Lecture 1

Basic Concepts in Game Theory

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Lecture 1: Basic Concepts in Game Theory

1. Games in Strategic Form
   - Best responses and Nash equilibria
   - Finite 2-player games
2. Existence & Non-existence of Equilibria
   - Mixed extension of finite games
   - Extension of Nash's Theorem
3. Potential Games
   - Examples: routing, congestion, connection, location
   - How to find a potential
4. Efficiency of Equilibria
   - Price-of-Anarchy and Price-of-Stability

Games in Strategic Form

Games in strategic form

A game consists of a set of players $i \in \{1, \ldots, n\}$*, each one with a strategy set* S_i *and a payoff function* $\ell_i : S \to \mathbb{R}$*, where* $S = \prod_{j=1}^{n} S_j$ *is the set of strategy profiles.*

Payoffs represent losses/utilities that each player i seeks to minimize/maximize

$$\min_{s_i \in S_i} \ell_i(s_i, s_{-i})$$

REMARK: For utility maximization games the payoffs are denoted $u_i(s_i, s_{-i})$.

The best-response map for player i associates to each tuple $s_{-i} = (s_j)_{j \neq i}$ *of strategies of his/her opponents, the set of optimal solutions*

$$BR_i(s_{-i}) = \operatorname*{Argmin}_{s_i \in S_i} \ell_i(s_i, s_{-i})$$

Nash equilibrium

Definition

A Nash equilibrium is a strategy profile $\bar{s} = (\bar{s}_1, \ldots, \bar{s}_n) \in S$ *with* $\bar{s}_i \in BR_i(\bar{s}_{-i})$ *for each player i, that is to say*

$$\ell_i(\bar{s}_i, \bar{s}_{-i}) \leq \ell_i(s_i, \bar{s}_{-i}) \text{ for all } s_i \in S_i$$

In words: No player i has incentive to deviate by choosing a different $s_i \neq \bar{s}_i$.

Remark: Nash equilibria are fully determined by the best-response maps
$\Rightarrow$ they are stable under transformations that preserve the best-responses.

Example: Bandwidth sharing

A communication channel with unit capacity is shared by n players who send information at rates $x_i \geq 0$. Because of packet losses, the transmission degrades linearly with the total rate, so that the utility for player i is

$$u_i(x_1, \ldots, x_n) = x_i \, [1 - \textstyle\sum_{j=1}^{n} x_j]_+$$

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Maximizing with respect to x_i yields the best response condition

$$x_i = 1 - \sum_{j=1}^{n} x_j$$

$\Rightarrow$ at equilibrium all players send the same rate $x_i = 1/(n+1)$ with utility

$$\bar{u}_i = 1/(n+1)^2.$$

Note that if every player sets a lower rate $x_i = 1/(2n)$, the utilities are larger

$$u_i(x) = 1/(4n) > 1/(n+1)^2.$$

Dominated strategies

A strategy $s_i \in S_i$ is said to be strictly dominated *by $s_i' \in S_i$ if*

$$\ell_i(s_i', s_{-i}) < \ell_i(s_i, s_{-i}) \quad \textit{for all } s_{-i} \in S_{-i}.$$

Such s_i will never be part of an equilibrum and can be dropped.

Under weak domination (with $\leq$ instead of $<$) one may still ignore it, though some equilibria might be lost.

Finite 2-player games

A 2-player game, where one player, called row player, has n strategies and the second, called column player, has m strategies, can be represented by a pair of $n \times m$ matrices, denoted (A, B), often called bimatrix game:

$$\begin{pmatrix} (a_{11}, b_{11}) & \dots & (a_{1j}, b_{1j}) & \dots & (a_{1m}, b_{1m}) \\ \dots & \dots & \dots & \dots & \dots \\ (a_{i1}, b_{i1}) & \dots & (a_{ij}, b_{ij}) & \dots & (a_{im}, b_{im}) \\ \dots & \dots & \dots & \dots & \dots \\ (a_{n1}, b_{n1}) & \dots & (a_{nj}, b_{nj}) & \dots & (a_{nm}, b_{nm}) \end{pmatrix}.$$

- The strategy spaces of the players are $S_1 = \{1, \dots, n\}$ and $S_2 = \{1, \dots, m\}$
- The choices of $i \in S_1$ and $j \in S_2$ yield the payoffs a_{ij} and b_{ij} respectively.

Example

$$\begin{pmatrix} (8,8) & (2,7) & (4,8) \\ (7,2) & (0,4) & (3,0) \\ (5,3) & (3,9) & (2,4) \end{pmatrix}$$

- The row player can eliminate...
- Knowing this the column player can eliminate...
- Knowing this the row player can eliminate...
- The outcome is...

