Étude combinatoire et probabiliste d’arbres

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Introduction
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- $V$ is a set, either finite or countable, called vertices;
- $E \subseteq V^2$ is a subset of pairs of elements in $V$, called edges.
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**Definition.** A **tree** is a graph which is **connected** and contains no **cycle**.
Graphs appear in many contexts of pure and applied math; trees

- can serve as a toy model of more complicated networks (due to the absence of cycle), or locally approximate general graphs with long cycles;
- in combinatorics, they are numerous bijections between trees and other objects;
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bonds, made of carbon atoms (C) with valency 4 and
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  H   H   H
 /   /   /
H --- C --- C --- C --- H
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\begin{array}{ccc}
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propanol:  
\[
\begin{array}{ccc}
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The question is then to count isomeres of alkanes
and alcohol with a given number of carbon atoms.
The former correspond to **unlabelled** trees with
constraints on the degrees; the latter correspond to
such rooted trees.
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**Definition.** A **rooted tree** is a tree with a distinguished vertex called its root.
We draw the edges as arrows pointing away from the root.

**Definition.** A **rooted plane tree** (also called “Catalan trees”) is a rooted tree in which all the oriented edges coming from the same vertex are ordered from left to right.
We represent such a tree as a genealogical tree.
I) Enumeration of trees

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**Theorem.** (Cayley’s formula) There are $n^{n-2}$ trees on 1, . . . , $n$.

**Proof.** (by Jim Pitman) Proof by double counting: Let us count in two different ways the number of rooted (but not plane) trees of size $n$ where the edges are labelled from 1 to $n - 1$; one way to relate it to Cayley trees and the other one to obtain the explicit cardinal.
On the one hand, take a Cayley tree, choose one of the $n$ vertices to be the root and choose one possible labelling of the edges, then the number $L_n$ of rooted labelled trees

$$L_n = \text{Cay}_n \times n \times (n - 1)!$$
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At the step \( k = 1, \ldots, n - 1 \), we have \( n - k + 1 \) rooted trees, we choose one of the \( n \) vertices and we draw the edge number \( k \) from this vertex to the root of another tree.
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This yields
\[
L_n = \prod_{k=1}^{n-1} n(n-k) = n^{n-1} \times (n - 1)!
\]

We conclude that \( \text{Cay}_n = \frac{L_n}{n!} = n^{n-2} \). \( \square \)
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First proof by **generating functions**: let $|T|$ denote the number of vertices of a tree, let $T$ be the set of all rooted plane trees. Proof by induction on the size? Removing the root of a tree produces a sequence of, say $k$ trees, with a smaller size. However one needs to put in the equation the precise size of these subtrees (OK, not so hard in this example).
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Instead define the formal power series

\[
F : x \mapsto \sum_{n \geq 0} x^n \text{Cat}_n = \sum_{T \in T} x^{|T|^{-1}}
\]

and try to determine its coefficients. Very powerful method which takes three steps:

1. Show that \( F \) satisfies some functional equation (usually easy);
2. Solve the equation to get a formula for \( F(x) \) (usually difficult, sometimes no closed formula);
3. Deduce the coefficient in the series development of \( F \) (not always possible to get a closed formula, sometimes asymptotic ones).
Remove the root of $T$ to obtain $k$ trees: $T_1, \ldots, T_k$ from left to right, then $|T| = 1 + \sum_{i=1}^{k} |T_i|$. 
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$$= x \sum_{k \geq 0} \sum_{t_1, \ldots, t_k \in T} x^{|t_i|}$$

$$= x \sum_{k \geq 0} (xF(x))^k$$

$$= \frac{x}{1 - xF(x)}.$$
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We obtain

$$xF(x)^2 - F(x) + 1 = 0, \quad \text{so} \quad F(x) = \frac{1 - \sqrt{1 - 4x}}{2x}.$$
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Recall that $(1 + z)^\alpha = 1 + \sum_{k \geq 1} \frac{\alpha(\alpha-1) \cdots (\alpha-k+1)}{k!} z^k$, then with $\alpha = 1/2$ and $z = -4x$ we get

$$\sqrt{1 - 4x} = 1 - 2x \sum_{k \geq 1} \frac{(2(k-1))!}{k!(k-1)!} x^{k-1} = 1 - 2x \sum_{k \geq 1} \frac{1}{k} \left( 2(k-1) \right) x^{k-1}.$$
3) Dissections of polygons

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It can be decomposed as a polygon with size \(k + 1\), to which we attach dissections of a \((p_1 + 1)\)-gon, ... \((p_k + 1)\)-gon, with \(p_1 + \cdots + p_k = n\).
Let $|D|$ be the number of vertices minus 1 of the dissection $D$ and let $f$ be the size of the face adjacent to the left of the segment between 1 and $\exp(2i \pi /(|D| + 1))$. 
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$$G(x) = \sum_{D \in D} x^{|D|}$$

