Théorie de Galois des équations aux différences, transcendance différentielle et combinatoire

(ou l'histoire du comment la n-ième preuve du Théorème de Hölder peut ouvrir des nouveaux horizons)

Lucia Di Vizio

Laboratoire de Mathématiques de Versailles

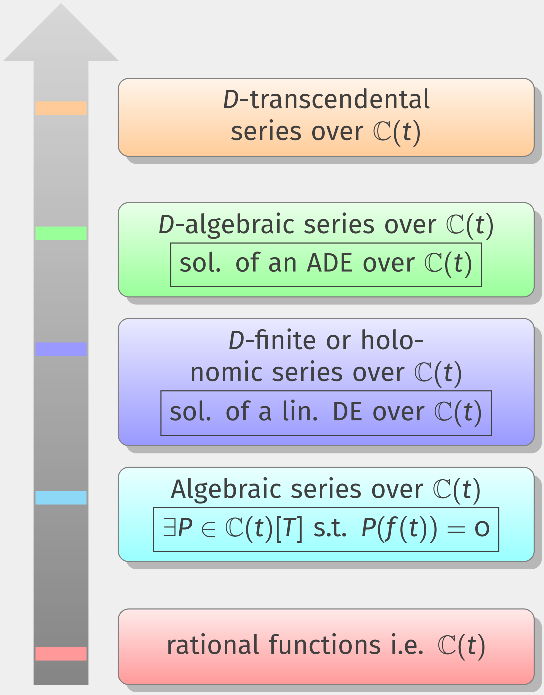
Orsay, 31/08/2020

Bell numbers	$B_n :=$ number of partitions of a set of cardinality $n \geq 1$
generating series	$B(t) := 1 + \sum_{n\geq 1} B_n t^n = 1 + t + 2t + 5t^3 + 15t^4 + 52t^5 + 203t^6 + 877t^6 + 4140t^7 + 21147t^8 + 115975t^9 + \cdots \in \mathbb{Z}[[t]]$
functional equation	$B\left(\frac{t}{1+t}\right) = tB(t) + 1$
Klazar, 2003	$B(t)$ is not the solution of an <u>ADE</u> with coefficients in $\mathbb{C}(\{t\})$

Algebraic differential equations (ADE)

$$P(t, y, y', \dots, y^{(n)}) \quad = \quad 0,$$

$$n \in \mathbb{Z}_{\geq 0},\ P(t, T_0, \dots, T_n) \in \mathbb{C}(\{t\})[T_0, \dots, T_n]$$



D-transcendental series over $\mathbb{C}(t)$
D-algebraic series over $\mathbb{C}(t)$
sol. of an ADE over $\mathbb{C}(t)$
D-finite or holonomic series over $\mathbb{C}(t)$
sol. of a lin. DE over $\mathbb{C}(t)$
Algebraic series over $\mathbb{C}(t)$
$\exists P \in \mathbb{C}(t)[T]$ s.t. $P(f(t)) = 0$
rational functions i.e. $\mathbb{C}(t)$

Examples of statements about D-transcendence of solutions of functional equations

Gamma function $\Gamma(t)$ (Euler, 1729)

$\rightsquigarrow$ meromorphic function over $\mathbb{C}$

Theorem (Bohr-Mollerup, 1922)

$\Gamma : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is the only function s.t.:
1. $\Gamma(t+1) = t\Gamma(t)$; 2. $\Gamma(1) = 1$; 3. $\log \circ \Gamma(t)$ is convex.

Theorem (Hölder, 1887)

$\Gamma(t)$ is not the solution of an ADE with coefficients in $\mathbb{C}(t)$.

Remark. Set $\Psi(t) = \frac{\Gamma'(t)}{\Gamma(t)}$. Then $\Psi(t+1) = \Psi(t) + \frac{1}{t}$.

Set $z(t) := \Gamma\left(\frac{1}{t}\right)^{-1}$. Then:

$$\Gamma(t+1) = t\Gamma(t) \Rightarrow z\left(\frac{t}{t+1}\right) = tz(t)$$

$\Rightarrow z(t)$ satisfies the homogeneous equation associated to

$$B\left(\frac{t}{t+1}\right) = tB(t) + 1$$

Klazar establishes the D-transcendence of $B(t)$ over $\mathbb{C}(\{t\})$, while Hölder theorem implies that $\Gamma\left(\frac{1}{t}\right)^{-1}$ is D-transcendental over $\mathbb{C}(t)$.