More examples

$$\begin{pmatrix} (3,2) & (3,6) \\ (4,2) & (2,1) \end{pmatrix}$$

Nash equilibria: $(3,2)$ and $(2,1)$. Likely, the players will agree on the latter.

$$\begin{pmatrix} (3,2) & (6,6) \\ (6,6) & (2,3) \end{pmatrix}$$

Nash equilibria: $(3,2)$ and $(2,3)$. Players have opposite preferences on these.

$$\begin{pmatrix} (1,1) & (0,0) \\ (0,0) & (1,1) \end{pmatrix}$$

Nash equilibria: $(0,0)$ and $(0,0)$. Players are indifferent but need to coordinate.

Prisoner's Dilemma I

Two prisoners face a trial.

- If they keep silent they are charged for a minor offense and get a sentence of 1 year in prison each (because of lack of evidence).
- If one decides to confess he is released and the other gets 6 years in jail.
- If both confess each one gets 5 years in jail.

$$\begin{pmatrix} (1,1) & (6,0) \\ (0,6) & (5,5) \end{pmatrix}$$

The unique rational outcome is not only Nash equilibrium, but also obtained with elimination of strictly dominated strategies!

Prisoner's Dilemma II

A father asks his two children:

Do you want me to give €1 to you, or €10 to your brother?

$$\begin{pmatrix} (10,10) & (0,11) \\ (11,0) & (1,1) \end{pmatrix}$$

Payoffs represent utilities to be maximized....

This game has exactly the same structure as the prisoner's dilemma.

Battle of the sexes

A couple plans to spend Sunday together, either by going to watch the Roma vs Lazio match, or to the movies to watch a recently released film. She loves movies while he prefers soccer, but both prefer being together rather than by their own.

$$\begin{pmatrix} (3,2) & (1,1) \\ (0,0) & (2,3) \end{pmatrix}$$

Maximize... Which are the equilibria?

Hawks and Doves

Two fighting birds can behave either aggressive as a Hawk or gentle as a Dove.

- If both are Hawks they get badly hurt and each one looses -100 utils.
- If both are Doves they get 1 util each.
- If a Hawk meets a Dove, the Hawk gets 10 and the Dove nothing.

$$\begin{pmatrix} (\text{-}100, \text{-}100) & (10, 0) \\ (0, 10) & (1, 1) \end{pmatrix}$$

There are two Nash equilibria in pure strategies.

Crossing game

Two drivers face a non-signalled intersection. If they both cross the crash is unavoidable. If one crosses and the other lends the first gets a slightly higher utility. If they both lend the passage they can wait forever...

$$\begin{pmatrix} (-100,-100) & (2,1) \\ (1,2) & (0,0) \end{pmatrix}$$

Again there are two Nash equilibria in pure strategies.

Higher payoffs need not be better...

Compare the following utility maximization games

The first game:

$$\begin{pmatrix} (10,10) & (3,15) \\ (15,3) & (5,5) \end{pmatrix}$$

The second game:

$$\begin{pmatrix} (8,8) & (2,7) \\ (7,2) & (0,4) \end{pmatrix}$$

In any outcome players are better off in the first game rather than in the second. Yet, it is more convenient for them to play the second !

More strategies might also hurt...

The first game:

$$\begin{pmatrix} (10,10) & (3,5) \\ (5,3) & (1,1) \end{pmatrix}$$

The game contains all possible outcomes of the first, and some additional ones

$$\begin{pmatrix} (1,1) & (11,0) & (4,0) \\ (0,11) & (10,10) & (3,5) \\ (0,4) & (5,3) & (1,1) \end{pmatrix}$$

Having less available actions can make the players better off!

Rock-Scissors-Paper

A 2-person game is called *zero-sum* if $a_{ij} + b_{ij} = 0$ for all i, j.

Clearly in such a case it suffices to display the payoff of player 1. In the popular game Rock-Scissors-Paper the payoff matrix is

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

In this case there is no equilibrium... at least in pure strategies...

Existence & Non-existence of Equilibria

Nash equilibrium in pure strategies might not exist

Consider the game

$$\begin{pmatrix} (4,0) & (3,1) \\ (3,5) & (5,0) \end{pmatrix}$$

There is no equilibrium... in pure strategies.

A player cannot use the same strategy all the time; this would be observed and the other player could take advantage from this.

- It makes sense to randomize among strategies.
- But the probabilities must be chosen strategically!