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We thus obtain $2G(x)^2 - (x + 1)G(x) + x = 0$ so

$$G(x) = \frac{x + 1 - \sqrt{x^2 - 6x + 1}}{4}.$$

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Asymptotic enumeration

**Transfer Lemma.** Suppose that $f$ is analytic in the open domain $\Delta$: $f(z) = \sum_{n \geq 0} a_n z^n$, and that for a certain $\alpha \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$, we have

$$f(z) \sim (1 - z)^{-\alpha} \quad \text{as} \quad z \to 1, z \in \Delta.$$ 

Then

$$a_n \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \quad \text{as} \quad n \to \infty,$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ is Euler’s Gamma function.

**Remark.** If for some $c, r > 0$ we have instead as $z \to r$,

$$f(z) \sim \frac{c}{(r - z)\alpha},$$

then by considering $g(z) = r^\alpha f(rz)/c$, we obtain

$$a_n \sim \frac{c}{r^\alpha \Gamma(\alpha)} r^{-n} n^{\alpha-1}.$$

(Also valid if $r \in \mathbb{C} \setminus \{0\}$ after a rotation.)
Application to dissections. Recall that $G(z) = \sum_{n\geq 1} z^n \#D_{n+1} = \frac{z+\sqrt{z^2-6z+1}}{4}$. Let

\[ f(z) = \sqrt{z^2 - 6z + 1}, \]
then if $f(z) = \sum_n a_n z^n$, then $\#D_n = -a_{n-1}/4$ for all $n \geq 3$.

Note that $z^2 - 6z + 1 \in \mathbb{R}_-$ if and only if $z \in [3 - 2\sqrt{2}, 3 + 2\sqrt{2}]$ and recall that the complex square-root is analytic outside non-positive numbers. Further for $\varepsilon \in \mathbb{C} \setminus \mathbb{R}_-$ small, we have

\[ f(3 - 2\sqrt{2} - \varepsilon) = \sqrt{\varepsilon^2 - 2(3 - 2\sqrt{2})\varepsilon + 6\varepsilon} \sim 2\sqrt{2\varepsilon}. \]

We have therefore $f(z) \sim \frac{c}{(r-z)^{\alpha}}$, where $c = 2\sqrt{2}$, $r = 3 - 2\sqrt{2}$, $\alpha = -1/2$, and so

\[ a_n \sim 2^{1/4} (3 - 2\sqrt{2})^{1/2} \frac{(3 - 2\sqrt{2})^{-n} n^{-3/2}}{\Gamma(-1/2)}. \]

Note that $(3 - 2\sqrt{2})(3 + 2\sqrt{2}) = 9 - 8 = 1$ and we know that $\Gamma(-1/2) = -2\sqrt{\pi}$ so

\[ \#D_n = -\frac{a_{n-1}}{4} \sim \frac{2^{1/4} (3 - 2\sqrt{2})^{1/2}}{4\sqrt{\pi}} \frac{(3 + 2\sqrt{2})^{n-1} n^{-3/2}}{(3 + 2\sqrt{2})^n n^{-3/2}} = \frac{\sqrt{99\sqrt{2} - 140}}{4\sqrt{\pi}} (3 + 2\sqrt{2})^n n^{-3/2}. \]
II) Coding Catalan trees by paths

1) Canonical ordering of Catalan trees

The **Depth-First Search algorithm**:

Visit the vertices as follows:
1. Start at the root of the tree.
2. When sitting at a vertex, move next to its left-most unvisited child if there is any; otherwise go back to its parent.
3. Stop when returning to the root after visiting all its children.

Then number the vertices according to their order of first visit.
2) Two bijections with Catalan trees

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2) Non-crossing partitions of 1, . . . , $n$:

$$= \{\{1, 2, 8, 9\}, \{3, 7\}, \{4, 5, 6\}, \{10\}, \{11\}, \{12, 13\}\}$$

All children with the same parent in the tree form a block of the partition.
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Given a rooted plane tree $T$ with $n$ vertices, for every $0 \leq i \leq n - 1$, let $w_T(i)$ denote the number of children minus 1 of the vertex number $i$. Then define $W_T(0) = 0$ and for $1 \leq k \leq n$,

$$W_T(k) = w_T(0) + \cdots + w_T(k-1).$$
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**Example.** Here $(w_T(i))_{0 \leq i \leq n-1}$ is given by

$$(3, -1, 1, 2, -1, -1, -1, -1, -1, 0, 0, 1, -1, -1),$$

so $W_T$ reads:
Proposition. Fix \( n \geq 1 \) and \( T \) a (Catalan) tree with \( n \) vertices; then

