Second order analysis of infinite dimensional bilinear optimal control

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The dynamics

Let $\mathcal{H}$ be a **reflexive Banach space.** Consider, for $t \in (0,T)$, **the bilinear equation**

$$\dot{\Psi}(t) + \mathcal{A}\Psi(t) = f(t) + u(t)(\mathcal{B}_1 + \mathcal{B}_2\Psi(t)); \quad \Psi(0) = \Psi_0, \tag{E}$$

where $\mathcal{A}$ is an **unbounded operator** on $\mathcal{H}$ that is the **infinitesimal generator** of the (strongly) continuous semigroup $e^{-t\mathcal{A}}$, that verifies

$$\|e^{-t\mathcal{A}}\|_{\mathcal{L}(\mathcal{H})} \leq c_{\mathcal{A}} e^{\lambda_{\mathcal{A}} t}, \quad \text{for } t > 0,$$

for $c_{\mathcal{A}}, \lambda_{\mathcal{A}} > 0$, and

$$\Psi_0 \in \mathcal{H}, \quad f \in L^1(0,T;\mathcal{H}), \quad \mathcal{B}_1 \in \mathcal{H}, \quad \mathcal{B}_2 \in \mathcal{L}(\mathcal{H}), \quad u \in L^1(0,T),$$

The operator $\mathcal{A}$

$\mathcal{A}$ has dense domain on $\mathcal{H}$ given by

$$\mathrm{dom}(\mathcal{A}) := \left\{ y \in \mathcal{H};\ \lim_{t \downarrow 0} \frac{e^{-t\mathcal{A}} y - y}{t} \text{ exists} \right\},$$

and, for $y \in \mathrm{dom}(\mathcal{A})$:

$$\mathcal{A} y = -\lim_{t \downarrow 0} \frac{e^{-t\mathcal{A}} y - y}{t}.$$

Solution in the semigroup sense

We define for the equation (E) the mild solution (or solution *in semigroup sense*): $\Psi \in C([0,T];\mathcal{H})$ such that

$$\Psi(t) = e^{-t\mathcal{A}}\Psi_0 + \int_0^t e^{-(t-s)\mathcal{A}}\big(f(s) + u(s)(\mathcal{B}_1 + \mathcal{B}_2\Psi(s))\big)\mathrm{d}s.$$

By a fixed-point argument:

Proposition (Existence & Uniqueness)

(E) has a unique mild solution in $C([0,T];\mathcal{H})$.

Notation: $\Psi[u]$ denotes the solution of (E) associated to u.

Theorem (Continuity with respect to data)

There exists $\gamma > 0$ *such that the solution* Ψ *of (E) satisfies*

$$\|\Psi[u]\|_{C([0,T];\mathcal{H})} \leq \gamma\left(\|\Psi_0\|_{\mathcal{H}} + \|f\|_{L^1(0,T;\mathcal{H})} + \|\mathcal{B}_1\|_{\mathcal{H}}\|u\|_1\right)e^{\gamma\|u\|_1}.$$

The optimal control problem

$$\mathcal{U} := L^1(0,T); \quad \mathcal{Y} := C([0,T];\mathcal{H}).$$

Optimal control problem (P):

$$\min \ J(u,\Psi) := \alpha \int_0^T u(t)\mathrm{d}t + \frac{a_1}{2}\int_0^T \|\Psi(t) - \Psi_d(t)\|^2_{\mathcal{H}}\mathrm{d}t + \frac{a_2}{2}\|\Psi_T - \Psi_{dT}\|^2_{\mathcal{H}},$$

$$\text{s.t.} \begin{cases} \dot{\Psi}(t) + \mathcal{A}\Psi(t) = f(t) + u(t)(\mathcal{B}_1 + \mathcal{B}_2\Psi(t)); \quad \Psi(0) = \Psi_0, \\ u \in \mathcal{U}_{ad} := \{u \in \mathcal{U};\ u_m \leq u(t) \leq u_M \text{ a.e. on } [0,T]\}, \end{cases}$$

where $\Psi_d \in L^\infty(0,T;\mathcal{H})$, $\Psi_{dT} \in \mathcal{H}$, $\alpha, a_1, a_2 \in \mathbb{R}$.

$\hat{u} \in L^1(0,T)$ is a weak minimum if $\hat{u}$ is minimum over

$$\{u \in \mathcal{U}_{ad} : \|u - \hat{u}\|_\infty < \varepsilon\}$$

for some $\varepsilon > 0$.