Mixed equilibria for finite games

Consider an n-person finite game with strategy sets S_i and payoffs $\ell_i(a_1, \ldots, a_n)$. In the *mixed extension* each player i choses a probability distribution $\pi_i \in \Delta(S_i)$, that is to say, $\pi_i(s) \geq 0$ for all $s \in S_i$ and $\sum_{s \in S_i} \pi_i(s) = 1$.

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The probability of an outcome $(s_1, \ldots, s_n)$ is $\prod_{i=1}^n \pi_i(s_i)$ with *expected payoffs*:

$$\bar{\ell}_i(\pi_1, \ldots, \pi_n) = \sum_{(s_1,\ldots,s_n)\in S} u_i(s_1, \ldots, s_n) \prod_{j=1}^{n} \pi_j(s_j) = \sum_{s_i \in S_i} \ell_i(s_i, \pi_{-i})\, \pi_i(s_i)$$

$$\ell_i(s_i, \pi_{-i}) = \sum_{s_j \in S_j, j \neq i} u_i(s_1, \ldots, s_n) \prod_{j \neq i} \pi_j(s_j)$$

Theorem (Nash 1951)

Every n-player finite game has at least one Nash equilibrium in mixed strategies.

Optimizing over a simplex

Letting $v_s \triangleq \ell_i(s, \pi_{-i})$, best response requires to solve

$$\min_{x \in \Delta(S_i)} \sum_{s \in S_i} v_s x_s$$

$\Rightarrow$ put all the weight on actions s with the smallest values v_s.

Thus, setting $v = \min_{s \in S_i} v_s$ we have that $x \in \Delta(S_i)$ is a best response iff

$$(\forall s \in S_i)\ x_s > 0 \Rightarrow v_s = v.$$

Player i should randomize only among actions of minimal expected cost.

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What about non-finite games ?

Example: Hotelling game

Several icecream vendors must place their cart in a beach 1 kilometer long. People are distributed uniformly on the beach and choose the closest cart to get icecream. People having several carts at the same distance are split evenly.

The following results are known:

1. If there are only two icecream vendors, the unique equilibrium is when they both stay in the center of the beach.
2. When there are three vendors, no Nash equilibrium exists.
3. With four or five vendors, there is one equilibrium (up to permutations).
4. With six or more carts there are infinitely many equilibria.

Extension of Nash's Theorem

Theorem (Debreu 1952, Glicksberg 1952, Fan 1952)

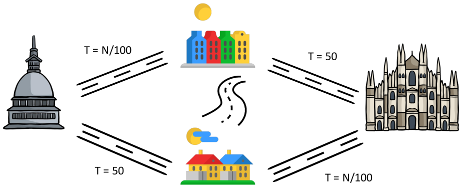
Given a n-player game with strategy sets S_i and payoffs $\ell_i : S \to \mathbb{R}$ where

- *each S_i is a closed bounded convex subset of $\mathbb{R}^{d_i}$*
- *each $\ell_i : S \to \mathbb{R}$ is continuous*
- *$s_i \mapsto \ell_i(s_i, s_{-i})$ is quasi-convex for each fixed $s_{-i} \in S_{-i}$*

Then there exists at least one Nash equilibrium.

A few additional examples

Braess Paradox



4.000 people travel from Rome to Milan, each one wants to minimize travel time. N is the number of people driving in the corresponding road.

- What are the Nash equilibria if the Up-Down street between the two small cities is closed for maintenance work?
- What if the Up-Down street is available with 5 minutes travel time?

El Farol bar

In Santa Fe there are 500 young people, happy to go to the El Farol bar. More people in the bar, happier they are, till they reach 300 people. They can also decide to stay at home. Utility is assumed to be 0 for people that stay at home, while people at the bar get $u(x) = x$ if $x \leq 300$ and $u(x) = 300 - x$ if $x > 300$.

There are multiple Nash equilibria where 300 people are in the bar, the other stay at home. An asymmetric situation, notwithstanding the players are symmetric.

A multitude of mixed equilibria exist, and a mixed symmetric equilibrium is also present in this case.

Auctions

Several types of auctions since ancient times: sequential offers, sealed offers, first price, second price,...

- There are n bidders, each one has a valuation v for the object, which is kept as private information. We assume that $v_1 > v_2 > \cdots > v_n$.
- Each bidder proposes a (non-negative) bid b_i, seen as strategy for the player. Thus the strategy space of the players is $[0, \infty)$.
- An assignment rule and the payment must be decided, including the rule for handling ties.
- Player i gets an utility $v_i - b_i$ if he wins the auction, and 0 otherwise.

We consider only auctions where the winner is the highest bidder. In case of tie in the highest bid the winner is the one who values more the object.

First price auction

In a first price auction the rule is: the player i offering the highest bid b_i gets the object and pays exactly her bid. The other players pay nothing.

