Quadratic Mean-Field Games and Entropy Minimization Part I: Theory

Guillaume Carlier [a]

Based on joint works with Jean-David Benamou, Simone Di Marino and Luca Nenna.

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[a]CEREMADE, Université Paris Dauphine and MOKAPLAN (Inria-Dauphine).

Introduction

Aim of this talk is to present a new viewpoint, related to the so-called entropic interpolation (a problem which goes back to Schrödinfer in the 1930's) on second-order MFGs with a quadratic Hamiltonian and a variational structure. In part II (Luca Nenna), you will see how this approach can be used for new efficient numerical methods.

More precisely, as we know from the seminal works of Lasry and Lions, that the second-order MFG system

$$\begin{cases} -\partial_t \varphi - \Delta\varphi + \frac{1}{2}|\nabla\varphi|^2 = \frac{\delta F}{\delta\rho}(\rho),\ \varphi_{|_{t=1}} = \frac{\delta G}{\delta\rho}(\rho_1) \\ \partial_t \rho - \Delta\rho - \operatorname{div}(\rho\nabla\varphi) = 0,\ \rho_{|_{t=0}} = \rho_0. \end{cases}$$

($t \in (0,1)$ and $x \in \mathbf{T^d}$, could be $\mathbf{R}^d$ as well taking some care of what's happening at infinity..) is, in some sense, the system of optimality conditions for the optimal control problem:

$$\inf_{(\rho,v)} \int_0^1 \int_{\mathbf{T^d}} \frac{1}{2}\rho|v|^2 dx dt + \int_0^1 F(\rho_t)dt + G(\rho_1)$$

subject to

$$\partial_t \rho - \Delta\rho + \operatorname{div}(\rho v) = 0,\ \rho_{|_{t=0}} = \rho_0.$$

This is a convex problem when F and G are local and convex. We shall assume that either F and G are convex and local (plus growth conditions) or that they are *regular*, this ensures existence of minimizers.

We shall see that this problem is a (Lagrangian) entropy minimization problem (whose discretization in time can be solved numerically efficiently).

Outline

① Schrödinger's problem

② The structure of entropic interpolation

③ MFGs by Entropy minimization

Schrödinger's problem

Let $\mathbf{T^d} := \mathbf{R^d}/\mathbf{2}\pi\mathbf{Z^d}$, $\Omega := C([0,1],\mathbf{T^d})$ and R be the reversible Wiener measure i.e. $R \in \mathcal{P}(\Omega)$ is given by

$$R = \frac{1}{(2\pi)^d}\int_{\mathbf{T^d}} \mathrm{Law}(x+B)\mathrm{d}x \tag{1}$$

where B is the standard Brownian motion starting at 0 (that is the Markov process whose generator is Δ on $\mathbf{T^d}$).

Given ρ_0 and ρ_1 probability measures on $\mathbf{T^d}$ (with finite entropy), the Schrödinger Bridge problem between ρ_0 and ρ_1 reads

$$S(\rho_0, \rho_1) := \inf \left\{ H(Q|R) \ : \ e_{0\#}Q = \rho_0, \ e_{1\#}Q = \rho_1 \right\}$$

where

$$H(Q|R) := \int_\Omega \ln\left(\frac{dQ}{dR}\right) dQ$$

and e_t is the evaluation at time t map, $e_t(\omega) = \omega(t)$.

Seems a complicated problem but can be brought down to a static problem (C. Léonard). Denote by $\Pi(\rho_0, \rho_1)$ the set of transport plans between ρ_0 and ρ_1 i.e. the set of joint probability measures having ρ_0 and ρ_1 as marginals.

Now disintegrate Q and R with respect to $Q_{0,1} := (e_0, e_1)_{\#}Q \in \Pi(\rho_0, \rho_1)$ and $R_{0,1} := (e_0, e_1)_{\#}R$ respectively:

$$Q = \int Q^{x,y} Q_{0,1}(dx, dy), \; R = \int R^{x,y} R_{0,1}(dx, dy)$$

Note that $R_{0,1}$ can be identified with its density $G_{0,1}(x-y)$ which is nothing but the heat kernel and $R^{x,y}$ is the Brownian Bridge between x and y.

Since

$$H(Q|R) = H(Q_{0,1}|R_{0,1}) + \int H(Q^{x,y}|R^{x,y})Q_{0,1}(dx, dy)$$

hence the entropy minimizing strategy consists in taking $Q^{x,y} = R^{x,y}$ and $Q_{0,1}$ solution of the static problem:

$$S(\rho_0, \rho_1) := \inf \left\{ H(Q_{0,1}|R_{0,1}),\ Q_{0,1} \in \Pi(\rho_0, \rho_1) \right\}.$$

The optimality conditon for this static problem (strictly convex minimization) reads

$$Q_{0,1}(x,y) = G_{0,1}(x-y) f_0(x) g_1(y)$$

and the fixed marginal conditions impose

$$(G_{0,1} \star g_1) f_0 = \rho_0, \; (G_{0,1} \star f_0) g_1 = \rho_0 \tag{2}$$

this is the so-called Schrödinger system. Here the log of f_0 and g_1 are Lagrange multipliers associated to the marginal constraints, f_0 and g_1 are called Schrödinger potentials. Existence of such potentials is not obvious at all but has been very much studied since the 1940's (Bernstein, Fortet, Beurling, Föllmer, Nussbaum, Borwein, Lewis, Léonard...). These potentials are unique up to $(f_0, g_1) \mapsto (\lambda f_0, g_1/\lambda)$.