1. \( w_T(k) \geq -1 \) for each \( 0 \leq k < n \);
2. \( W_T(k) \geq 0 \) for each \( 0 \leq k < n \);
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Moreover the the construction can be inverted so the map \( T \mapsto w_T \) is a bijection.
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**Proof.** The first point is immediate since $w_T(k)$ is the number of children minus 1 of the vertex number $k$. Next, for each $k < n$, $W_T(k) = w_T(0) + \cdots + w_T(k-1)$ is the sum of the number of children of the vertices $0, \ldots, k-1$ minus $k$; but the vertices $1, \ldots, k$ are amongst these children so indeed $W_T(k) \geq 0$. For $k = n$, we sum the number of children of every individual, minus $n$; now every every individual is the child of another one, except the root, so indeed $W_T(n) = -1$. 
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Finally, the fact that $T \mapsto w_T$ is a bijection can be checked by induction on $n$. For
$n = 1$ we have only one tree and one Łukasiewicz path. Then for any $n \geq 2$, assuming
that the maps are bijective for any size $k \leq n - 1$, then, first the value $w_T(0) = W_T(1)$
gives the number of children of $T$ (just add $1$); further, setting $i_0 = 1$ and
$i_k = \inf\{j > i_{k-1} : W_T(j) < W_T(i_{k-1})\}$ for $1 \leq k \leq w_T(0)$, then the sequence
$(w_T(j); i_{k-1} \leq j < i_k)$ codes the subtree of the descendants of the $k$’th child of the
root. □
4) The cycle lemma
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Notation. For \( n \geq 1 \) set

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\mathcal{S}_n = \{(w_0, \ldots, w_{n-1}) \in \{-1, 0, 1, 2, \ldots\}^n : w_0 + \cdots + w_{n-1} = -1\},
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and

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so we just proved that \( \overline{\mathcal{S}}_n \) is in one-to-one correspondence with the set \( \mathcal{T}_n \) of Catalan trees with \( n \) vertices.
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Proposition. The cardinal of \( S_n \) equals \( \binom{2n-2}{n-1} \) for each \( n \geq 1 \).
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Proposition. The cardinal of $S_n$ equals $\left(\binom{2n-2}{n-1}\right)$ for each $n \geq 1$.

Proof. By adding 2 to each $w$ we have that

$$\#S_n = \#\{(x_0, \ldots, x_{n-1}) \in \{1, 2, \ldots\}^n : x_0 + \cdots + x_{n-1} = 2n - 1\}.$$

Then by considering $X_k = x_0 + \cdots + x_{k-1}$ we obtain

$$\#S_n = \#\{0 = X_0 < X_1 < \cdots < X_{n-1} < X_n = 2n - 1\} = \binom{2n-2}{n-1},$$

since we only need to fix the value of $1 \leq X_1 < \cdots < X_{n-1} \leq 2n - 2$. □
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Then for every $i \in \mathbb{Z}$, define the cyclic shift of $w$ at time $i$ by

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![Diagram of a sequence of points](image-url)
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![Image of a graph with a red line labeled $W^{(3)}$]
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$$i_\ast = \min\{1 \leq i \leq n : w_0 + \cdots + w_{i-1} = \min_{1 \leq k \leq n} w_0 + \cdots + w_{k-1}\}.$$
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\[ w_0 + \cdots + w_{k-1} > w_0 + \cdots + w_{i^*-1}, \quad \text{for} \quad 1 \leq k \leq i^*-1, \]

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\[ w_{i^*} + \cdots + w_{k-1} \geq 0, \quad \text{for} \quad i^* + 1 \leq k \leq n. \]
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This shows that if $1 \leq j \leq n - i^*$, then

$$w_0^{(i^*)} + \cdots + w_{j-1}^{(i^*)} = w_{i^*} + \cdots + w_{i^*+j-1} \geq 0,$$

and if $n - i^* + 1 \leq j \leq n - 1$, then

$$w_0^{(i^*)} + \cdots + w_{j-1}^{(i^*)} = w_{i^*} + \cdots + w_{n-1}^{(i^*)} + w_0 + \cdots + w_{i^*+j-n-1} - 1 > -1.$$

> $w_0 + \cdots + w_{i^*-1}$
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\begin{align*}
\sum_{i=0}^{k-1} w_i &> \sum_{i=0}^{i_*-1} w_i, \quad \text{for } 1 \leq k \leq i_* - 1, \\
\text{and} \quad w_{i_*} + \cdots + w_{k-1} &\geq 0, \quad \text{for } i_* + 1 \leq k \leq n.
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\sum_{i=0}^{j-1} w_{i_*} > \sum_{i=0}^{i_*-1} w_{i_*} - (w_{i_*} + \cdots + w_{i_*-1}) \leq -1.
\]