1. For player i bidding more than v_i is weakly dominated.
2. One Nash equilibrium is $(v_2, v_2, v_3, \ldots, v_n)$.
3. In all equilibria the winner is player 1.
4. The two highest bids are the same and one is made by player 1. The highest bid b_1 satisfies $v_2 \leq b_1 \leq v_1$. All such bid profiles are Nash equilibria.

Second price auctions

In a second price auction the rule is: the player i offering the highest bid b_i gets the object and pays the second highest bid. The other players pay nothing,

1. One Nash equilibrium is $(v_1, v_2, v_3, \ldots, v_n)$
2. Other equilibria: $(v_1, 0, 0, \ldots, 0)$, $(v_2, v_1, v_3, \ldots, v_n)$
3. A player's bid equalizing her evaluation is a weakly dominant strategy

Potential Games

Two (trivial) observations

1. Suppose that all the players have exactly the same payoff

$$\ell_i(s_1, \ldots, s_n) = p(s_1, \ldots, s_n).$$

Then, any strategy profile $\bar{s} \in S$ that minimizes $p(\bar{s})$ is a Nash equilibrium in pure strategies (there might be more).

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2. Suppose that we add some constants c_i to the payoffs

$$\tilde{\ell}_i(s_1,\ldots,s_n)=\ell_i(s_1,\ldots,s_n)+c_i$$

Best responses and equilibria are unchanged !

The same holds if $\tilde{\ell}_i(s)-\ell_i(s)$ does not depend on s_i but only on s_{-i}.

Potential games

Definition

A finite game is called a potential game *if there exists a* potential function $p : S \to \mathbb{R}$ *such that* $p(s) - \ell_i(s)$ *does not depend on* s_i.

Proposition

Every finite potential game has at least one pure Nash equilibrium.

Defining the increments $\Delta\ell_i(s_i', s_i, s_{-i}) = \ell_i(s_i', s_{-i}) - \ell_i(s_i, s_{-i})$, the condition for a potential can be expressed by

$$\Delta p(s_i', s_i, s_{-i}) = \Delta\ell_i(s_i', s_i, s_{-i}).$$

Best response dynamics

Consider the following payoff-improving procedure:

1. Start from an arbitrary strategy profile $(s_1, \ldots, s_n) \in S$
2. If any player has a strategy s_i' that strictly decreases her payoff

$$\ell_i(s_i', s_{-i}) < \ell_i(s_i, s_{-i})$$

   replace s_i with s_i' and repeat, otherwise Stop.

Proposition

In a finite potential game every best response iteration attains a pure Nash equilibrium after finitely many steps.

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Proof. Since

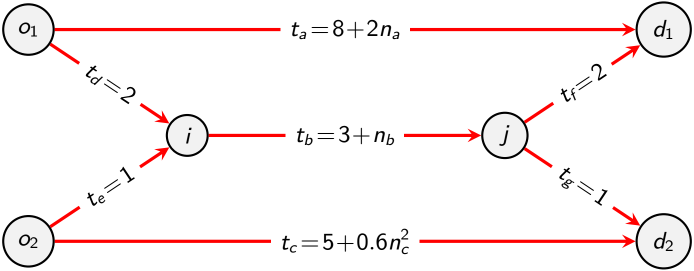
$$\Delta p(s'_i, s_i, s_{-i}) = \Delta \ell_i(s'_i, s_i, s_{-i}) < 0,$$

each iteration strictly decreases the potential $p(s)$ so that no strategy profile $s \in S$ is visited twice $\Rightarrow$ we stop after at most $|S|$ steps at a pure equilibrium. $\square$

REMARK: The procedure may fail to reach the global minimum $\bar{s}$.

Example 1: Routing games

Consider n drivers traveling between different origins and destinations in a city. The transport network is modeled as a graph (V, E) with node set V and edges E. Because of congestion, the travel time of an edge $e \in E$ is a non-negative increasing function $t_e = c_e(n_e)$ of the load $n_e = \#$ of drivers using the edge.



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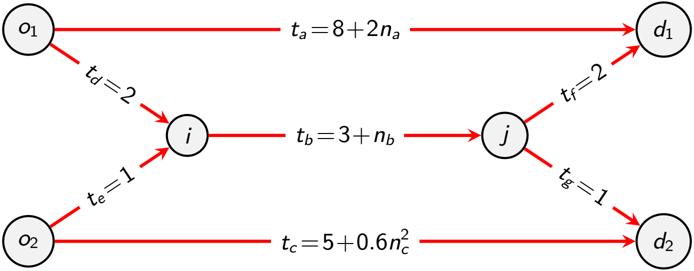

Each driver i selects a route $r_i = e_1 e_2 \cdots e_\ell$, that is, a sequence of edges connecting her origin $o_i \in N$ to her destination $d_i \in N$. Her total travel time is

$$\ell_i(r_1, \ldots, r_n) = \sum_{e \in r_i} c_e(n_e) \quad ; \quad n_e = \#\{j : e \in r_j\}$$

Example 1: Routing games

To minimize travel time, drivers may restrict to *simple paths* with no cycles: nodes are visited at most once. Hence, the strategy set for player i is the *finite* set S_i of all simple paths connecting o_i to d_i.

