Mining and Proving Conjectures: Discovering Invariants on Integer Sequences

(with an application for short term replanification)

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PGMO

The Question Motivating this Work

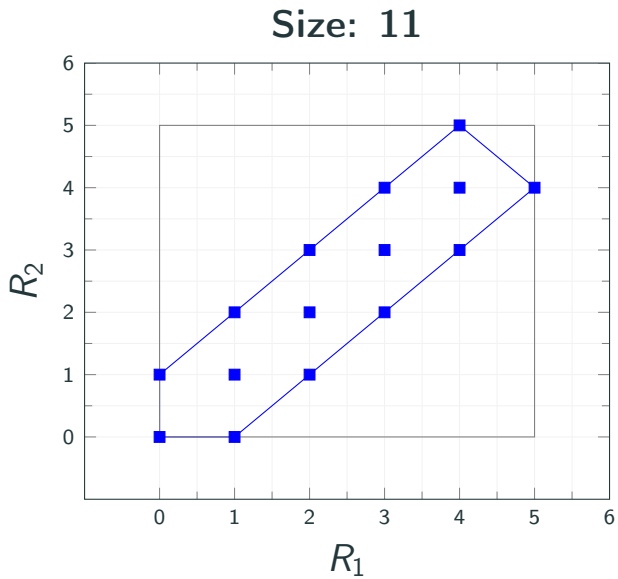
Consider two constraints $\gamma_1(\langle X_1, \ldots, X_n \rangle, R_1)$ and $\gamma_2(\langle X_1, \ldots, X_n \rangle, R_2)$, where each X_i is in $\{1, \ldots, n\}$.

The variables R_1 and R_2 are constrained to be the result of some computations over $\langle X_1, \ldots, X_n \rangle$ depending only on the relations between $X_1, X_2, \ldots, X_k$.

For example, R_1 is the number of peaks in $\langle X_1, \ldots, X_n \rangle$ and R_2 is the number of valleys in $\langle X_1, \ldots, X_n \rangle$.

What is the set of feasible pairs of R_1 and R_2?

Example of Sets of Feasible Pairs of R_1 and R_2: Convex Case

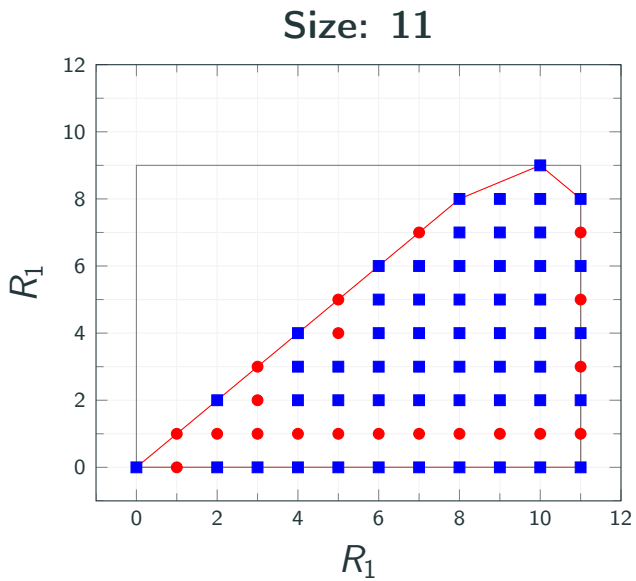


$\gamma_1 = \text{nb\_peak}$

$\gamma_2 = \text{nb\_valley}$

- The set of feasible (blue) points is convex.
- A conjunction of linear inequalities of R_1, R_2 and n.

Example of Sets of Feasible Pairs of R_1 and R_2: Non-Convex Case



$\gamma_1 = \text{sum\_width\_decreasing\_sequence}$

$\gamma_2 = \text{sum\_width\_zigzag}$

- The set of feasible (blue) points is non-convex.
- A conjunction of linear inequalities of is **not enough**.
- A need for **non-linear** characterisation.

Two Emerging Problems for Characterising Infeasible Combinations

1. Generate linear inequalities depending on R_1, R_2 and parameterised by $f(n) \in \{n, n \mod p, \sqrt{n}, \dots\}$, which represent the facets of the convex hull.
2. Generate non-linear invariants eliminating infeasible points on and/or inside of the convex hull.

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How to solve these two problems in a systematic way for a large family of constraints?

Main Insight...

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Main Insight...

Use **register automata** and parameterised characterisation.

Take-Away Message

- **Convex Case:** A compositional way of generating cuts from register automata.

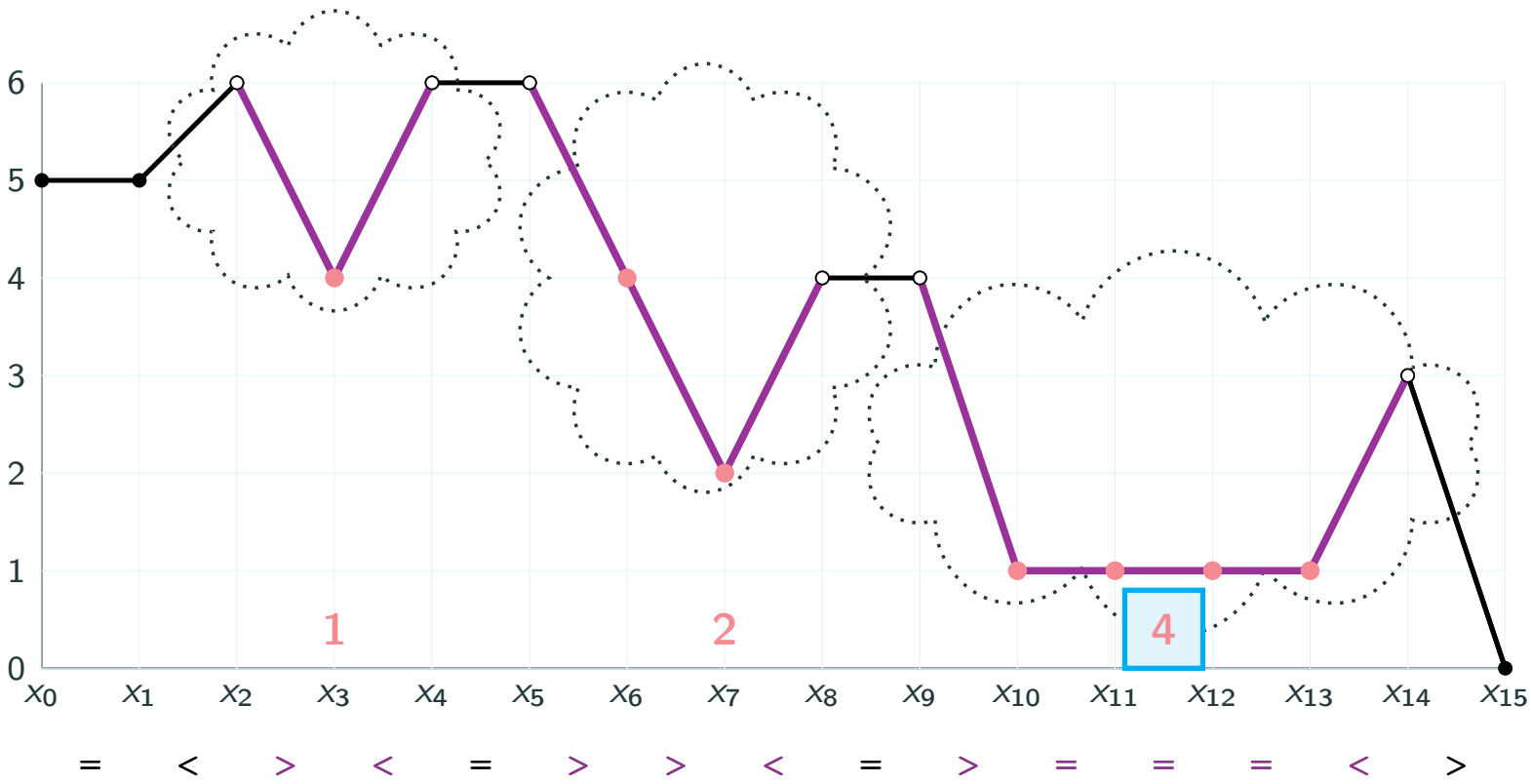
- **Non-Convex Case:** Data Mining + Proof using automata characterising infeasible combinations of points for conjunction of constraints.