Literature

Control-affine problems, second order sufficient conditions for **singular arcs**:

- A.V. Dmitruk, *Soviet Math. Dokl.* 1977
- M.S. Aronna, J.F. Bonnans, A.V. Dmitruk, P.A. Lotito, *Numer. Algebra Control Optim.* 2012.
- J.F. Bonnans, *Opt. Methods & Software* 2014

Control-affine **PDEs**, second order sufficient conditions for **bang-bang case:**

- M. Bergounioux, D. Tiba, D., *SIAM J. Control Optim.* 1996
- E. Casas, F. Tröltzsch, *Jahresber. Dtsch. Math. Ver.* 2015
- J.F. Bonnans, D Tiba, D., *Math. Rep. (Bucur.)* 2009

Optimal control in the **semigroup setting:**

- X. Li, J. Yong, in *Systems & Control: Foundations & Applications,* Birkhäuser, 1995
- H.O. Fattorini, H. Frankowska, *MCSS,* 1991
- H. Goldberg, F. Tröltzsch, *SICON,* 1993
- K. Ito, K. Kunisch, *SICON,* 2007

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The linearized state equation

Linearized equation at $(\hat{\Psi}, \hat{u})$: for $v \in \mathcal{U}$,

$$\dot{z}(t) + \mathcal{A}z(t) = \hat{u}(t)\mathcal{B}_2 z(t) + v(t)(\mathcal{B}_1 + \mathcal{B}_2\hat{\Psi}(t)); \quad z(0) = 0,$$

Notation: $z[v]$ is the associated solution

Theorem

The mapping $u \mapsto \Psi[u]$ from $\mathcal{U}$ to $\mathcal{Y}$ is of class C^∞ and we have that

$$D\Psi[u]v = z[v], \qquad \forall v \in \mathcal{U}.$$

Existence of optimal control

Let $y[y_0, g]$ denote the solution of

$$\dot{y}(t) + \mathcal{A}y(t) = g(t), \quad t \in (0,T), \quad y(0) = y_0.$$

We consider the compactness hypothesis:

$$\begin{cases} \text{For given } y_0 \in \mathcal{H}\text{, the mapping } g \mapsto y[y_0, g] \\ \text{is compact from } L^2(0,T;\mathcal{H}) \text{ to } L^2(0,T;\mathcal{H}). \end{cases}$$

Theorem (Existence of optimal solution)

The problem (P) has a nonempty set of solutions.

The costate equation

Let $\mathcal{A}^*$ be the infinitesimal generator of the strongly continuous semigroup $e^{-t\mathcal{A}^*}$ in $\mathcal{H}^*$ called dual (backward) strongly continuous semigroup given by

$$e^{-t\mathcal{A}^*} = (e^{-t\mathcal{A}})^*.$$

The costate equation reads

$$-\dot{p} + \mathcal{A}^* p = a_1(\Psi - \Psi_d) + u\mathcal{B}_2^* p; \quad p(T) = a_2(\Psi(T) - \Psi_{dT}).$$

and in the mild (semigroup) sense,

$$p(t) = \ e^{(t-T)\mathcal{A}^*} a_2(\Psi(T) - \Psi_{dT}) + \int_t^T e^{(t-s)\mathcal{A}^*} \Big(a_1(\Psi(s) - \Psi_d(s)) + u(s)\mathcal{B}_2^* p(s) \Big) \mathrm{d}s.$$

First order necessary conditions

Theorem

Set $F(u) := J(u, \Psi[u])$ *and* $\Lambda(t) := \alpha + \langle p(t), \mathcal{B}_1 + \mathcal{B}_2 \hat{\Psi}(t) \rangle$. *Then,*

$$DF(u)v = \int_0^T v(t)\Lambda(t)\mathrm{d}t, \qquad \text{for all } v \in \mathcal{U}.$$

Λ plays the role of the *"switching function"*.

Theorem (First order necessary optimality condition)

If $\hat{u}$ *is a weak minimum, then there holds*

$$\begin{cases} \hat{u}(t) = u_m, & \text{if } \Lambda(t) > 0, \\ \hat{u}(t) = u_M, & \text{if } \Lambda(t) < 0, \\ \Lambda(t) = 0, & \text{if } u_m < u(t) < u_M. \end{cases}$$

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Second order expansion

For $(\hat{\Psi}, \hat{u})$ state-control pair, and $\hat{p}$ the corresponding costate, set

$$\mathcal{Q}(z,v) := \int_0^T \Big(a_1 \|z(t)\|_{\mathcal{H}}^2 + 2v(t)\langle \hat{p}(t), \mathcal{B}_2 z(t)\rangle \Big) \mathrm{d}t + a_2 \|z(T)\|_{\mathcal{H}}^2.$$

Appropriate estimations lead to

Proposition

For any $u \in \mathcal{U}$,

$$F(u) = F(\hat{u}) + DF(\hat{u})v + \tfrac{1}{2}\mathcal{Q}(z[v],v) + O(\|v\|_1^3).$$