Theorem (Rosenthal'73)

A routing game admits the potential

$$p(r_1,\ldots,r_n)=\sum_{e\in E}\sum_{k=1}^{n_e}c_e(k) \quad ; \quad n_e=\#\{j\colon e\in r_j\}.$$

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Proof: It suffices to note that for $r=(r_1,\dots,r_n)$ we have

$$p(r)-\ell_i(r)=\sum_{e\in E}\sum_{k=1}^{n_e}c_e(k)-\sum_{e\in r_i}c_e(n_e)=\sum_{e\in E}\sum_{k=1}^{n_e^{-i}}c_e(k)$$

where $n_e^{-i}=\#\{j\neq i\colon e\in r_j\}$ is the number of drivers other than i using the edge e. Hence, the difference $p(r)-\ell_i(r)$ depends only on r_{-i} and not on r_i. $\square$

Example 2: Congestion games

A routing game is a special case of the more general class of *Congestion games*. Here each player $i = 1, \ldots, n$ has to perform a certain task which requires some resources taken from a set R. The strategy set S_i for player i is the family of all subsets $s_i \subseteq R$ that allow her to perform the task.

Each resource $r \in R$ has a cost $c_r(n_r)$ which depends on the number of players that use the resource. We no longer assume $c_r(\cdot)$ non-negative nor increasing. Player i only pays for the resources she uses

$$\ell_i(s_1, \ldots, s_n) = \sum_{r \in s_i} c_r(n_r) \quad ; \quad n_r = \#\{j \colon r \in s_j\}.$$

Exercise: Prove that $p(s_1, \ldots, s_n) = \sum_{r \in R} \sum_{k=1}^{n_r} c_r(k)$ is a potential.

Example 3: Network connection games

A telecommunication network (V, E) is under construction. Each player i wants a route r_i to be built between a certain origin o_i and a destination d_i. The cost v_e of building a link $e \in E$ is shared evenly among the players who use it.

Hence, the cost for player i is

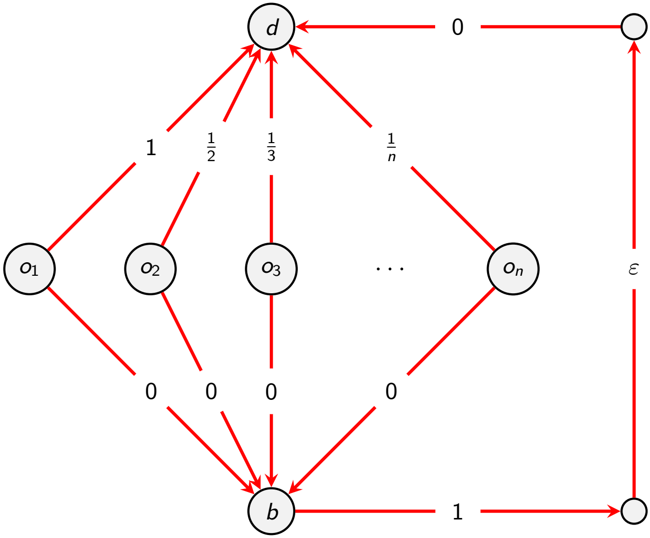
$$\ell_i(r_1, \ldots, r_n) = \sum_{e \in r_i} \frac{v_e}{n_e} \quad ; \quad n_e = \#\{j\colon a \in r_j\}.$$

In this case there is an incentive to use congested arcs as this reduces the cost.