Case Study: Time-Series Constraints

- Family of constraints arising from **power systems** modelling.
- Declaratively described by **regular expressions**.
- Baseline implementation as **register automata**.
- Lots of missing propagation, not good enough for use case.
- Work on improving propagators for **all** constraints
  at the same time.

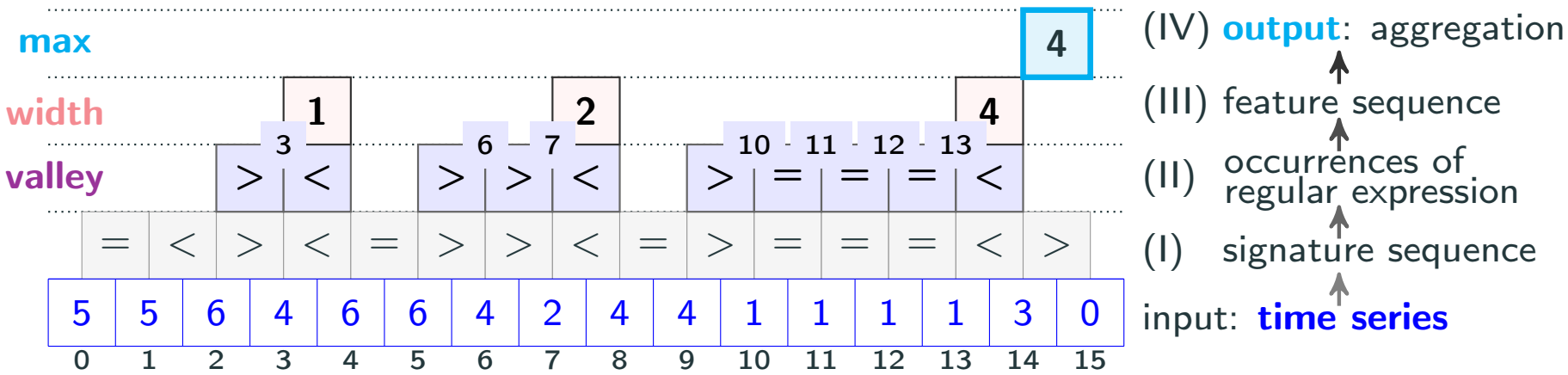
Example of a Time-Series Constraint

Constrain the **maximum** of the **widths** of the **valleys**
in the time series $X = \langle 5, 5, 6, 4, 6, 6, 4, 2, 4, 4, 1, 1, 1, 1, 3, 0 \rangle$.



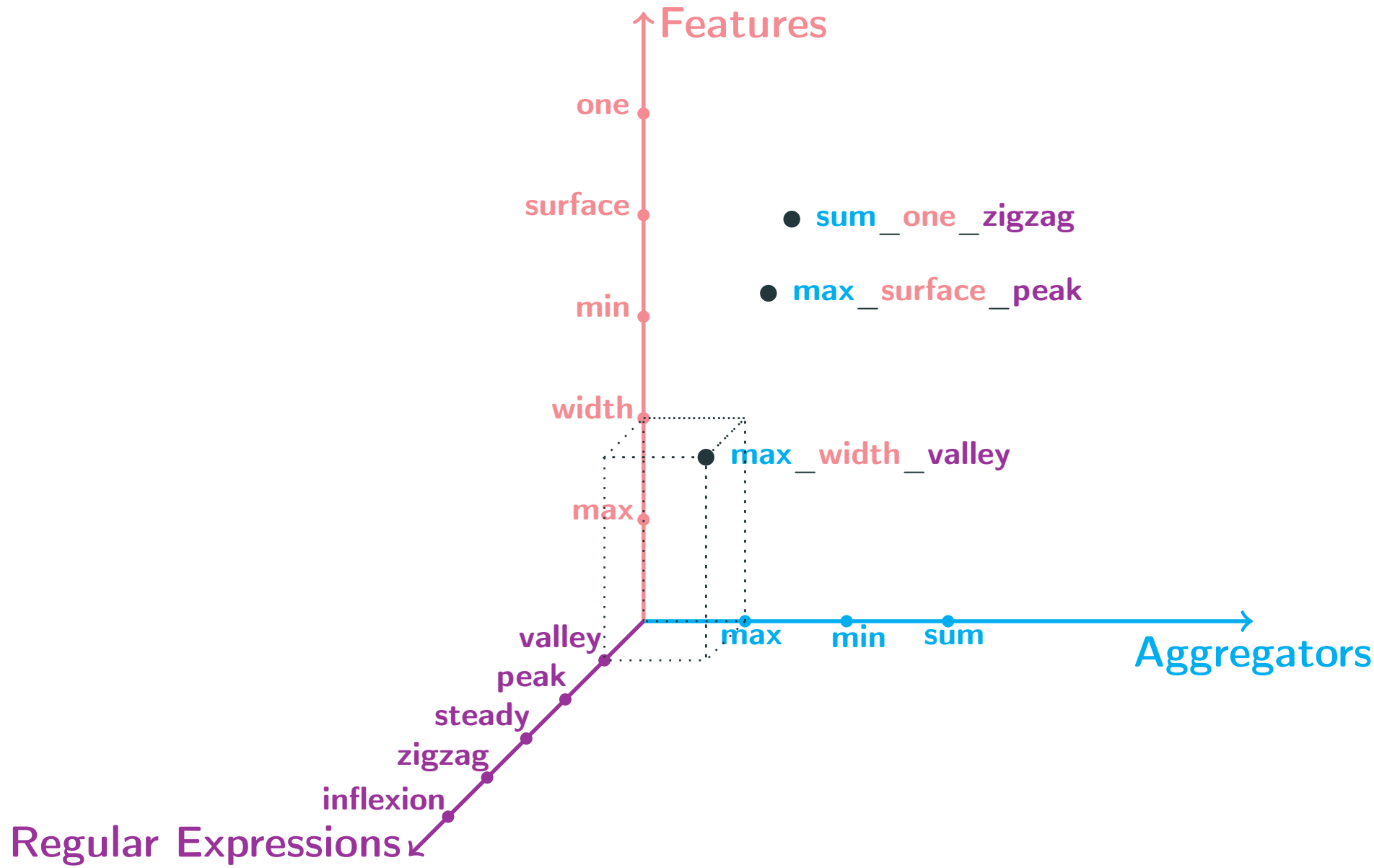
A subsequence $\langle X_i, \ldots, X_j \rangle$ of $\langle X_0, \ldots, X_m \rangle$ is a **valley** if the signature of $\langle X_{i-1}, \ldots, X_{j+1} \rangle$ is a maximal word matching '>(>|=)*(<|=)*<'.