Similarly, if \( i < i_* \), then

\[
\sum_{i=0}^{i_*-1} w_{i_*-i-1} = w_i + \cdots + w_{i_*-1} \leq -1,
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and if \( i > i_* \), then

\[
\sum_{i=0}^{n-i+i_*-1} w_{i_*-i-1} = w_i + \cdots + w_{n-1} + w_0 + \cdots + w_{i_*-1} = -1 - (w_{i_*} + \cdots + w_{i-1}) \leq -1.
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Proof. Let us prove that the map

$$w \mapsto (w^{(i*)}, i*)$$

is a bijection between $S_n$ and $\overline{S}_n \times \{1, \ldots, n\}$. Since $w = (w^{(i)})^{(n-i)}$ for every $i$, then we only need to prove that for any $(w, i) \in \overline{S}_n \times \{1, \ldots, n\}$ we have

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Since $w \in \overline{S}_n$, then for every $1 \leq k \leq n - 1$ we have

$$w_0 + \cdots + w_{k-1} \geq 0, \quad \text{and} \quad w_k + \cdots + w_{n-1} < 0.$$ 

Thus, if $j > i$, then

$$w_0^{(n-i)} + \cdots + w_{j-1}^{(n-i)} = \underbrace{w_{n-i} + \cdots + w_{n-1}}_{\geq 0} + \underbrace{w_0 + \cdots + w_{j-i-1}}_{\geq 0} = w_0^{(n-i)} + \cdots + w_{i-1}^{(n-i)} \geq 0$$

and if $j < i$, then

$$w_0^{(n-i)} + \cdots + w_{j-1}^{(n-i)} = w_{n-i} + \cdots + w_{n-(i-j)-1} > w_{n-i} + \cdots + w_{n-1} = w_0^{(n-i)} + \cdots + w_{i-1}^{(n-i)}.$$ 

$\square$
5) Application in probability
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A **binary** tree is a tree in which every individual has either 0 or 2 offsprings. Note that there necessarily are \( n \) individuals with 2 children and \( n + 1 \) with 0 child for some \( n \geq 0 \).
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Then its Łukasiewicz is obtained by taking the cyclic shift of a path containing $n$ increments $+1$ and $n + 1$ increments $-1$. Since there are ${2n+1 \choose n}$ ways of placing these increments, we see that the number of such binary trees is

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Exercise. Find an explicit bijection between plane trees with \( n + 1 \) vertices and binary trees with \( n + 1 \) leaves.
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How to sample a **binary** tree uniformly at random?

1. Write a list containing “+1” \( n \) times and “−1” \( n + 1 \) times in a uniformly random order (via a random permutation for example).
2. Take the cyclic shift to get an element of \( S_{2n+1} \).
3. Build the tree via the DFS algorithm.
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How to sample a general plane tree with \( n \) vertices uniformly at random? Here the \( d_i \)’s are not fixed in advance so not only the order but also the values of the increments of the first path in \( S_n \) have to be sampled randomly, but then we can follow Step 2 and Step 3.
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It turns out uniform random trees are particular cases of **Bienaymé–Galton–Watson** trees!
III) Bienaymé–Galton–Watson trees
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1) Probability of extinction
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Take a French or British lord, is his family name going to disappear or is there a chance that it remains forever?

Model of BGW: fix $\mu$ a probability on \{0, 1, 2, $\ldots$\} and suppose that, independently of each other, every (male) individual has a random number of offsprings sampled according to $\mu$. The genealogical tree is called a **BGW tree**; if $Z_n$ denotes the number of individual at generation $n$, then the sequence $(Z_n)_{n \geq 1}$ is called a **BGW process**.

The previous question is: can $Z_n$ remain $\geq 1$ forever?
III) Bienaymé–Galton–Watson trees

1) Probability of extinction

Take a French or British lord, is his family name going to disappear or is there a chance that it remains forever?

Model of BGW: fix \( \mu \) a probability on \( \{0, 1, 2, \ldots \} \) and suppose that, independently of each other, every (male) individual has a random number of offsprings sampled according to \( \mu \). The genealogical tree is called a **BGW tree**; if \( Z_n \) denotes the number of individual at generation \( n \), then the sequence \( (Z_n)_{n \geq 1} \) is called a **BGW process**.

The previous question is: can \( Z_n \) remain \( \geq 1 \) forever?