This is again a congestion game with potential

$$p(r_1, \ldots, r_n) = \sum_{e \in E} v_e(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n_e}).$$

Example 3: Network connection games



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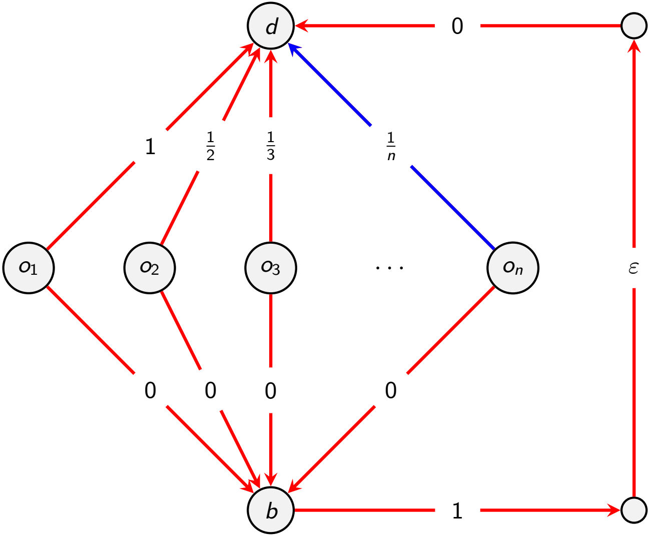

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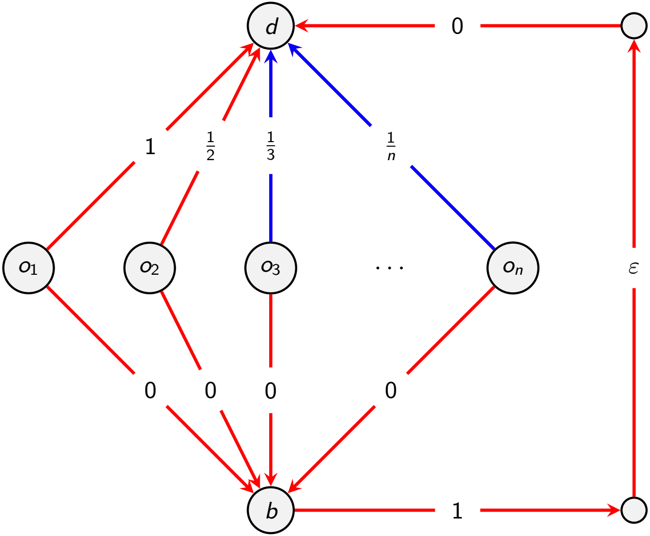

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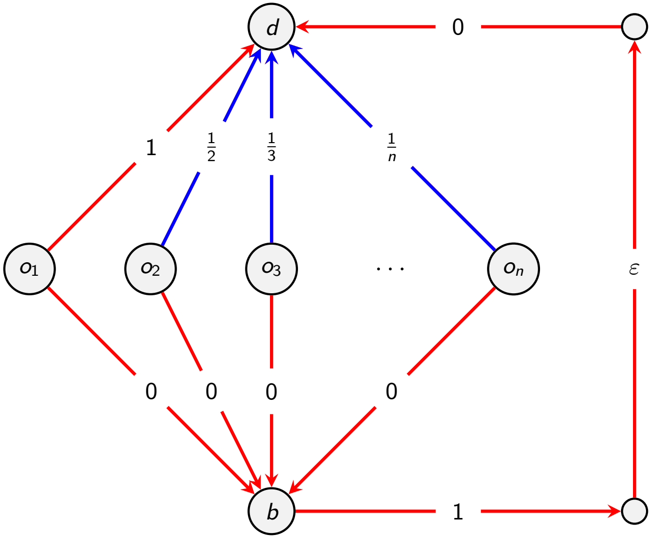

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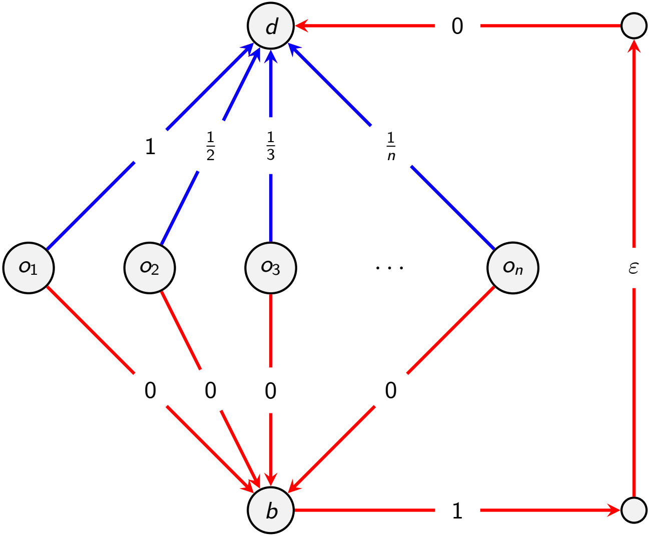

Example 4: Location games

A group of Internet Service Providers (ISPs) $i = 1, \ldots, n$ compete for providing connectivity to a finite set of customers $k \in K$. Each firm i has to decide where to locate its Data Center, chosen among a finite set of possible sites S_i.