Compositional Time-Series Definition by Multiple Layers of Functions

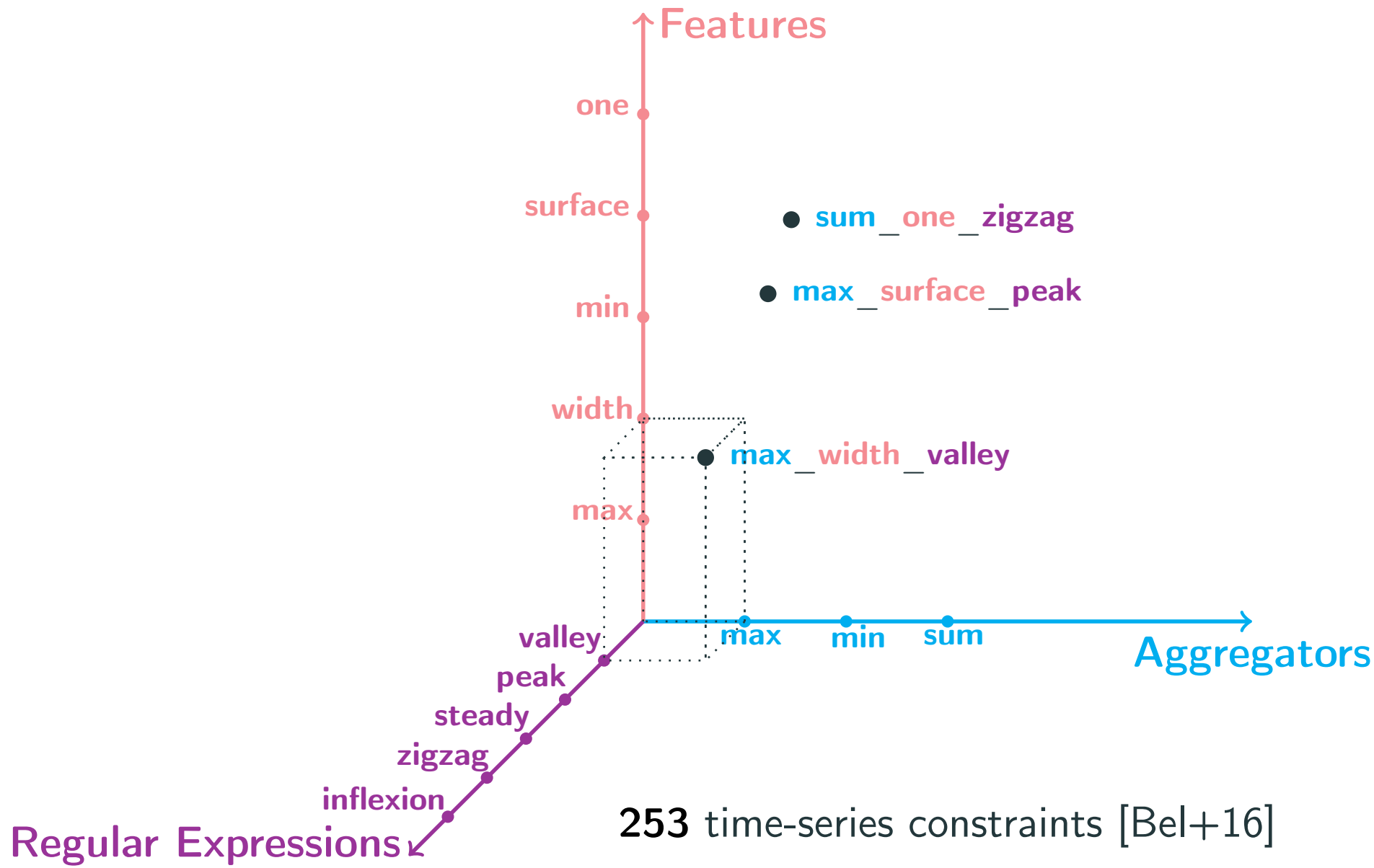


$$\texttt{max\_width\_valley}(\langle 5,5,6,4,6,6,4,2,4,4,1,1,1,1,3,0\rangle, 4)$$

Space of Time-Series Constraints



Space of Time-Series Constraints



253 time-series constraints [Bel+16]

Pratical Motivation of Time-Series Constraints: Unit Commitment Problem for a Power Plant

Let $X = \langle X_1, \ldots, X_n \rangle$ be a production curve of a power plant.

Time-series constraints can be used to implement missing constraints, e.g. **business rules**.

Examples of Business Rules:

- After reaching a production plateau a production unit should stay at the same level for at least k period of times.
  min_width_plateau(X, k).
- No zigzag of production level spanning over more than k periods of times.
  max_width_zigzag(X, k).

Time-Series Constraints Families of This Work

- Only topological constraints,
  i.e. nb_$\sigma(X, R)$ and sum_width_$\sigma(X, R)$.
- R depends only on the relations between X variables,
  but not on their values themselves.
- Representation as register automata
  with linear register updates.
- 35 constraints in the two families.