Although not so realistic for humans, this model describes well e.g. the growth of a population of cells or other simple organisms, or the spread of an epidemic, or the replication of DNA molecules, etc. It also arises often in phenomena that exhibit a “branching property” (see later).
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**Trivial cases.** We will discard two trivial cases:

- If $\mu(0) = 0$, then the tree is always infinite and $(Z_n)$ is nondecreasing.
- If $\mu(0) \neq 0$ and $\mu(0) + \mu(1) = 1$, then the tree is always finite: each individual has only one children until (after a geometric number of generations), one has no child.
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In general, the value is related to the moment generating function of $\mu$ given by

\[ \varphi(s) = E[s^{Z_1}] = \sum_{k \geq 0} s^k \mu(k), \quad s \in [0, 1], \]

as well as its mean

\[ m_\mu = E[Z_1] = \sum_{k \geq 0} k \mu(k) = \varphi'(1-). \]
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**Theorem.** The probability of extinction $p_e$ is the smallest fixed point of $\varphi$:
\[ p_e = \inf\{s \in [0, 1] : s = \varphi(s)\}. \]
It equals 1 if and only if $m_\mu \leq 1$. 

First, about the fixed points: with the trivial cases discarded, the function $\phi$ is continuous, strictly convex and increasing, with $\phi(0) = \mu(0) > 0$ and $\phi(1) = 1$.

Therefore $1$ is a fixed point and there can be at most one other fixed point.

Furthermore, one can check that this occurs if and only if $m \mu = \phi'(1-) > 1$.

We denote the smallest fixed point by $s_\star$. 

\[
\begin{align*}
\text{If } m \mu < 1 & \quad \text{then } s_\star = 1 \\
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Proof Let \( \varphi_n \) be the moment generating function of \( Z_n \) for every \( n \geq 1 \). Since \( Z_1 \) has the law \( \mu \), we have

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\varphi_1(s) = \mathbb{E}[s^{Z_1}] = \varphi(s), \quad s \in [0, 1].
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Further, each of the $Z_n$ individuals at generation $n$ gives birth independently to a random number of offsprings with the same law as $Z_1$. This yields for $s \in [0, 1]$,

$$\varphi_{n+1}(s) = \sum_{k \geq 0} \mathbb{P}(Z_n = k) \mathbb{E}[s^{Z_1}]^k = \sum_{k \geq 0} \mathbb{P}(Z_n = k) \varphi(s)^k = \mathbb{E}[\varphi(s)^{Z_n}] = \varphi_n(\varphi(s)).$$
Proof

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By induction we deduce that

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\varphi_n = \varphi \circ \cdots \circ \varphi \quad n \text{ times}.
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Let $u_n = \varphi_n(0) = P(Z_n = 0)$ for every $n$. Since $\{Z_n = 0\} \subset \{Z_{n+1} = 0\}$, then $(u_n)$ is increasing, so it converges to a limit $s$, and

$$p_e = P\left(\bigcup_{n \geq 1} Z_n = 0\right) = \lim_{n \to \infty} P(Z_n = 0) = s.$$
**Proof** Let $\varphi_n$ be the moment generating function of $Z_n$ for every $n \geq 1$. Since $Z_1$ has the law $\mu$, we have

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Since $\varphi_{n+1} = \varphi \circ \varphi_n$, by continuity, necessarily $s = \varphi(s)$. Note that $\varphi(0) > 0$ so the smallest fixed point is $s_\star > 0$; since $\varphi$ is increasing, then $u_1 = \varphi(0) < \varphi(s_\star) = s_\star$. By induction, this implies that $u_n < s_\star$ for all $n \geq 1$ and therefore $s = s_\star$.  \[\square\]
2) The Erdős–Rényi random graphs.
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The degree of 1 (or any other vertex) has the binomial $\text{Bin}(n - 1, p)$ distribution, so if $p = p_n \sim c/n$, in the large $n$ limit we get the Poisson distribution with parameter $c$. 
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Next, for each neighbour of 1, its number of other neighbours has the law $\text{Bin}(n - 1, c/n) \approx \text{Po}(c)$. Further, there is little chance that one of these neighbours are also neighbours of 1.
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Next, for each neighbour of \( 1 \), its number of other neighbours has the law \( \text{Bin}(n - 2, c/n) \approx \text{Po}(c) \). Further, there is little chance that one of these neighbours are also neighbours of \( 1 \).

Conclusion: we see appearing the model of BGW tree with offspring distribution \( \text{Po}(c) \) to describe locally the geometry of \( G(n, c/n) \). Related to the previous theorem, the behaviour is very different according as whether \( c < 1 \), or \( c = 1 \), or \( c > 1 \).
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If $\mu$ is an offspring distribution with mean $m_\mu \leq 1$, then the law $P_\mu$ of a $\mu$-BGW tree is as follows: for every finite rooted plane tree $T$, with, say $n$ vertices whose offspring numbers in DFS order are $c_0, \ldots, c_{n-1}$, we have

