Optimal Control Techniques for Sampled-Data Control Systems with Medical Applications

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Control system of the form

\[ \dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0, \quad x(T) = x_T \]

and an optimal control of the Mayer type

\[ \min_{u(\cdot)} \varphi(x(T)) \]

with \( \varphi : \mathbb{R}^n \rightarrow \mathbb{R} \).

- **Permanent Control** : \( u : [0, T] \rightarrow U, \ u \in \mathcal{U} \) where \( \mathcal{U} \) are the absolutely continuous maps valued in \( U \).

- **Sampled Data Case** \( u \in \mathcal{U}_{sampled} : \) fix \( n \in \mathbb{N} \),
  - \( n \) sampling times
  - \( (n + 1) \)-amplitudes

\[ \eta = (\eta_0, \ldots, \eta_n) \in [0, 1]^{n+1} \]

The control is constant over \([t_i, t_{i+1}]\).
Numerical schemes

In the **permanent case**, the optimal control can be computed using

- **Direct scheme**: the problem is transformed into a finite dimensional optimization problem using
  - discretization scheme for the dynamics
  - discretization scheme for the control

- **Indirect scheme**: the problem can be analyzed using Pontryagin Maximum Principle which leads to Hamiltonian equations

$$
\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}
$$

$$
H(x, p, u) = \max_{v \in U} H(x, p, v)
$$

with $H(x, p, v) = p \cdot f(x, v)$, $p$ being the adjoint vector satisfying

$$
p(T) = p_0 \frac{\partial \phi}{\partial x}(x(T)) \quad \text{(transversality condition)}.
$$

This necessary optimality condition can be handled using a **shooting method**.
Shooting method

\[ p(t_2) \]

\[ p(T) \]

\[ x(t_1) \]

\[ x(T) \]

\[ x(0) \]
**Force-Fatigue muscular model**

**FES input** \( i \). Dirac impulses \( \delta \) at times \( t = 0, t_1, t_2, \ldots, t_N \).

\[
i(t) = \sum_{i=0}^{N} R_i \eta_i \delta(t - t_i), \quad \eta_i \in [0,1]
\]

where \( R_i := \begin{cases} 
1, & \text{for } i = 0, \\
1 + (\bar{R} - 1) \exp\left(-\frac{t_i - t_{i-1}}{\tau_c}\right), & \text{for } i = 1, \ldots, N,
\end{cases} \)

takes into account the *tetanic* contraction.

**FES signal** \( E_s \).

\[
E_s(t) = \frac{1}{\tau_c} \sum_{i=0}^{N} R_i \eta_i H(t - t_i) \exp\left(-\frac{t - t_i}{\tau_c}\right)
\]

\( H \) : Heaviside

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2. based on the Ding et al. / Hill-Huxley model
The FES signal drives the evolution of the dynamics:

\[
\dot{C}_N(t) = -\frac{C_N(t)}{\tau_c} + E_s(t; t_i, \eta_i),
\]
\[
\dot{F}(t) = -F(t) \gamma(t) + A \beta(t).
\]

where the Hill functions are given by

\[
\beta(t) := \frac{C_N(t)}{K_m + C_N(t)}, \text{ and } \gamma(t) := \frac{1}{\tau_1 + \tau_2 \beta(t)}.
\]

\((A, K_m, \tau_1, \tau_2)\) are the fatigue parameters.
The problem fits in the sampled data control frame with:

- \( u_0 = \eta_0 e^{-t/\tau_c} \) on \([0, T]\)
- \( u_1 = u_0(t_1) + \eta_1 R_1 e^{-(t-t_1)/\tau_c} \) on \([t_1, T]\)

Note that in this form each control splits into

- **Head**: restricting to \([t_i, t_{i+1}]\)
- **Tail**: restricting to \([t_i, T]\)

**Optimal control problems considered.**

- **(OCP1)** \( \max_{t_i, \eta_i} F(T) \)
- **(OCP2)** \( \min_{t_i, \eta_i} \int_0^T |F(t) - F_{ref}|^2 dt \) \((F_{ref} \text{: reference force})\).
Main theoretical results

\[ c_N(T) = \frac{1}{\tau_c} \sum_{i=0}^{n} e^{-(t-t_i)/\tau_c} (T - t_i) R_i \]

\( F(T) \) is a piecewise \( C^\infty \) mapping with respect to \( \eta_i, t_i \) and the problem fits in a finite dimensional optimization problem.

The non-fatigue model for a sequence of train, fatigue parameters \( P \) satisfy a dynamics of the form

\[
\dot{P}(t) = \frac{P(t) - P_{rest}}{\tau_p} + \alpha_p F(t).
\]

First order necessary optimality conditions can be obtained adapting [5] and are described in [2]. They were numerically implemented and compared with a direct optimization scheme in [3].
Numerical results using direct method

Direct method: \( F(T) \), \( \Delta t = 10 \, ms, \quad N = 10 \).
Theorem (see [2])

If \((\eta_0^*, \eta_1^*, \ldots, \eta_N^*, t_1^*, \ldots, t_N^*)\) is optimal, then there exists \(p\) satisfying the co-state equation and the transversality condition.

Moreover, the necessary conditions are:

(i) the inequality

\[
\left( \int_{t_i^*}^{T} p_1(s) b(s) \, ds \right) \tilde{\eta}_i \leq 0,
\]

for all \(i = 0, \ldots, n\) and all admissible perturbation \(\tilde{\eta}_i\) of \(\eta_i^*\);

(ii) and the inequality

\[
NC_i := \left( -p_1(t_i^*) b(t_i^*) G(t_{i-1}^*, t_i^*) \eta_i^* + b(-t_i^*) \eta_i^* \int_{t_i^*}^{T} p_1(s) b(s) \, ds \\
+ b(-t_i^*) (\tilde{R} - 1) \eta_{i+1}^* \int_{t_{i+1}^*}^{T} p_1(s) b(s) \, ds \right) \tilde{t}_i \leq 0,
\]

for all \(i = 1, \ldots, n\) and all admissible perturbation \(\tilde{t}_i\) of \(t_i^*\).
We need efficient algorithms (real-time application):

- Explicit expression of

\[(t_1, \ldots, t_n, \eta_0, \ldots, \eta_n) \rightarrow F(T)\]

to apply a direct optimization scheme

- \textit{LQ} methods
- \textit{MPC} methods

- Online parameter estimation coupled with optimization methods [1]

- Extension to the non-isometric case with \textit{joint angle variable} to produce a motion [4]