Generating Linear Inequalities for Two Families of Time-Series Constraints

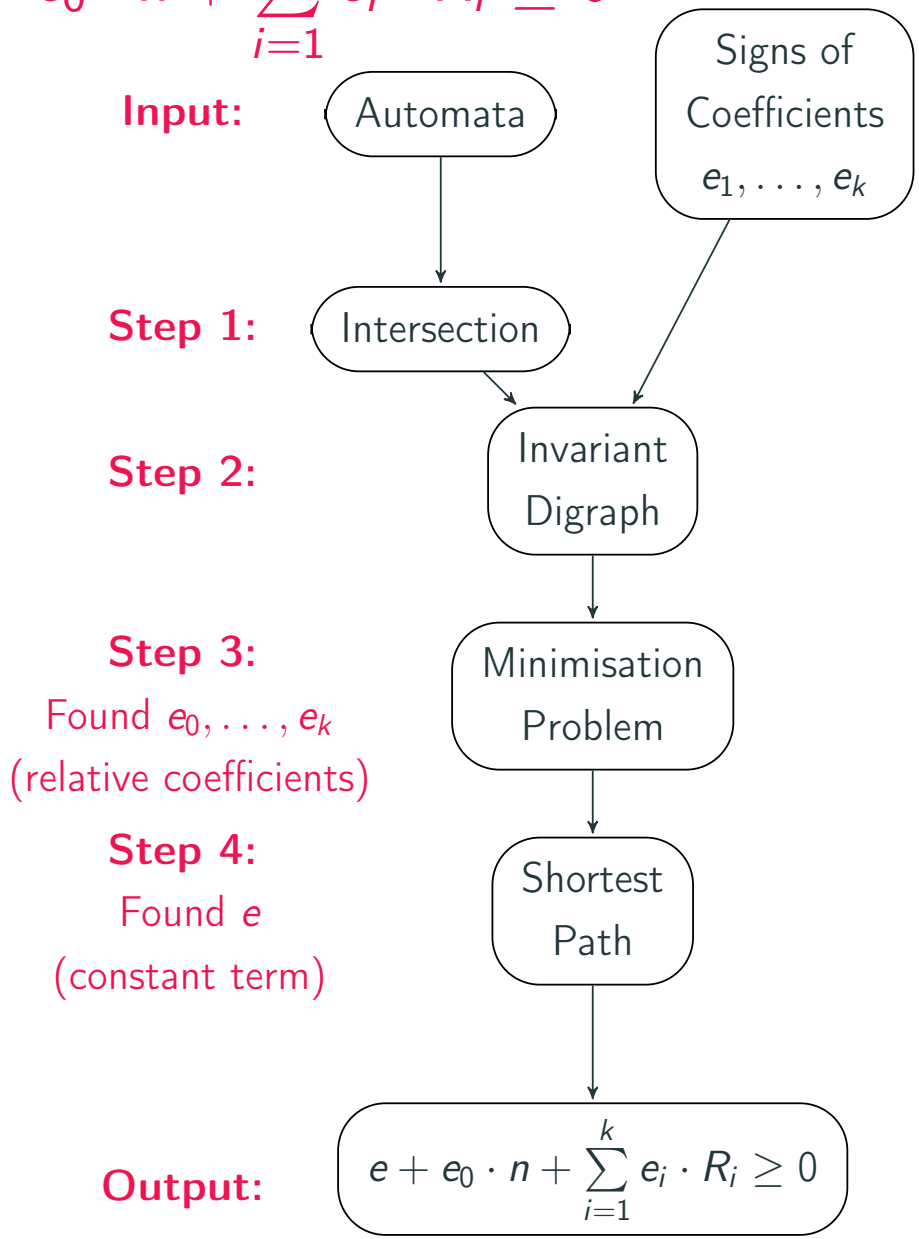
Consider a conjunction of time-series constraints $\gamma_1(X, R_1), \ldots, \gamma_k(X, R_k)$, with X being $\langle X_1, \ldots, X_n \rangle$.

Every γ_i has a representation by a register automaton $\mathcal{M}_i$ satisfying some conditions.

In [ABS17] we proposed a systematic and uniform technique to generate 2^k linear inequalities $e + e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i \geq 0$, with $e, e_0, \ldots, e_k \in \mathbb{Z}$.

These inequalities are true for any sequence recognised by **all** the automata for $\gamma_1, \gamma_2, \ldots, \gamma_k$.

Our Method for Generating Linear Inequalities of the form $e + e_0 \cdot n + \sum_{i=1}^{k} e_i \cdot R_i \geq 0$



Example of Generated Invariants for a Conjunction of Two Constraints

$\text{nb\_peak}(X, R_1) \wedge \text{nb\_valley}(X, R_2)$ with $X = \langle X_1, \ldots, X_n \rangle$

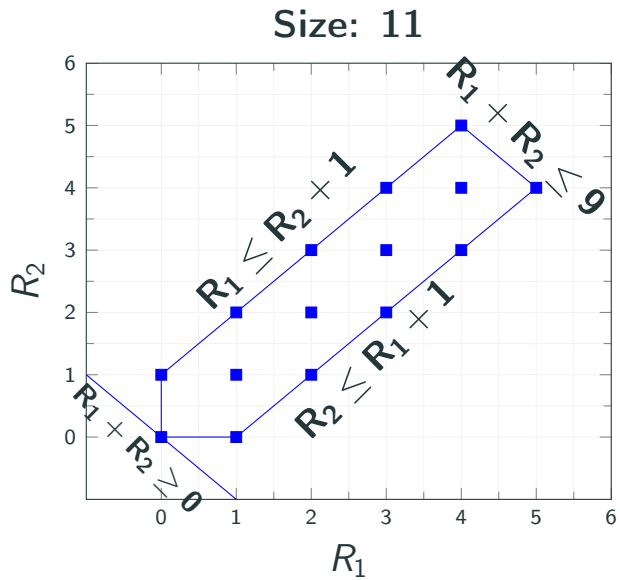
Our method finds the following invariants:

$$R_2 \leq R_1 + 1$$

$$R_1 \leq R_2 + 1$$

$$R_1 + R_2 \leq n - 2$$

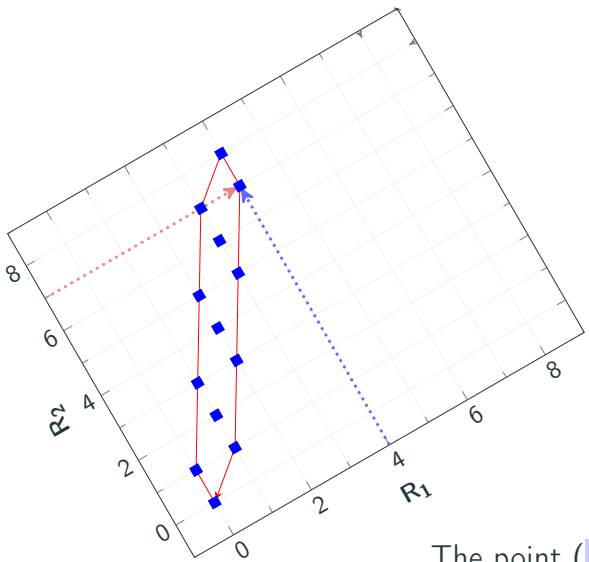
$$R_1 + R_2 \geq 0$$

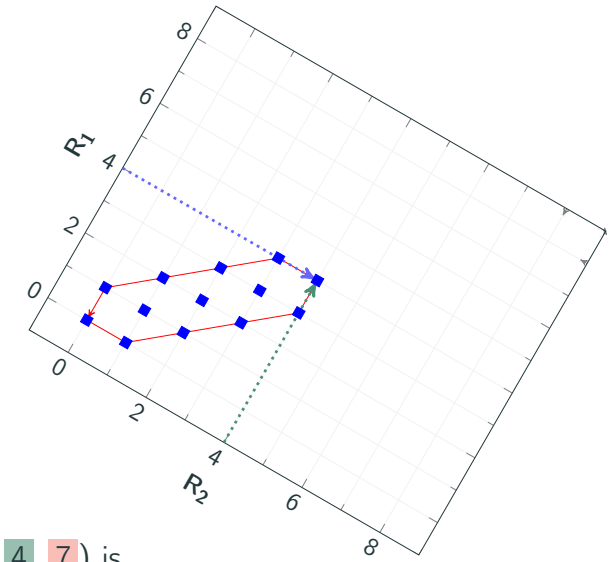


3 out of the 4 found inequalities are facet-defining!