Ishizaki-Ogawara theorem

Let $q \in \mathbb{C}$, s.t. $q^n \neq 0, 1 \; \forall n \in \mathbb{Z}$,

$a(t), b(t) \in \mathbb{C}(t)$,

$f(t) \in \mathbb{C}((t)) := \mathbb{C}[[t]]\left[\frac{1}{t}\right]$ s.t. $f(qt) = a(t)f(t) + b(t)$.

Theorem (Ishizaki (1998), Ogawara (2015))

If $f(t) \notin \mathbb{C}(t)$, then $f(t)$ is D-transcendental over $\mathbb{C}(t)$.

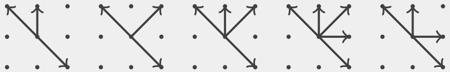
Application to lattice walks with small steps in the quarter plan

Walks in $\mathbb{N}^2$ starting at $(0,0)$, with steps in $\mathcal{S} \subset \{\rightarrow, \leftarrow, \uparrow, \downarrow, \searrow, \swarrow, \nearrow, \nwarrow\}$

$q_{\mathcal{S}}(i,j;n) :=$ number of walks of length n ending at (i,j)

$$Q_{\mathcal{S}}(x,y;t) = \sum_{i,j,n=0}^{\infty} q_{\mathcal{S}}(i,j;n) x^i y^j t^n \in \mathbb{Z}[[x,y,t]]$$

In the 5 cases (called genus zero or singular models) below:



they satisfies a functional equation as the one in Ishizaki-Ogawara theorem (after some manipulations...).

Using Ishizaki-Ogawara theorem :

- for $t \in (0,1) \setminus \overline{\mathbb{Q}}$, $Q_{\mathcal{S}}(x,y;t)$ is D-transcendental w.r.t. x and y (Dreyfus-Hardouin-Roques-Singer, 2017)

- $Q_{\mathcal{S}}(x,y;t)$ is D-transcendental with respect to t (Dreyfus-Hardouin, 2019)

- generalization to the D-transcendency for some $t \in (0,1) \cap \overline{\mathbb{Q}}$ (Bostan, 2019)

[It is only a small fragment of a long story…]

General result on D-transcendence (Adamczewski-Dreyfus-Hardouin, 2019)

Theorem (Adamczewski-Dreyfus-Hardouin, 2019)

Let $f \in \mathbb{C}((t))$ be a formal power series satisfying a linear functional equation of the form

$$\alpha_0 y + \alpha_1 \tau(y) + \cdots + \alpha_n \tau^n(y) = 0,$$

where $\alpha_i \in \mathbb{C}(t)$ and τ is one of the following operators:

- $\tau(f(t)) = f\left(\frac{t}{t+1}\right)$;
- $\tau(f(t)) = f(qt)$ for some $q \in \mathbb{C}^*$, not a root of unity;
- $\tau(f(t)) = f(t^m)$ for some positive integer m.

Then either $f \in \mathbb{C}(t^{1/n})$ for some positive integer n or it is D-transcendental over $\mathbb{C}(t)$.

Galois theory of functional equations

$\mathbb{K}$ field (of characteristic zero)

$\tau : \mathbb{K} \to \mathbb{K}$ automorphism

$C := \mathbb{K}^\tau := \{f \in \mathbb{K} : \tau(f) = f\}$ algebraically closed (τ-constants)

∂ derivative of $\mathbb{K}$ commuting to τ.

Example

$\mathbb{K} = \mathbb{C}((t))$ and $\tau : f(t) \mapsto f\left(\frac{t}{t+1}\right) \Rightarrow \mathbb{C}((t))^\tau = \mathbb{C}$.
$\Rightarrow \partial := t^2 \frac{d}{dt}$

Remark.

- τ induces an automorphism of $\mathbb{C}(t)$ and of $\mathbb{C}(\{t\})$, hence $\mathbb{C}(t)^\tau = \mathbb{C}(\{t\})^\tau = \mathbb{C}$.
- $\tau\left(\frac{1}{t}\right) = \frac{1}{t} + 1$.