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Customer $k \in K$ can be served from the different ISP sites $s_i \in S_i$ at a cost $c^k_{s_i}$. Then, firm i will propose to k the competitive price

$$p^k_i(s) = \max\{c^k_{s_i}, \min_{j \neq i} c^k_{s_j}\}.$$

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Hence k is served by the ISP with minimal cost and pays the second lowest cost. The utility for firm i is therefore

$$u_i(s_1, \dots, s_n) = \sum_{k \in K} [p^k_i(s) - c^k_{s_i}].$$

We assume that the value v^k that customer k gets from the service is higher that all the costs $c^k_{s_i}$, so that customers are always willing to buy the service.

Example 4: Location games

Proposition

The location game admits the potential (to be maximized)

$$p(s_1,\dots,s_n)=\sum_{k\in K}[v^k-\min_{j=1\dots n} c^k_{s_j}]$$

which corresponds to the sum of excess utilities for customers and providers.

Example 4: Location games

Proposition

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Proof: Considering separately the customers k for which firm i is the minimum cost provider, and the k's for whis it is not, in both cases we get

$$\begin{aligned} p(s)-u_i(s) &= \sum_{k\in K}[v^k-\min_{j=1\ldots n} c^k_{s_j}-p^k_i(s)+c^k_{s_i}] \\ &= \sum_{k\in K}[v^k-\min_{j\neq i} c^k_{s_j}] \end{aligned}$$

where the latter depends only on s_{-i} and not on s_i. □

How to find a potential

A potential $p : A \to \mathbb{R}$ is characterized by

$$\Delta p(a_i', a_i, a_{-i}) = \Delta \ell_i(a_i', a_i, a_{-i}).$$

Adding a constant to $p(\cdot)$ changes nothing so that $p(\cdot)$ is never unique. However, we may choose an arbitrary profile $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n)$ to be used as a reference point and set $p(\bar{a}) = 0$. Once this is done, the potential $p(\cdot)$ *is determined uniquely*:

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$$\begin{array}{rcl}
p(a_1, a_2, \ldots, a_n) - p(\bar{a}_1, a_2, \ldots, a_n) & = & \ell_1(a_1, a_2, \ldots, a_n) - \ell_1(\bar{a}_1, a_2, \ldots, a_n) \\
p(\bar{a}_1, a_2, \ldots, a_n) - p(\bar{a}_1, \bar{a}_2, \ldots, a_n) & = & \ell_2(\bar{a}_1, a_2, \ldots, a_n) - \ell_2(\bar{a}_1, \bar{a}_2, \ldots, a_n) \\
& \vdots & \\
p(\bar{a}_1, \bar{a}_2, \ldots, a_n) - p(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) & = & \ell_n(\bar{a}_1, \bar{a}_2, \ldots, a_n) - \ell_n(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n) \\
\hline
\end{array}$$

$$\Rightarrow \quad p(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} [\ell_i(\bar{a}_1 \ldots a_i \ldots a_n) - \ell_i(\bar{a}_1 \ldots \bar{a}_i \ldots a_n)]$$

Existence of a potential

Note that instead of changing $a_i \rightsquigarrow \bar{a}_i$ in the order $i = 1, 2, \ldots, n$ we could proceed backwards from n down to 1, or using an arbitrary ordering. After all, using integers to name the players is inmaterial and any order is equally valid.

A game is potential iff the sum on the right hand side is *independent of the particular order used*.

However, checking that all these orders yield the same answer is impractical for more than 2 or 3 players, so this is not of much help.

Example: computing a potential

Is the following a potential game?

$$\begin{pmatrix} 2,5 & 2,6 & 3,7 & 8,9 & 5,7 \\ 1,4 & 1,5 & 3,7 & 2,3 & 0,2 \\ 6,5 & 2,2 & 0,0 & 6,3 & 3,1 \end{pmatrix}$$

Potential:

$$\begin{pmatrix} 5 & 6 & 7 & 9 & 7 \\ 4 & 5 & 7 & 3 & 2 \\ 9 & 6 & 4 & 7 & 5 \end{pmatrix}$$

Efficiency of Equilibria

Social cost and efficiency

Nash equilibria can be bad for all players as in the Braess' paradox, the Prisoner's dilemma, or the Tragedy of the commons. But how *"bad"* can be the outcome of a game ?

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First we must agree on what *"bad"* means. We measure the quality of a strategy profile $s = (s_1, \ldots, s_n)$ through a *social cost* function $C : S \to \mathbb{R}_+$. The smaller $C(s)$ the better the outcome $s \in S$. We use as a benchmark the minimal value that a benevolent social planner could achieve

$$Opt = \min_{s \in S} C(s).$$

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The quotient $\frac{C(s)}{Opt}$ measures how far is s from being optimal. A large value implies a big loss in social welfare, while a quotient close to 1 implies that s is almost as efficient as an optimal solution.