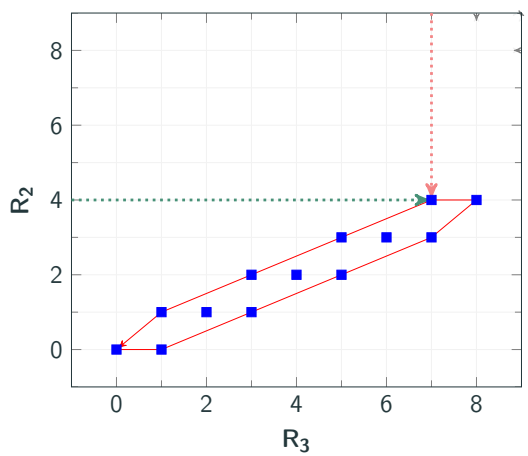
Example of a Generated Invariant for a Conjunction of Three Constraints

$$\text{nb\_peak}(X, R_1) \wedge \text{nb\_valley}(X, R_2) \wedge \text{nb\_inflexion}(X, R_3)$$





The point (4 , 4 , 7) is discarded by $R_1 + R_2 \leq R_3$



Discarded by $R_1 + R_2 \leq R_3$:

- $(4, 4, 7)$
- $(3, 3, 5)$
- $(2, 2, 3)$
- $(1, 1, 1)$

Using Linear Invariants in the Context of Linear Programming

$\text{nb\_peak}(X, P) \wedge \text{nb\_valley}(X, V)$,
where X is a sequence of 100 variables ranging over $[0, 1000]$.

Objective Function: maximise $P - 2 \cdot V$

	LP	LP + cuts
Time	300s	0s
Best objective	1	1
Best bound	49	1
Gap	4800%	0%

LP is a pure linear reformulation [Ara+16]

LP + cuts is a linear reformulation with cuts on every prefix

Generating Non-Linear Invariants that Deal With Missing, Infeasible Cases

Three Phases of our Method:

1. **Generation of Data:** generate all feasible combinations of $R_1, \ldots, R_k$ for a given range of n values.
2. **Mining Phase:** generate hypothesis covering subsets of infeasible points using the generated data.
3. **Proving Phase:** prove the generated hypothesis using register automata.

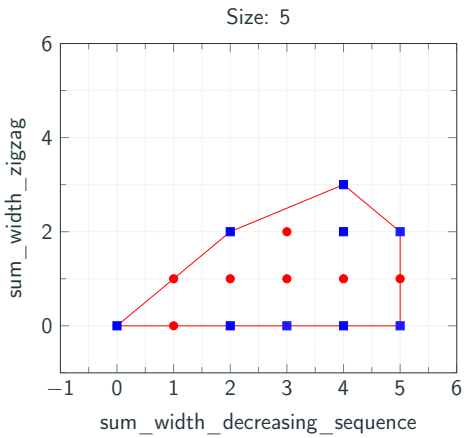
The three phases are **offline** and solver/system **independent**.

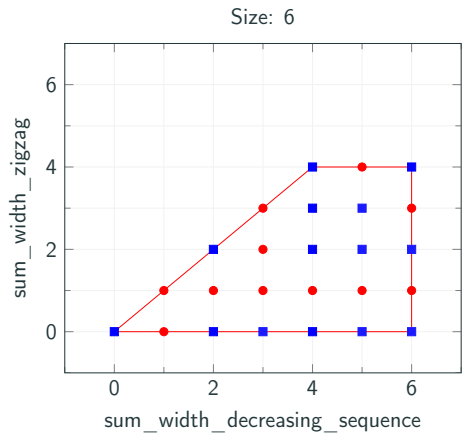
Generation of Data: Our Case

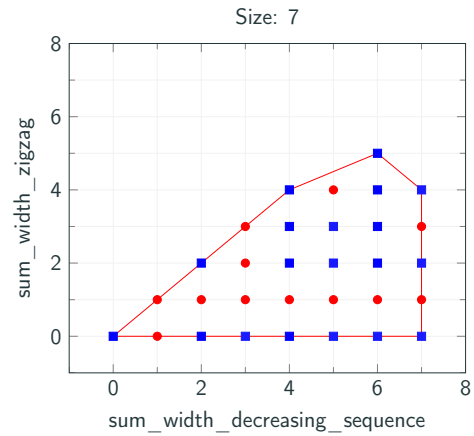
- Pairs of different time-series constraints $\gamma_1(\langle X_1, \ldots, X_n \rangle, R_1)$ and $\gamma_2(\langle X_1, \ldots, X_n \rangle, R_2)$.
- Generate all feasible pairs (R_1, R_2) for $n \in \{1, 2, \ldots, 12\}$.
- Compute the convex hull using Graham's scan.
- Collect all infeasible points inside the convex hull.

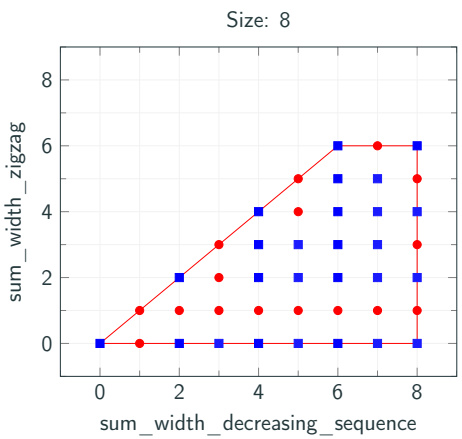
Example of Samples of the Generated Data

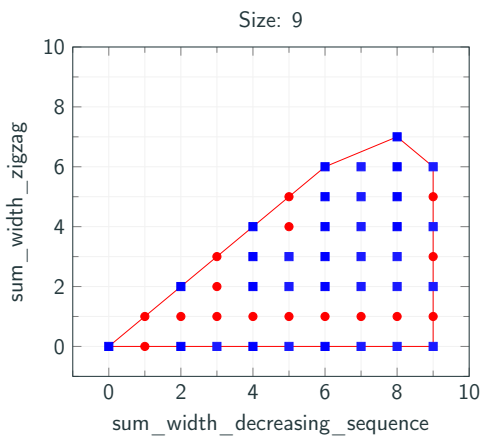
$\gamma_1 = \text{sum\_width\_decreasing\_sequence}$, $\gamma_2 = \text{sum\_width\_zigzag}$