$\tau(\vec{y}) = A\vec{y}$, $A \in \mathrm{GL}_\nu(\mathbb{K})$, $\vec{y}$ vector of unknown functions

∂-Picard-Vessiot ring for $\tau(\vec{y}) = A\vec{y}$ over $\mathbb{K}$

$\exists!$ a $\mathbb{K}$-algebra $\mathcal{R}$ such that

1. $\exists$ an extension of τ and ∂, preserving the commutation;
2. there exists $Y \in \mathrm{GL}_\nu(\mathcal{R})$ such that $\tau(Y) = AY$;
3. $\mathcal{R} = \mathbb{K}\left[Y, \partial(Y), \partial^2(Y), \ldots, \det Y^{-1}\right]$;
4. $\mathcal{R}$ does not have any non-trivial proper ideal invariant by τ.

$\Rightarrow$ ■ $\mathcal{R}^\tau = C$.

Notation. $R_0 = \mathbb{K}[Y, \det Y^{-1}] \subset \mathcal{R}$

The Galois group

$$Gal(R_0/\mathbb{K}) := \{\varphi : R_0 \to R_0 : \varphi|\mathbb{K} = id_{\mathbb{K}}, \varphi \circ \tau = \tau \circ \varphi\}$$

$$\Rightarrow \; \tau(\varphi(Y)) = A\varphi(Y)$$

$$\Rightarrow \; Gal(R_0/\mathbb{K}) \to \mathrm{GL}_\nu(C), \varphi \mapsto \varphi(Y)^{-1}Y.$$

"Theorem"

When there are many algebraic relations among the entries of Y the image of $Gal(R_0/\mathbb{K})$ in $\mathrm{GL}_\nu(C)$ is small.

$Gal(R_0/\mathbb{K})$ is an algebraic subgroup of $\mathrm{GL}_\nu(C)$,
$\dim_C Gal(R_0/\mathbb{K}) = \mathrm{deg.tr.}_{\mathbb{K}} R_0$.

Theorem

$r \in R_0$ is such that $\varphi(r) = r \; \forall \varphi \in Gal(R_0/\mathbb{K}) \Leftrightarrow r \in \mathbb{K}$.

The case of an inhomogeneous order 1 equation

$\tau(y) = y + b, b \in \mathbb{K} \rightsquigarrow \mathcal{R} := \mathbb{K}[w, \partial(w), \dots]$, with $\tau(w) = w + b$

$$\begin{cases} \tau(w) = w + b \\ \tau(\partial(w)) = \partial(w) + \partial(b) \\ \dots \\ \tau(\partial^n(w)) = \partial^n(w) + \partial^n(b) \end{cases} \rightsquigarrow R_n = \mathbb{K}[w, \partial(w), \dots, \partial^n(w)] \subset \mathcal{R}$$

$\Rightarrow \forall \varphi \in Gal(R_n/\mathbb{K})$, $\exists c_{\varphi,h} \in C$ s.t. $\varphi(\partial^h(w)) = \partial^h(w) + c_{\varphi,h}$

$\Rightarrow Gal(R_n/\mathbb{K}) \subset C^{n+1}$

w is a D-algebraic solution of $\tau(w) = w + b$

$\Rightarrow$ $\exists n \geq 0$ s.t. $w, \partial(w), \ldots, \partial^n(w)$ are algebraically dependent

$\Rightarrow$ $\dim_C Gal(R_n/\mathbb{K}) \leq n$ and $Gal(R_n/\mathbb{K}) \subset \mathbb{C}^{n+1}$

$\Rightarrow$ $\exists \alpha_0, \ldots, \alpha_n \in C$ s.t. $\sum_{h=0}^{n} \alpha_h c_{\varphi,h} = 0 \ \forall \varphi$

$\Rightarrow$ $\exists \alpha_0, \ldots, \alpha_n \in C$ s.t. $\varphi(\sum_{h=0}^{n} \alpha_h \partial^h(w)) = \sum_{h=0}^{n} \alpha_h \partial^h(z) \ \forall \varphi$

$\Rightarrow$ $\exists \alpha_0, \ldots, \alpha_n \in C$ s.t. $g := \sum_{h=0}^{n} \alpha_h \partial^h(w) \in \mathbb{K}$

$\Rightarrow$ $\exists \alpha_0, \ldots, \alpha_n \in C$ and $g \in \mathbb{K}$ s.t.

$$\tau(g) - g = \alpha_0 b + \alpha \partial(b) + \cdots + \alpha_n \partial^n(b)$$

