mathbb{F}_{p^n}$, so every finitely generated subfield is finite
$\longrightarrow$ $\overline{\mathbb{F}_p}$ is locally finite

The conclusion by transfer

We know Ax's theorem is true for all locally finite fields (by case 2) , so
Ax's theorem is true in $\overline{\mathbb{F}_p}$ for every p

Now, invoke (classical) model theory
Conclude it is also true for $\mathbb{C}$
Why?
A particular case of a classical transfer theorem of Model Theory nearly says
"Every property which is true in the algebraic closure of $\mathbb{F}_p$ for nearly all p (= all except a finite number), is also true of $\mathbb{C}$ "
Of course not "every property" !
– for every p , $\mathbb{F}_p$ satisfies : "is not of characteristic 0"
– for every p $\overline{\mathbb{F}_p}$ is countable and $\mathbb{C}$ is not.

So the real transfer theorem is

Theorem

Every property which can be expressed by a first-order formula in the language of rings and which is true in the algebraic closure of $\mathbb{F}_p$ for all except finitely many prime p's is also true of $\mathbb{C}$

And " being of characteristic 0" is not expressible by such a formula !

Theorem

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Need two ingredients

1. the definition of first-order formula in the language of rings

This is the first fundamental concept at the basis in Model theory : studying mathematical structures (rings, fields, groups , ordered sets, graphs...) by studying their subsets which can be defined by first order formulas

and then need to check that Ax's theorem can be expressed by first order formulas.

2. Prove the transfer principle

a consequence of the first essential theorem in model theory the compactness theorem

or equivalently the fundamental notion of "limit" of structures = ultraproducts of structures..

And the "limit" or ultraproduct of the $\overline{\mathbb{F}_p}$'s, when $p \mapsto \infty$, is $\mathbb{C}$!!

Other classical Theorem of transfer about fields:

Every first order formula which is true in ONE algebraically closed field is true in ALL algebraically closed fields of the same characteristic.

This transfer principle is often used informally in algebra or in algebraic geometry under the name "Lefschetz Principle" : " To show some algebraic property in an algebraically closed field of characteristic 0 it suffices to show it for $\mathbb{C}$.

Here we will NOT use this transfer principle and also NOT directly the compactness theorem.

Will concentrate on 2: the transfer.

Will just very quickly and informally describe what is a formula and what is NOT!

First order formulas :

very very quickly....

First fundamental concept at the basis of Model theory :we study abstractly mathematical structures (rings, fields, groups , ordered sets, graphs...) by studying their subsets which can be defined by first order formulas = definable subsets

A structure is a (non empty set) with some operations or relations. Here we will NOT define , just examples of classes of structures. The type of operations chosen gives the language

Structures and languages

– For commutative rings :
R a ring : work in the language of rings : basic ring operations;
the structure is $\langle R, +, -, ., 0, 1\rangle$ where $+, .$ are maps from RxR to$R$,
$-$ is a map from R to R, $0, 1$ are distingushed constants.
– For ordered rings, add the ordering $\langle R, +, -, ., 0, 1, <\rangle$
– If interested in the class of all groups : $\langle G, ., ^{-1}, 1\rangle$
– ordered groups $\langle G, ., ^{-1}, 1, <\rangle$
– Graphs : $\langle S, E\rangle$ S the set of vertices (sommets) and E the binary relation linking two vertices.

Operations —— $>$ the language associated to the structure

First-order formulas

for rings R

What you are used to with some important rules

basic formulas :$f(x_1, \dots, x_n) = 0$ for$f \in R[X_1, \dots, X_n]$

Note : f of degree 0 possible so : for a prime number p "$p = 0$" is a formula true in R if and only if R has characteristic p.

then the conjonctions (and, $\wedge$) , disjunctions (or, $\vee$), negation and quantifiers, $\exists$,$\forall$ with the usual rules

$\forall x(x = 0) \vee \exists y \; x.y = y.x = 1$ is true in fields.