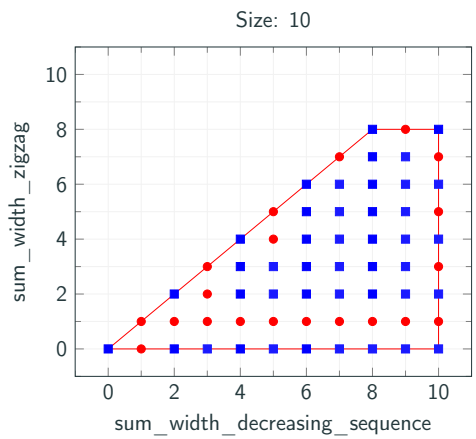








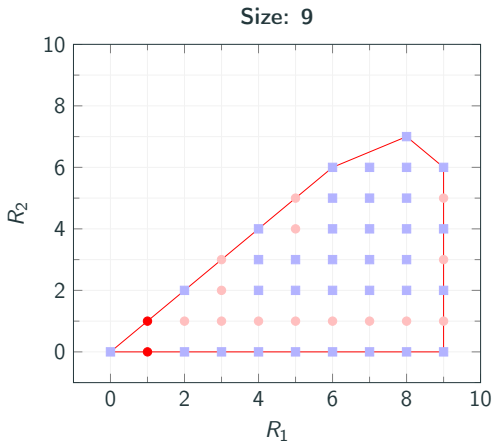




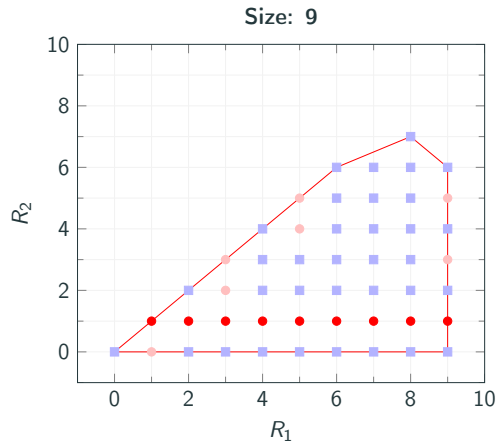
Mining Phase: Generation of Hypothesis

- Consider only samples of sizes from 7 to 12.
- Hypothesis of type $C_1 \wedge C_2 \wedge \cdots \wedge C_p$ to cover infeasible points inside the convex hull.
- Every C_k is a relation from our bias.
- Examples of relations in our bias
  - $R_i = c,\ c \in \mathbb{Z}$
  - $R_i = upper\_bound(R_i, n)$
  - R_i is odd (even)
  - $R_i = R_j$
  - etc.
- Every infeasible point on/inside of the convex hull must be covered by at least one hypothesis.

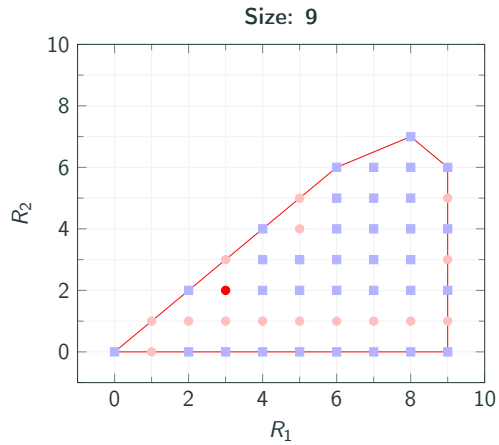
Mining Phase: Example



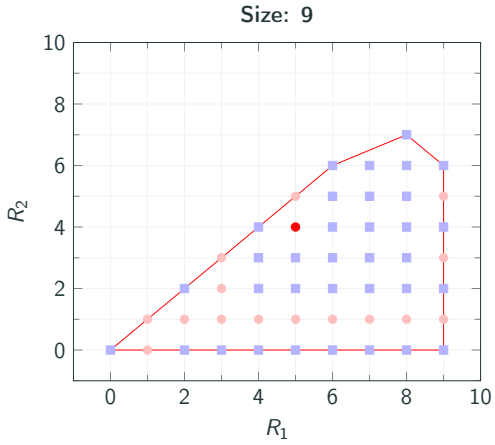
(Group ①) $R_1 = 1$



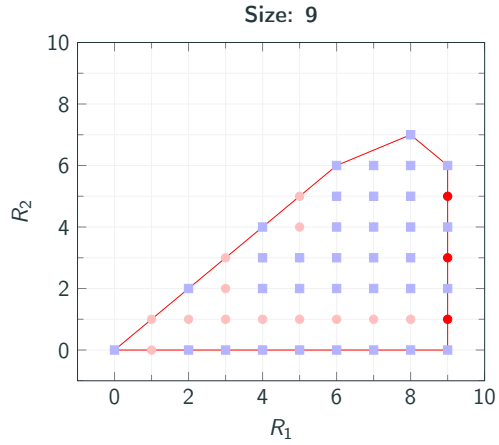
(Group ②) $R_2 = 1$



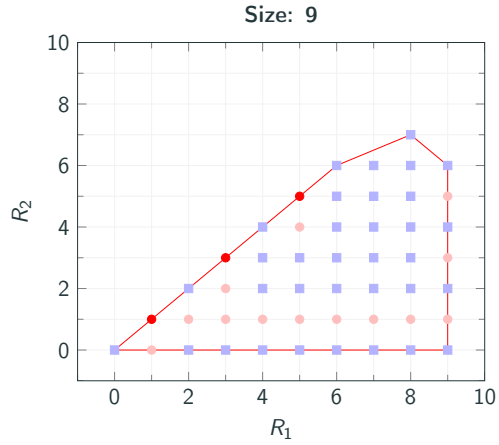
(Group ③) $R_1 = 3 \wedge R_2 = 2$



(Group ④) $R_1 = 5 \wedge R_2 = 4$



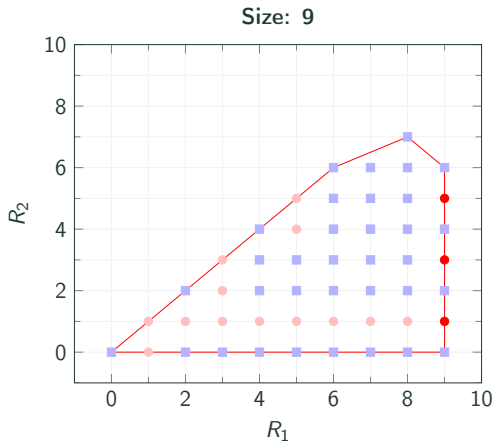
(Group ⑤) $R_1 = up(R_1, n) \wedge R_2 \bmod 2 = 1$



(Group ⑥) $R_1 = R_2 \wedge R_1 \bmod 2 = 1$

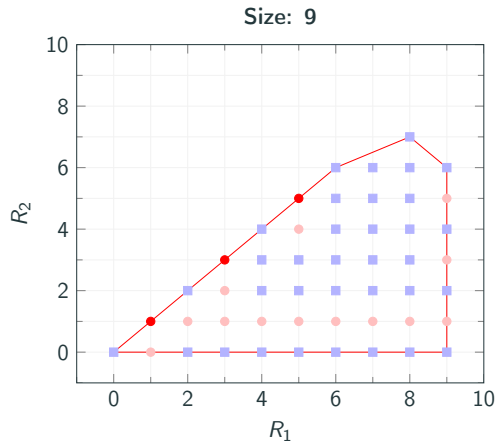
Classification of Groups of Points

1. **Independent Groups**: $H = C_1 \wedge C_2 \wedge \cdots \wedge C_p$, every C_k depends only on one R_i.
2. **Dependent Groups**: $H = C_1 \wedge C2 \wedge \cdots \wedge C_p$, at least one C_k depends on more than one R_i.