Hölder theorem

$\tau : f(t) \mapsto f(t+1)$, $\partial = \frac{d}{dt}$, $b = \frac{1}{t}$, $g \in \mathbb{C}(t) \Rightarrow \ldots$

$$\tau(y) = ay + b \text{ with } a, b \in \mathbb{K} \quad \Rightarrow \quad \tau(\vec{y}) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \vec{y}$$

$$\exists w, z \in \mathcal{R} \text{ s.t. } \begin{cases} \tau(z) = az \\ \tau(w) = aw + b \end{cases} \quad \Rightarrow \quad Y = \begin{pmatrix} z & w \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathcal{R})$$

Corollary (Hardouin-Singer, 2008)

If w is D-algebraic over $\mathbb{K}$ then z is D-algebraic over $\mathbb{K}$.

$\Leftrightarrow$

If z is D-transcendental over $\mathbb{K}$ then w is D-transcendental over $\mathbb{K}$.

Hölder theorem $\Rightarrow$ D-transcendence of $B(x)$ over $\mathbb{C}(t)$

Klazar theorem is an instance of a general behavior

Theorem (Bostan, D.V., Rachel, 2020)

Let $a(t), b(t) \in \mathbb{C}(t)$ and let $w(t) \in \mathbb{C}((t)) \setminus \mathbb{C}(t)$ satisfy

$$w\left(\frac{t}{1+t}\right) = a(t)w(t) + b(t)$$

$\Rightarrow$ w is D-transcendental over $\mathbb{C}(\{t\})$.

In particular $B(t)$ is D-transcendental over $\mathbb{C}(\{t\})$.

$F(x,t)=\sum_{n\geq 0} P_n(x)t^n \in \mathbb{C}[x][[t]]$ s.t. $\exists \widetilde{a},\widetilde{b} \in \mathbb{C}(x,t)$:

$$F(x,t)=\widetilde{a}\left(x,\frac{t}{1+t}\right)\cdot F\left(x,\frac{t}{1+t}\right)+\widetilde{b}\left(x,\frac{t}{1+t}\right)$$

polynomial $P_n(x)$	EGF $\sum_{n\geq 0} P_n(x)\frac{t^n}{n!}$	$\widetilde{a}$	$\widetilde{b}$
Bernoulli $B_n(x)$	$\frac{t}{\mathrm{e}^t-1}\cdot\exp(xt)$	$1-t$	$\frac{t(1-t)}{(xt-1)^2}$
Imschenetsky $S_n(x)$	$\frac{1}{\mathrm{e}^t-1}\cdot(\exp(xt)-1)$	$1-t$	$\frac{xt(1-t)}{1-xt}$
Euler $E_n(x)$	$\frac{2}{\mathrm{e}^t+1}\cdot\exp(xt)$	$-(1-t)$	$\frac{2(1-t)}{1-xt}$
Fubini $F_n(x)$	$1/(1-x(\mathrm{e}^t-1))$	$\frac{x+1}{x}\cdot(1-t)$	$\frac{t-1}{x}$
Bell-Touchard $\phi_n(x)$	$\exp\left(x(\mathrm{e}^t-1)\right)$	$\frac{1}{x}\cdot\frac{1-t}{t}$	$\frac{t-1}{xt}$
Mahler $s_n(x)$	$\exp\left(x(1+t-\mathrm{e}^t)\right)$	$\frac{(xt-1)(1-t)}{xt}$	$\frac{1-t}{xt}$
Actuarial $a_n^{(\beta)}(x)$	$\exp\left(-x\mathrm{e}^t+\beta t+x\right)$	$\frac{(1-t)(\beta t-1)}{xt}$	$\frac{1-t}{xt}$

EGF := exponential generating functions

Conjecture (Pak-Yeliussizov, 2018)

$\sum_{n\geq 0} a_n t^n, \sum_{n\geq 0} a_n \frac{t^n}{n!} \in \mathbb{C}[[t]]$ are D-algebraic over $\mathbb{C}(t)$

$\Rightarrow$ both are D-finite

[$\Leftrightarrow$ both $(a_n)_{n\geq 0}$ and $\left(\frac{a_n}{n!}\right)_{n\geq 0}$ satisfiy a lin. recurrence with polynomial coeff. in n]

All the series in the previous table are D-transcendental and have D-algebraic, but not D-finite, EGF!!
$\rightsquigarrow$ they are instances of the above conjecture.

Thanks!

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