Second order necessary optimality condition

Critical cone

$$C(u) := \left\{ \begin{array}{l} v \in L^1(0,T)\,;\; \Lambda(t)v(t) = 0 \text{ a.e. on } [0,T], \\ v(t) \geq 0 \text{ a.e. on } \{\hat{u}(t) = u_m\},\; v(t) \leq 0 \text{ a.e. on } \{\hat{u}(t) = u_M\} \end{array} \right\}.$$

Theorem (Second order necessary condition)

Let $\hat{u} \in \mathcal{U}$ be a weak minimum, $\hat{p}$ corresponding costate. Then,

$$\mathcal{Q}(z[v], v) \geq 0 \quad \textit{for all } v \in C(\hat{u}).$$

Goh transform on a linear equation

Given a **linear equation,** of the form:

$$\dot{y} + \mathcal{A}y = ay + b^0 v, \qquad y(0) = 0,$$

with $a \in L^\infty(0,T;\mathcal{L}(\mathcal{H}))$; $\quad b^0 \in C([0,T];\mathcal{H})$.

The Goh transform:

$$w(t) := \int_0^t v(s)\mathrm{d}s, \quad \xi := y - wb^0; \tag{GOH}$$

where $b^1 := (a - \mathcal{A})b^0 - \dot{b}^0$.

Lemma

ξ is a mild solution of

$$\dot{\xi} + \mathcal{A}\xi = a\xi + wb^1; \quad \xi(0) = 0.$$

Goh transform on the linearized equation

Set $\mathcal{B} := \mathcal{B}_1 + \mathcal{B}_2 \hat{\Psi}$. For the linearized equation

$$\dot{z}(t) + \mathcal{A}z(t) = \hat{u}(t)\mathcal{B}_2 z(t) + v(t)\mathcal{B}; \quad z(0) = 0,$$

we set

$$\xi := z - w\mathcal{B}, \quad w := \int_0^t v(s)\mathrm{d}s.$$

Lemma

ξ is a mild solution of

$$\dot{\xi} + \mathcal{A}\xi = \hat{u}\mathcal{B}_2\xi + wb^1;$$

where $b^1 := -\mathcal{B}_2 f - [\mathcal{A}, \mathcal{B}_2]\hat{\Psi} - \mathcal{A}\mathcal{B}_1$.

Goh transform of the second variation Q

Let

$$\Omega(\xi, w, h) = \Omega_T(\xi, h) + \Omega_a(\xi, w) + \Omega_b(w),$$

where

$$\Omega_b(w) := \int_0^T w^2(t) R(t) \mathrm{d}t$$

$$\Omega_T(\xi, h) := a_2 \|\xi(T) + h\mathcal{B}(T)\|^2_{\mathcal{H}} + h^2 \langle \hat{p}(T), \mathcal{B}_2\mathcal{B}_1 + \mathcal{B}_2^2 \hat{\Psi}(T)\rangle + h\langle \hat{p}(T), \mathcal{B}_2\xi(T)\rangle,$$

$$\Omega_a(\xi, w) := \int_0^T \Big(a_1 \|\xi\|_{\mathcal{E}} + 2w\langle Q\xi, \mathcal{B}\rangle + 2w\langle \hat{\Psi} - \Psi_d, \mathcal{B}_2\xi\rangle - 2w\langle \hat{p}, [\mathcal{A}, \mathcal{B}_2]\xi\rangle \Big) \mathrm{d}t,$$

with $R \in L^\infty(0, T)$ given by

$$R(t) := a_1 \big(\|\mathcal{B}\|_{\mathcal{H}} + \langle \hat{\Psi} - \Psi_d, \mathcal{B}_2\mathcal{B}\rangle \big) + \langle \hat{p}(t), r(t)\rangle,$$

$$r(t) := \mathcal{B}_2^2 f(t) - \mathcal{A}\mathcal{B}_2\mathcal{B}_1 + 2\mathcal{B}_2\mathcal{A}\mathcal{B}_1 - \big[[\mathcal{A}, \mathcal{B}_2], \mathcal{B}_2\big]\hat{\Psi}.$$

Second variation in the transformed variables

Theorem

For $v \in L^1(0,T)$ and $w \in AC(0,T)$ given by Goh transformation, there holds

$$\mathcal{Q}(z[v], v) = \Omega(\xi[w], w, w(T)).$$

Write $PC_2(\hat{u})$ for the closure in the $L^2 \times \mathbb{R}$–topology of

$$\{(w,h) \in W^{1,\infty}(0,T) \times \mathbb{R}, \dot{w} \in C(\hat{u});\ w(0) = 0,\ w(T) = h\}.$$

Theorem (Second order necessary condition)

We have that

$$\Omega(\xi[w], w, h) \geq 0 \quad \textit{for all } (w,h) \in PC_2(\hat{u}).$$

Necessary conditions on singular arcs

Singular arc: $(t_1, t_2) \subset [0, T]$ such that for all $\theta > 0$, there exists $\varepsilon > 0$ such that

$$\hat{u}(t) \in [u_m + \varepsilon, u_M - \varepsilon], \quad \text{for a.a. } t \in (t_1 + \theta, t_2 - \theta).$$

Lower bound arc: (t_1, t_2) such that $\hat{u}(t) = u_m$ for a.a. $t \in (t_1, t_2)$,

Upper bound arc: (t_1, t_2) such that $\hat{u}(t) = u_M$ for a.a. $t \in (t_1, t_2)$.