(Group ⑤) $R_1 = up(R_1, n) \wedge R_2 \bmod 2 = 1$

Independent Group



(Group ⑥) $R_1 = R_2 \wedge R_1 \bmod 2 = 1$

Dependent Group

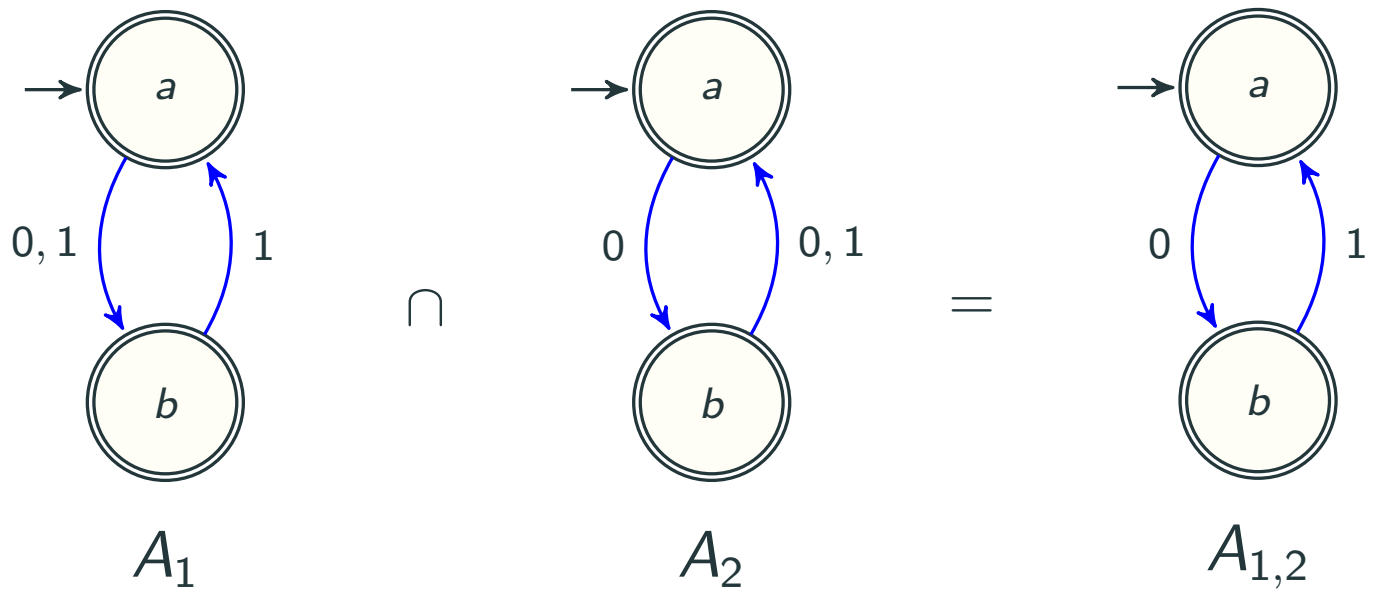
The proof scheme depends on the group type!

Proving Phase: Independent Groups

- For every hypothesis $C_1 \wedge C_2 \wedge \cdots \wedge C_p$,
  generate a **constant size automaton** for each C_i relation.
- Do the **intersection** of the automata for all $C_1, C_2, \ldots C_p$.
- The intersection is an automaton that recognises **all and only**
  sequences satisfying the conjunction $C_1 \wedge C_2 \wedge \cdots \wedge C_p$.
- If the intersection is empty,
  then $C_1 \wedge C_2 \wedge \cdots \wedge C_p$ is not feasible.
- Otherwise, a counter example violating the hypothesis.

Example: Intersection of Two Automata

- Two automata consuming sequences of '0' and '1'.
- A_1: all '0' can appear only in odd positions, e.g. $\langle 0, 1, 0, 1, 1 \rangle$.
- A_2: all '1' can appear only in even positions, e.g. $\langle 0, 1, 0, 1, 0 \rangle$.
- Intersection $A_{1,2}$ of A_1 and A_2 recognises sequences of alternating '0' and '1'.

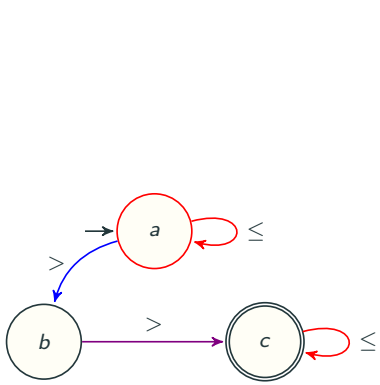


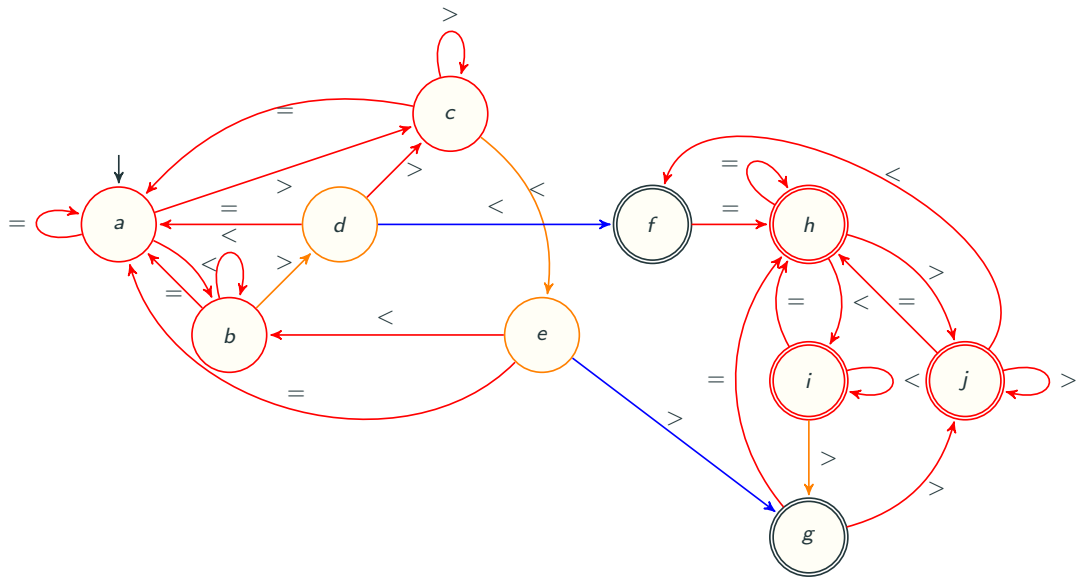
A_1 A_2 $A_{1,2}$

Proving Phase: Independent Group Example

$\text{sum\_width\_decreasing\_sequence}(X, R_1) \wedge \text{sum\_width\_zigzag}(X, R_2)$

An independent group is described by $R_1 = 3 \wedge R_2 = 2$





$R_1 = 3$ $R_2 = 2$

The intersection of two automata is **empty**!