The two fundamental RULES :

First-order: Variables represent elements of the structure, not subsets

Finitary : only finitely many variables and only finite number of conjunctions or disjucntions and quantifiers in a formula

So can say R characteristic $p > 0$ ϕ_p :" $p = 0$"
cannot say R has characteristic 0 by one formula
But can find an infinite set of formulas:

$$\{\neg\phi_p) : p \text{ prime } \}.$$

$\neg(\phi_p)$: "$p \neq 0$"
such that R has characteristic 0 if and only every $\neg\phi_p$ is true in R.

One can say :
R is a ring, a field.
K is algebraically closed: need again infinitely many formulas
for each $n \geq 1$ "every unitary polynomial in one variable of degree n has a root"
$\forall y_1 \ldots \forall y_n \exists x \quad x^n + y_1 x^{n-1} + \ldots + y_{n-1} x + y_n = 0$
Can say a set is finite of cardinality n , but cannot say a set is infinite
...
No more!

Must check how to express Ax's theorem
An infinite list of formulas
$\psi_{n,m}$ every polynomial map in n variables and of degree less than m which is injective is surjective.
An easy exercise , but " long"
Now for the transfer principle : $\mathbb{C}$ is the "limit" of the $\overline{\mathbb{F}_p}$'s

Ultraproducts

We will use the "limit" construction via ultraproducts

A very useful construction! used in many different domains of mathematics

- analysis (non standard analysis)
- geometric group theory

Here to simplify :

Will only consider ultraproducts of commutative rings.

Consider $(A_i)_{i\in I}$ where each A_i is a commutative ring.
$\Pi_{i\in I}A_i$, = the cartesian product of the A_i's = the set of maps a from I in the union $\bigcup_{i\in I} A_i$ such that for every $i \in I$, $a(i) \in A_i$.
An element a of $\Pi_{i\in I}A_i$, $a = (a(i))_{i\in I}$.
Define on $\Pi_{i\in I}A_i$, addition and multiplication coordinate by coordinate:

$$(a+b)(i) := a(i) + b(i) \text{ and } (a.b)(i) := a(i).b(i).$$

$(\Pi_{i\in I}A_i, +, .)$ is a commutative ring , with $0 = (0, ..., 0, ...)$ and $1 = (1, ..., 1, ...)$.

Suppose all the A_i are fields?
$\Pi_{i \in I} A_i$ is never a field if I has at least 2 elements:
$a := (0, 1, \ldots, 1, \ldots)$ and $b := (1, 0, 0, \ldots, 0, \ldots)$ then $a.b = 0$.
To avoid this we want to identify to $0 = (0, 0 \ldots, 0, \ldots)$ all a such that $a(i) = 0$ for almos all i .
So we quotient by an ultrafilter on the set I

Definition: Let I be a non empty set an ultrafilter $\mathbb{F}$ on I is a family of subsets of I such that :

1. - $I \in \mathbb{F}$, $\emptyset \notin \mathbb{F}$
2. - if $X, Y \in \mathbb{F}$, then $X \cap Y \in \mathbb{F}$
3. - if $X \in \mathbb{F}$ and $X \subset Z \subset I$, then $Z \in \mathbb{F}$.
4. - if $X \subset I$, $X \in \mathbb{F}$ or $I \setminus X \in \mathbb{F}$.

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Remark : if $\mathbb{F}$ satisfies 1,2,3 it is a filter
If $\mathbb{F}$ satisfies also 4. it is an ultrafilter
Fact : Every filter is contained in an ultrafilter (uses Zorn's Lemma)
Very important example if I is infinite the Frechet filter $\mathfrak{F}$ = the family of cofinite subsets of I (X is cofinite if $I \setminus X$ is finite) .
If $\mathbb{F}$ is an ultrafilter, we say that $\mathbb{F}$ is non principal if $\mathbb{F} \supset \mathfrak{F}$, i.e. $\mathbb{F}$ contains the Frechet filter.