Price-of-Anarchy and Price-of-Stability

Definition

Let $NE \subseteq S$ be the set of pure Nash equilibria of the game. The Price-of-Anarchy and the Price-of-Stability are defined respectively by

$$PoA = \max_{\bar{s} \in NE} \frac{C(\bar{s})}{Opt} \qquad ; \qquad PoS = \min_{\bar{s} \in NE} \frac{C(\bar{s})}{Opt}$$

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Clearly $1 \leq PoS \leq PoA$. Having $PoA \leq \alpha$ means that in every possible pure equilibrium the social cost $C(\bar{s})$ is no worse than $\alpha\, Opt$. When $PoS \leq \alpha$ we can only say that there exists some equilibrium with social cost at most $\alpha\, Opt$.

Social cost – Egalitarian function

In games where $\ell_i(s)$ represent costs, a natural choice for a social cost is the egalitarian function which simply aggregates the costs of all the players

$$C(s) = \sum_{i=1}^{n} \ell_i(s).$$

This presupposes that the individual costs $\ell_i(s)$ are expressed in some comparable units and scale (monetary, time, weight,...).

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Examples:

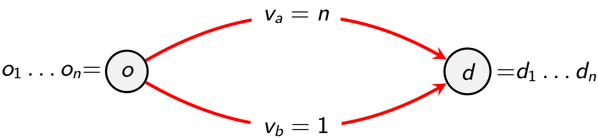
- In the routing game the egalitarian function is the total time traveled by all the players, and can be expressed as

$$C(r_1, \ldots, r_n) = \sum_{e \in E} n_e c_e(n_e) \quad ; \quad n_e = \#\{j \colon e \in r_j\}.$$

- In the network connection game the egalitarian function gives the total investment required to connect all the players

$$C(r_1, \ldots, r_n) = \sum_{e \in E} v_e \mathbb{1}_{\{n_e > 0\}} \quad ; \quad n_e = \#\{j \colon e \in r_j\}.$$

Example: PoA and PoS — Network connection game



$$Opt = 1$$

$$PoS = 1$$

$$PoA = n \to \infty$$

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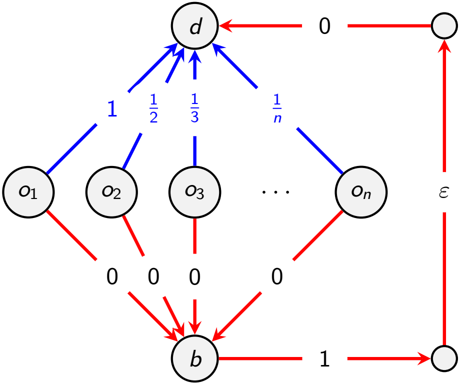

$Opt = 1 + \varepsilon$

$C(\bar{a}) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = H_n$

$PoA = PoS = \frac{H_n}{1+\varepsilon} \sim \ln(n) \to \infty$

An estimate for PoS

Proposition

Consider a cost minimization finite potential game with potential $p : S \to \mathbb{R}$, and suppose that the social cost $C : S \to \mathbb{R}_+$ is such that there exist strictly positive constants α, β such that

$$\frac{1}{\alpha} C(s) \le p(s) \le \beta C(s) \quad \forall\, s \in S.$$

Then $PoS \le \alpha\beta$.

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Then $PoS \leq \alpha\beta$.

Proof: Let $\bar{s}$ be a minimum of $p(\cdot)$ so that $\bar{s}$ is a Nash equilibrium. It follows that for all $s \in S$ we have

$$\tfrac{1}{\alpha} C(\bar{s}) \leq p(\bar{s}) \leq p(s) \leq \beta C(s)$$

hence $C(\bar{s}) \leq \alpha\beta\, Opt$. □

Application: PoS in network connection games

Proposition

Consider a network congestion game with n players on a general graph (N, A) *with arc construction costs* $v_a \geq 0$*. Then* $PoS \leq H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

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Proof: In this case the potential and the social cost are

$$p(r_1, \ldots, r_n) = \sum_{a \in A} \sum_{k=1}^{n_a} \frac{v_a}{k}$$

$$C(r_1, \ldots, r_n) = \sum_{a \in A} v_a \mathbb{1}_{\{n_a > 0\}}$$

so that $C(r) \leq p(r) \leq H_n C(r)$ and the previous result yields $PoS \leq H_n$. □