The combination $R_1 = 3$ and $R_2 = 2$ is indeed **infeasible**.

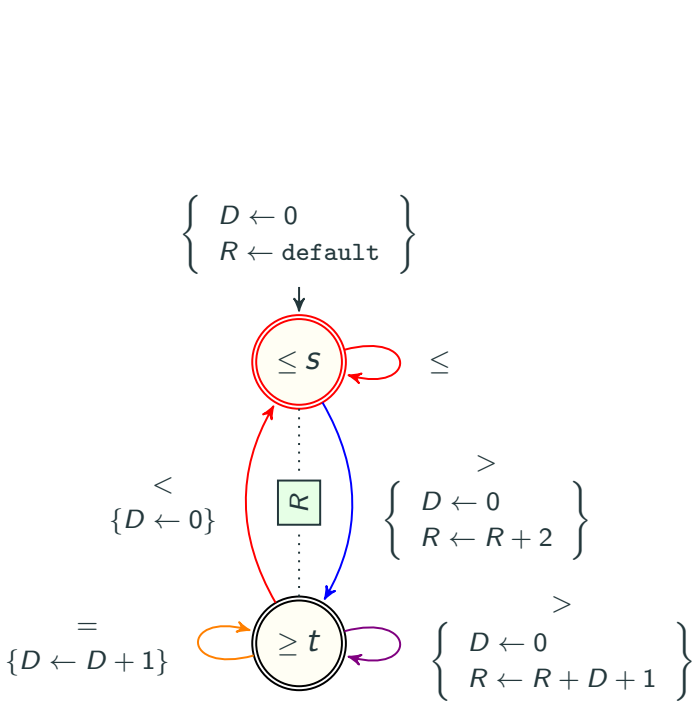
Systematic Generation of Automata for Proving Independent Groups

For two considered families of time-series constraints, we can generate systematically automata for:

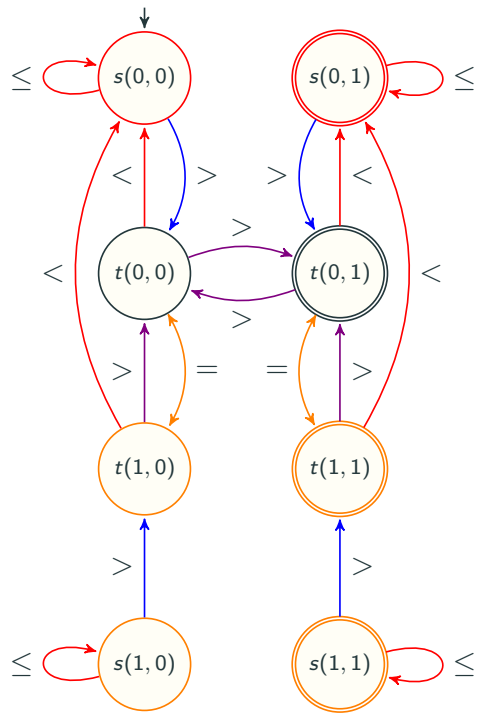
- $R_i = c$, $c \in \mathbb{Z}$,
- $R_i = up(R_i, n) - c$, $c \geq 0 \in \mathbb{Z}$, and γ_i is nb_σ,
- $R_i = up(R_i, n)$, and γ_i is sum_width_σ,
- R_i is odd/even.

Example of Automaton for the 'R is odd' Rule

sum_width_decreasing_sequence(X, R)



(a)

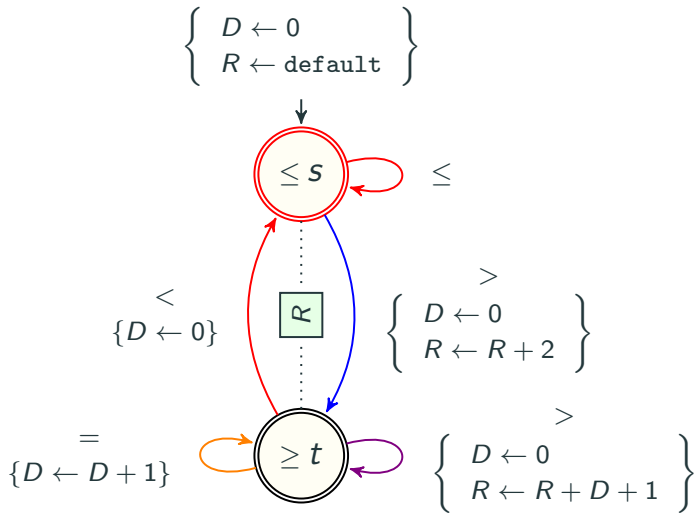


(b)

(a) Automaton for the sum_width_decreasing_sequence time-series constraint; (b) Automata for the 'R is *odd*' rule, constructed from (a)

Example of Automaton for the $R = up(R, n)$ Rule

sum_width_decreasing_sequence(X, R)



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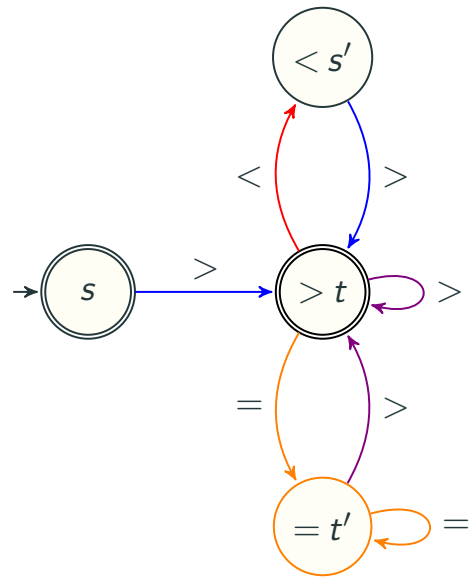

(b)

(a) Automaton for the sum_width_decreasing_sequence time-series constraint; (b) Automata for the $R = up(R, n)$ rule, constructed from (a)

Proving Phase: Dependent Groups

- Proof of dependent groups requires case by case consideration.
- The proof consists of verifying a certain property using our cut-generation technique.
- Often, this property is only a sufficient, but not necessary condition, for proving a hypothesis.

Conclusion (Take-Away Message)

- **Convex Case:** A compositional way of generating cuts from register automata. Already evaluated in [ABS17].
- **Non-Convex Case:** Data Mining + Proof using automata characterising infeasible combinations of points for conjunction of constraints. Currently evaluated.
- Our method is offline and solver/system independent.

References I

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Nicolas Beldiceanu et al. "Using finite transducers for describing and synthesising structural time-series constraints". In: *Constraints* 21.1 (2016). Journal fast track of CP 2015: summary on p. 723 of LNCS 9255, Springer, 2015, pp. 22–40.