First and second order necessary conditions in optimal control of evolution systems

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An infinite dimensional optimal control problem

Minimize $\quad g_0(x(1)) \quad$ over solutions of

$$\dot{x}(t) = \mathbb{A}x(t) + f(t, x(t), u(t)), \quad t \in [0,1],$$

$$x(0) = x_0, \qquad x(1) \in \bigcap_{i=1,\dots,k} \{x \in X : g_i(x) \leq 0\},$$

$$x(t) \in \bigcap_{j=1,\dots,q} \{x \in X : \varphi_j(x) \leq 0\}, \quad \forall\, t \in [0,1].$$

- ⋆ X is an **infinite dimensional** separable Banach space;
- ⋆ $\mathbb{A}$ generates a **strongly continuous** semigroup $S(t)$ on X;
- ⋆ $f : [0,1] \times X \times Z \to X$;
- ⋆ u **measurable** with values in a subset U of a separable Banach space Z;
- ⋆ $g_i : X \to \boldsymbol{R}$, $\varphi_j : X \to \boldsymbol{R}$.

We provide:

- necessary optimality conditions in the form of a **maximum principle** and a **second order variational inequality**;
- **representations** for the multipliers associated to the end point and running constraints;
- sufficient conditions guaranteeing **normality** of the maximum principle;
- applications to concrete **models involving PDEs**. These include both hyperbolic and parabolic problems;
- indications on how the same arguments can be adapted to problems involving constraints of a different nature, in particular in the case of **pointwise state constraints**.

Necessary optimality conditions

Let

- $(\bar{x}, \bar{u})$ be **locally optimal**; $[t] = (t, \bar{x}(t), \bar{u}(t))$;
- $(\bar{y}, \bar{v})$ be a **critical** trajectory/control pair for the linearized system.

Then $\exists\, \lambda_i \geq 0$, $\exists\, \psi_j \in \mathcal{M}([0,1], \boldsymbol{R})$ positive, not vanishing simultaneously, such that the solution p to

$$\begin{cases} dp(t) = -\left(\mathbb{A}^* + f_x[t]^*\right) p(t)\mathrm{d}t - \sum_{j=1}^{q} \nabla\varphi_j(\bar{x}(t)) d\psi_j(t)\,, & t \in [0,1] \\ p(1) = \sum_{i=0}^{k} \lambda_i \nabla g_i(\bar{x}(1)) & \text{satisfies:} \end{cases}$$

- the **minimum principle**

$$\langle p(t), f[t]\rangle = \min_{u \in U} \langle p(t), f(t, \bar{x}(t), u)\rangle\,, \quad \text{a.e. } t \in [0,1]$$

- the **second order condition**

$$\frac{1}{2}\sum_{i=0}^{k} \lambda_i \langle g_i''(\bar{x}(1))\bar{y}(1), \bar{y}(1)\rangle + \frac{1}{2}\sum_{j=1}^{q} \int_0^1 \langle \varphi_j''(\bar{x}(s))\bar{y}(s), \bar{y}(s)\rangle d\psi_j(s)$$

$$+\frac{1}{2}\int_0^1 \langle p(s), f''[s](\bar{y}(s), \bar{v}(s))^2\rangle ds + \int_0^1 \langle p(s), f_u[s] w(s)\rangle ds \geq 0$$

for every selection $w(t) \in T_U^{\flat(2)}(\bar{u}(t), \bar{v}(t))$.