Lower and upper bound arcs: bang arcs

Corollary

If $\hat{u}$ is a weak minimum then

$$R(t) \geq 0 \quad \textit{for a.a. } t \in [t_1, t_2], \quad \textit{for all } (t_1, t_2) \textit{ singular arc.}$$

It follows from classical results of [Hestenes, M.R., *Applications of the theory of quadratic forms in Hilbert space to the calculus of variations.* Pacific J. Math., 1951]

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Pontryagin minimum

$\hat{u}$ is a Pontryagin minimum if it is minimum over $\{u \in \mathcal{U}_{ad} : \|u - \hat{u}\|_1 < \varepsilon\}$ for some $\varepsilon > 0$.

$(v_k) \subset L^\infty(0,T)$ bounded converges to 0 in the Pontryagin sense if $\|v_k\|_1 \to 0$

Proposition (Expansion of the cost with remainder in w)

Let (v_k) converge to 0 in the Pontryagin sense. Then

$$F(\hat{u}+v_k) = F(\hat{u}) + \int_0^T \Lambda(t) v_k(t) dt + \tfrac{1}{2}\Omega(\xi[w_k], w_k, w_k(T)) + o(\|w_k\|_2^2 + w_k(T)^2),$$

where $(\xi[w_k], w_k)$ is obtained by the Goh transform.

Structural hypotheses on the control:

1. finite structure: there are finitely many boundary and singular maximal arcs and the closure of their union is $[0, T]$,
2. strict complementarity for the control constraint: Λ has nonzero values over the interior of bang arcs
3. letting $\mathcal{T}_{BB}$ denote the *set of bang-bang junctions,* we assume

$$R(t) > 0, \quad t \in \mathcal{T}_{BB}.$$

Proposition (Characterization of the critical cone)

$$PC_2(\hat{u}) = \left\{ \begin{array}{l} (w, h) \in L^2(0, T) \times \mathbb{R};\ w \textit{ is constant over boundary arcs,} \\ w = 0 \textit{ over initial bang arc,} \\ w = h \textit{ over terminal bang arc} \end{array} \right\}.$$

Second order sufficient condition

Uniform positivity condition:

$$\exists \alpha > 0: \ \Omega\big(\xi[w], w, h\big) \geq \alpha\big(\|w\|_2^2 + h^2\big), \quad \text{for all } (w,h) \in PC_2(\hat{u}). \qquad \text{(SC)}$$

w-quadratic growth condition: $\exists \beta > 0$ such that for any $u \in \mathcal{U}_{ad}$, setting $v := u - \hat{u}$ and $w(t) := \int_0^T v(s)\mathrm{d}s$:

$$J(u) \geq J(\hat{u}) + \beta\big(\|w\|_2^2 + w(T)^2\big), \quad \text{if } \|v\|_1 \text{ is small enough.}$$

Theorem (Sufficient condition)

Let $\hat{u}$ satisfy the first order conditions.
Then uniform positivity condition holds **iff** *$\hat{u}$ is a Pontryagin minimum for which the $w-$quadratic growth condition is satisfied.*

Conclusions and remarks

- We proved necessary and sufficient conditions for a class of abstract bilinear optimal control systems in infinite dimension
- These results are applicable to the heat and wave equations
- We have extended this to the complex setting and applied it to the Schrödinger equation
- Possible extensions: multiple controls, and state constraints
- **In progress: second order analysis for the bilinear heat equation with vector control subject to control bounds and state constraints**

References

- M.S. Aronna, F. Bonnans, A. Kröner, *Optimal Control of Infinite Dimensional Bilinear Systems: Application to the Heat and Wave Equations,* **Mathematical Programming, 2016**

- M.S. Aronna, F. Bonnans, A. Kröner, *Optimal control of bilinear systems in a complex space setting,* **Proceedings of the 20th IFAC World Congress, 2017**

- M.S. Aronna, F. Bonnans, A. Kröner, *Optimal control of PDEs in a complex space setting; application to the Schrödinger equation,* **submitted 2016**

- J.F. Bonnans, *Optimal control of a semilinear parabolic equation with singular arcs,* **Optim., Meth. & Software, 2014**

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