Equality almost everywhere

If $\mathbb{F}$ is an ultrafilter on I, We define an equivalence relation on $\Pi_{i\in I} A_i$, $\equiv_{\mathbb{F}}$:

$$a \equiv_{\mathbb{F}} b \text{ iff } \{i \in I;\ a(i) = b(i)\} \in \mathbb{F}.$$

Easy to check $\equiv_{\mathbb{F}}$ is an equivalence relation . The meaning : a and b are equal almost everywhere, $a(i)$ and $b(i)$ are equal for almost all i , for all i except a "small set"
$X \in \mathbb{F} = X$ is big ....
Denote $a_{\mathbb{F}}$ the class of a modulo the equivalence relation $equiv_{\mathbb{F}}$ and $\Pi A_i/\mathbb{F}$ the quotient of $(\Pi A_i)_{i\in I}$ by $equiv_{\mathbb{F}}$;
$= \Pi A_i/\mathbb{F}$ is an ultraproduct of the A_i's

Fields

The equivalence relation $\equiv_{\mathbb{F}}$ is compatible with addition and multiplication on $\Pi_{i\in I}A_i$, so $\Pi_{i\in I}A_i/\mathbb{F}$ is a commutative ring.
But now if all A_i's are fields, $\Pi_{i\in I}A_i/\mathbb{F}$ is also a field.
Can be checked by hand
But follows from a general theorem about ultraproducts and formulas

Theorem

(Théorème de Łos, for the case of commutative rings)
Let $(A_i)_{i\in I}$ be a family of commutative rings and $\mathbb{F}$ an ultrafilter on I. Let $\phi(x_1, \dots, x_n)$ be a formula (in the language of rings) and $a_1, \dots, a_n \in \Pi_{i\in I}A_i/\mathbb{F}$. Then $\phi(a_{1_{\mathbb{F}}}, \dots, a_{n_{\mathbb{F}}})$ is true in $\Pi_{i\in I}A_i/\mathbb{F}$ iff the set of $i \in I$ such that $\phi(a_1(i), \dots, a_n(i))$ is true in in A_i is an element of $\mathbb{F}$.

So something is true in the ultraproduct $\Pi_{i\in I}A_i/\mathbb{F}$ iff it is true in almost all A_i.

Note The real strength of the ultraproduct construction and of Łos theorem is that it is true in general for all first order structures and all languages

Application to Ax's theorem

Let I = the set of all primes
Let $\mathbb{F}$ be any non principal ultrafilter on I.
(If $\mathbb{F}$ is principal then the ultraproduct is isomoprhic to some A_j)
Consider $\mathcal{K} := \Pi_{p \in I} \overline{\mathbb{F}_p}/\mathbb{F}$.
Then $\mathcal{K}$ is a field, algebraically closed because every $\overline{\mathbb{F}_p}$ is algebraically closed.
What is the characteristic of $\mathcal{K}$?
Let p be any prime , for $q > p$, $\overline{\mathbb{F}}_q$ satisfies the formula $p \neq 0$,
so by Łos, $\mathcal{K}$ also satisfies $p \neq 0$,
so $\mathcal{K}$ has characteristic 0 .
$\mathcal{K}$ is an algebraically closed field of characteristic zero!
By Łos again, Ax's theorem is true in $\mathcal{K}$ because it is true in all the $\overline{\mathbb{F}_q}$.

Important Fact $\mathcal{K}$ is NOT countable, it has cardinality continuum (the proof is an interesting exercice! in fact $\Pi_{p \in I} \mathbb{F}_p$ also has cardinality continuum!)
Conclusion: $\mathcal{K}$ is an algebraically closed field of characteristic 0, of cardinality continuum :
So it is isomorphic to $\mathbb{C}$.
And so Ax's theorem is true in $\mathbb{C}$.