$$P_\mu(T) = \prod_{i=0}^{n-1} \mu(c_i).$$
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We will focus on offspring distributions with mean $m_\mu = 1$, which are called critical. Note that the size of such a $\mu$-BGW is random, one is then interested in conditioning the tree to have size $|T| = n$, and study its properties as $n \to \infty$. 
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Depending on $\mu$, we cannot always condition the tree to have size $n$ for every $n$; for example, if $\mu(0) + \mu(2) = 1$ (binary tree), then $n$ must be odd. Nevertheless, this is not an issue if $\mu(1) \neq 0$, and more generally one can show the following result.
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**Proposition.** Suppose that $\gcd(\{k \geq 1 : \mu(k) \neq 0\}) = 1$, then there exists $n_0$ such that for every $n \geq n_0$, one can condition the tree to have size $n$. 
1. Take $\mu(0) = \mu(2) = 1/2$, fix $2n + 1 \geq 1$. What is the conditional law $P_{\mu}(\cdot | \cdot \in T_{2n+1})$?
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Here if \( T \in T_n \), then

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Again, this does not depend on \( T \), so now the law \( P_\mu(\cdot \mid \cdot \in T_n) \) is the uniform distribution on Catalan trees with \( n \) vertices.
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Again, this does not depend on \( T \), so now the law \( P_\mu(\cdot \mid \cdot \in T_n) \) is the uniform distribution on Catalan trees with \( n \) vertices.

3. Slightly different: take \( \mu(k) = e^{-1}/k! \) the Poisson distribution and sample \( T_n \) from the law \( P_\mu(\cdot \mid \cdot \in T_n) \). Then label the vertices uniformly at random from 1 to \( n \) and forget both the root and the orientation around each vertex, then the resulting tree is a uniform random Cayley tree!

Hint: if a vertex has \( k \) children, there are \( k! \) possible ways to order them.
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Define another random path \((W(k))_{k \geq 0}\) such that \( W(0) = 0 \) and the increments \( W(k + 1) - W(k) \) are i.i.d. with the law \( P(W(1) = i) = \mu(i + 1) \) for all \( i \geq -1 \). Let \( \zeta = \inf\{k \geq 1 : W(k) = -1\} \).
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**Lemma.** The two paths $(W_T(k))_{0 \leq k \leq |T|}$ and $(W(k))_{0 \leq k \leq \zeta}$ have the same law. In particular, $|T|$ and $\zeta$ have the same law.
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**Lemma.** The two paths $(W_T(k))_{0 \leq k \leq |T|}$ and $(W(k))_{0 \leq k \leq \zeta}$ have the same law. In particular, $|T|$ and $\zeta$ have the same law.

**Proof.** Fix any possible sequence $(w_0, \ldots, w_{n-1}) \in \overline{S}_n$, and let $t$ be the tree associated with this path, so its $i$'th vertex has $c_i = w_i + 1$ children. Then on the one hand

$$P_\mu(W_T(k) - W_T(k - 1) = w_k \text{ for all } k) = P_\mu(T = t) = \prod_{i=0}^{n-1} \mu(c_i) = \prod_{i=0}^{n-1} \mu(w_i + 1),$$

which coincides with $P_\mu(W(k) - W(k - 1) = w_k \text{ for all } k)$. \qed
Corollary. For every $n \geq 1$,

$$\Pr_\mu(|T| = n) = \Pr_\mu(\zeta = n) = \frac{1}{n} \Pr_\mu(W(n) = -1).$$
Corollary. For every $n \geq 1$,

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Proof. Let us write $\Delta W(k)$ for $W(k + 1) - W(k)$ for all $0 \leq k \leq n - 1$, then

$$\mathbb{P}_\mu(\zeta = n) = \mathbb{P}_\mu(\Delta W \in \mathcal{S}_n),$$

whereas

$$\mathbb{P}_\mu(W(n) = -1) = \mathbb{P}_\mu(\Delta W \in \mathcal{S}_n).$$
Corollary. For every $n \geq 1$,

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Proof. Let us write $\Delta W(k)$ for $W(k + 1) - W(k)$ for all $0 \leq k \leq n - 1$, then $P_\mu(\zeta = n) = P_\mu(\Delta W \in \mathcal{S}_n)$, whereas $P_\mu(W(n) = -1) = P_\mu(\Delta W \in \mathcal{S}_n)$.

Since the variables $\Delta W(k)$ are i.i.d. then the cyclicly shifted sequences $\Delta W^{(i)}$ have the same law for every $i \in \{1, \ldots, n\}$. Recall that if $\Delta W \in \mathcal{S}_n$, then by the cycle lemma, for only one such $i$ the sequence $\Delta W^{(i)}$ belongs to $\mathcal{S}_n$. 

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**Corollary.** For every \( n \geq 1 \),

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P_\mu(|T| = n) = P_\mu(\zeta = n) = \frac{1}{n} P_\mu(W(n) = -1).
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**Proof.** Let us write \( \Delta W(k) \) for \( W(k + 1) - W(k) \) for all \( 0 \leq k \leq n - 1 \), then
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We infer that

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n P_\mu(\Delta W \in \mathcal{S}_n) = \sum_{i=1}^{n} P_\mu(\Delta W^{(i)} \in \mathcal{S}_n) = P_\mu\left( \bigcup_{i=1}^{n} \Delta W^{(i)} \in \mathcal{S}_n \right) = P_\mu(\Delta W \in \mathcal{S}_n).
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Corollary. For every $n \geq 1$,

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$$nP_\mu(\Delta W \in \bar{S}_n) = \sum_{i=1}^{n} P_\mu(\Delta W^{(i)} \in \bar{S}_n) = P_\mu\left(\bigcup_{i=1}^{n} \Delta W^{(i)} \in \bar{S}_n\right) = P_\mu(\Delta W \in S_n).$$

□

Extension to forest. Let us write $P_{\mu,k}$ for the law of $k$ independent $\mu$-BGW trees and let $|F|$ denote the total size of such a forest. Then more generally

$$P_{\mu,k}(|F| = n) = P_\mu(\inf\{i : W(i) = -k\} = n) = \frac{k}{n} P_\mu(W(n) = -k).$$
Fix once and for all $\mu$ an offspring distribution such that $\gcd(\{k \geq 1 : \mu(k) \neq 0\}) = 1$ and with mean $m_\mu = 1$. Sample $T_n$ from the conditional law $P_\mu(\cdot | \cdot \in T_n)$. How does $T_n$ behave as $n \to \infty$?
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**Theorem.** (Strong Ratio Theorem) For every $\ell \in \mathbb{Z}$, we have

$$\lim_{n \to \infty} \frac{P_\mu(W(n) = \ell)}{P_\mu(W(n) = 0)} = 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{P_\mu(W(n + 1) = 0)}{P_\mu(W(n) = 0)} = 1.$$
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This implies in particular that for any \( k \geq 1 \) fixed, as \( n \to \infty \), we have
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P_{\mu,k}(|F| = n - 1) = \frac{k}{n - 1} P_\mu(W(n - 1) = -k) \sim \frac{k}{n} P_\mu(W(n) = -1) = k P_\mu(|T| = n).
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First question about $T_n$: on the law of the number of offsprings $c_0$ of the root:
\[
P_{\mu}(c_0 = k | T \in T_n) = \frac{P_{\mu}(c_0 = k \text{ and } T \in T_n)}{P_{\mu}(T \in T_n)} = \mu(k) \frac{P_{\mu,k}(|F| = n-1)}{P_{\mu}(|T| = n)} \to k \mu(k).
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We see that the law of the offspring number of the root of $T_n$ converges to $\hat{\mu}(k) = k\mu(k)$, called the **size-biased** version of $\mu$ (indeed a probability since $m_\mu = 1$).
5) Convergence of trees
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For a (possibly infinite) rooted plane tree $T$, and $r \geq 0$, let us denote by $[T]_r$ the subtree containing only the vertices at generation $r$ or lower, called the **ball** of radius $r$ (centred at the root).
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We say that a sequence of random trees $(T_n)_{n \geq 1}$ converges to a limit $T_\infty$ if, for every $r \geq 1$, the sequence of trees $([T_n]_r)_{n \geq 1}$ converges in distribution towards $[T_\infty]_r$ in the sense that for every possible deterministic tree $t$, one has

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Take $T_n$ our $\mu$-BGW tree conditioned to have size $n$. The preceding results concerns the case $r = 1$, for which the tree $[T_n]_1$ is nothing but the root of $T_n$ together with its offsprings. By the preceding result, if $T_n \rightarrow T_\infty$ in distribution, then the offspring number of the root of $T_\infty$ has the size-biased law $\hat{\mu}$.
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First guess: is $T_\infty$ a $\hat{\mu}$-BGW tree? Well $\hat{\mu}(0) = 0$ so this tree has a geometry much different from a $\mu$-BGW, e.g. each generation is at least as large as the previous one.
The correct construction of $T_\infty$ is as follows:
1. the root reproduces according to the law $\hat{\mu}$;
2. choose one of its children uniformly at random and let it reproduce according to the law $\hat{\mu}$;
3. choose one child of the latter uniformly at random and again let it reproduce according to the law $\hat{\mu}$, etc.
4. all other individual reproduce according to the original distribution $\mu$.
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Since $\hat{\mu}(0) = 0$, the tree never gets extinct; it is called the $\mu$-BGW tree \textbf{conditioned to survive}. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{tree_diagram.png}
\end{figure}
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**Theorem.** Suppose that $\mu$ is an offspring distribution such that $\gcd(\{k \geq 1 : \mu(k) \neq 0\}) = 1$ and with mean $m_{\mu} = 1$. Then $T_n \to T_\infty$ in distribution.
**Proof.** Fix $r \geq 1$, we aim at showing that $[T_n]_r \rightarrow [T_\infty]_r$ in distribution. Let $t$ be a deterministic tree such that $P([T_\infty]_r = t) \neq 0$. In particular $t$ has no vertex at generation $r + 1$ or higher, but it has at least one vertex at generation $r$; let $\ell$ denote the number of such vertices.
Proof. Fix $r \geq 1$, we aim at showing that $[T_n]_r \rightarrow [T_\infty]_r$ in distribution. Let $t$ be a deterministic tree such that $P([T_\infty]_r = t) \neq 0$. In particular $t$ has no vertex at generation $r + 1$ or higher, but it has at least one vertex at generation $r$; let $\ell$ denote the number of such vertices.

The tree $t$ is a realisation of $[T_n]_r$ if $T_n$ is obtained by replacing each of these $\ell$ leaves at generation $r$ by a forest $F$, with total size $n - |t| + \ell$. Therefore

$$P_\mu([T_n]_r = t) = P_\mu([T]_r = t \mid |T| = n) = \frac{P_\mu(t)}{P_\mu(|T| = n)} \frac{P_{\mu,\ell}(|F| = n - |t| + \ell)}{P_\mu(|T| = n)} \rightarrow \ell \text{ (strong ratio)}$$
**Proof.** Fix \( r \geq 1 \), we aim at showing that \([T_n]_r \to [T_\infty]_r\) in distribution. Let \( t \) be a deterministic tree such that \( P([T_\infty]_r = t) \neq 0 \). In particular \( t \) has no vertex at generation \( r + 1 \) or higher, but it has at least one vertex at generation \( r \); let \( \ell \) denote the number of such vertices.

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\[
P_\mu([T_n]_r = t) = P_\mu([T]_r = t \mid |T| = n) = \frac{P_\mu(t) \cdot P_{\mu,\ell}(|F| = n - |t| + \ell)}{\mu(0)^\ell} \cdot \frac{P_\mu(|T| = n)}{P_\mu(|T| = n)} \to \ell \text{ (strong ratio)}
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It remains to prove that \( P_\mu([T_\infty]_r = t) = \ell P_\mu(t)/\mu(0)^\ell \).
**Proof.** Fix $r \geq 1$, we aim at showing that $[T_n]_r \rightarrow [T_\infty]_r$ in distribution. Let $t$ be a deterministic tree such that $P([T_\infty]_r = t) \neq 0$. In particular $t$ has no vertex at generation $r + 1$ or higher, but it has at least one vertex at generation $r$; let $\ell$ denote the number of such vertices.

The tree $t$ is a realisation of $[T_n]_r$ if $T_n$ is obtained by **replacing** each of these $\ell$ leaves at generation $r$ by a forest $F$, with total size $n - |t| + \ell$. Therefore

$$P_\mu([T_n]_r = t) = P_\mu([T]_r = t \mid |T| = n) = \frac{P_\mu(t) P_{\mu,\ell}(|F| = n - |t| + \ell)}{\mu(0)^\ell \mu(|T| = n)} \rightarrow \ell \text{ (strong ratio)}.$$ 

It remains to prove that $P_\mu([T_\infty]_r = t) = \ell P_\mu(t)/\mu(0)^\ell$.

Note that amongst the $\ell$ leaves of $t$ at generation $r$, one of them, say $v_*$, must belong “special”. Choose one of these leaves, say $u$, then conditionally given the event $[T_\infty]_r = t$, the probability that $u = v_*$ is given by the inverse of the product of the offspring number of each of its ancestors. This cancels the extra factor given by $\hat{\mu}$, so that

$$P_\mu([T_\infty]_r = t \text{ and } u = v_*) = P_\mu(t)/\mu(0)^\ell.$$ 

Since this holds for each of the $\ell$ possible leaves, the claim follows $\square$
**Remark.** The proof only relies on the convergence of the ratio of two probabilities and the theorem extends to other conditioning on the tree such as e.g. conditioning to reach height $n$ (i.e. survive for at least $n$ generations), or conditioning to have $n$ leaves, etc. Similarly, one can also condition these quantities to be at least $n$. 
Remark. The proof only relies on the convergence of the ratio of two probabilities and the theorem extends to other conditioning on the tree such as e.g. conditioning to reach height $n$ (i.e. survive for at least $n$ generations), or conditioning to have $n$ leaves, etc. Similarly, one can also condition these quantities to be at least $n$.

Remark. On the other hand, the assumption that the mean of $\mu$ equals one is quite crucial.
If the mean is smaller than 1, then the behaviour can be drastically different and $T_\infty$ may possess an individual with infinitely many offsprings.