The structure of entropic interpolation

Assume that we have a pair of Schrödinger potentials, f_0, g_1, set then $f_1 := G_{0,1} \star f_0$, $g_0 := G_{0,1} \star g_1$ so that $\rho_0 = f_0 g_0$, $\rho_1 = f_1 g_1$. Interpolation: $\rho_t := f_t g_t$ where f_t and g_t respectively solve a forward and backward heat equation

$$\partial_t f = \Delta f,\ f|_{t=0} = f_0,\ \partial_t g = -\Delta g,\ g|_{t=1} = g_1. \qquad (3)$$

This (Eulerian) entropic interpolation ρ solves Fokker-Planck with the drift $\nabla\varphi$ where $\varphi = -\log g$ (this is Hopf-Cole...):

$$\partial_t \rho - \Delta\rho - \mathrm{div}(\rho\nabla\varphi) = 0, \tag{4}$$

whereas φ soves the backward Hamilton-Jacobi-Bellman equation

$$-\partial_t\varphi - \frac{1}{2}\Delta\varphi + \frac{1}{2}|\nabla\varphi|^2 = 0 \tag{5}$$

At least formally, $(\rho, -\nabla\varphi)$ solves

$$\mathrm{FP}(\rho_0, \rho_1) := \inf\left\{\frac{1}{2}\int_0^1 \int_{\mathbf{R}^d} |v_t(x)|^2 \mu_t(x)dxdt\right\},$$

subject to

$$\partial_t \mu - \Delta\mu + \mathrm{div}(\mu v) = 0,\ \mu_{t=0,1} = \rho_0, \rho_1$$

And one has

$$S(\rho_0, \rho_1) = \mathrm{FP}(\rho_0, \rho_1) + \int_{\mathbf{T^d}} \rho_0 \ln(\rho_0).$$

where we recall that

$$S(\rho_0, \rho_1) := \inf\left\{H(Q|R)\ :\ e_{0\#}Q = \rho_0,\ e_{1\#}Q = \rho_1\right\}$$

The solution of $S(\rho_0, \rho_1)$ is Markovian it is the law of the diffusion process X:

$$dX_t = -\nabla \varphi_t(X_t)dt + dW_t,\ X_0 \sim \mu_0.$$

Conversely if Q solves the dynamic Schrödinger problem, $\rho_t := e_{t\#}Q$ is an optimal trajectory for $\mathrm{FP}(\rho_0, \rho_1)$.

The manipulations above are slightly formal but can be made rigorous (juste under a finite entropy condition):

- By Girsanov theory (Léonard),
- By stochastic control arguments (Mikami).

Nice references (connections with Nelson's stochastic mechanics, with optimal transport, large deviations, functional inequalities....): Beurling 1960, Föllmer, 1988, Léonard 2012, 2014, Gentil, Léonard, Ripani 2016, Mikami 1991, Zambrini 1986, Jamison 1975. Incompressible fluids: Yasue 1982, Arnaudon, Cruzeiro, Léonard, Zambrini, 2017, Benamou-C.-Nenna 2017....

MFGs by Entropy minimization

In the entropic interpolation, as in the optimal transport problem, we prescribe the two end-points measures ${e_0}_\# Q = \rho_0$ and ${e_1}_\# Q = \rho_1$. Instead, we can penalize/put constraints on the marginals $((e_t)_\# Q)_{t\in[0,1]}$. Important example: incompressibility i.e. ${e_t}_\# Q = (2\pi)^{-d}\mathcal{L}^d$, such Q's are called generalized incompressible flows, following the seminal contributions of Yann Brenier. See the recent work of Arnaudon, Cruzeiro, Léonard, Arnaudon, Benamou-C.-Nenna.

Consider the (Eulerian) variational formulation of the MFG system we started with:

$$\text{MFG} := \inf_{\rho, v} \int_0^1 \int_{\mathbf{T^d}} \frac{1}{2} \rho |v|^2 dx dt + \int_0^1 F(\rho_t) dt + G(\rho_1)$$

subject to

$$\partial_t \rho - \Delta \rho + \operatorname{div}(\rho v) = 0, \;\; \rho_{|_{t=0}} = \rho_0.$$

Consider now its (Lagrangian) Schrödinger version

$$\text{SMFG} := \inf_{Q \,:\, e_{0\#} Q = \rho_0} H(Q|R) + \int_0^1 F(e_{t\#} Q) dt + G(e_{1\#} Q)$$

Equivalence with Mean-Field Games

Theorem 1 *Under the assumptions that F and G are either convex or regular and $\int_{\mathbf{T^d}} \rho_0 \ln(\rho_0) < +\infty$ we have*

$$\mathrm{SMFG} = \mathrm{MFG} + \int_{\mathbf{T^d}} \rho_0 \ln(\rho_0)$$

and when Q solves SMFG, $\rho_t := e_{t\#}Q$ *solves* MFG.

Ideas of the proof (which does not use fine Girsanov-like arguments). First discretize both problems in time, $h = 1/N$ a time step discretization, given $(\mu_0, \cdots, \mu_N) \in \mathcal{P}(\mathbf{T^d})^{\mathbf{N+1}}$ set

$$J_N(\mu_0, \cdots, \mu_N) := \frac{1}{N} \sum_{i=1}^{N} F(\mu_i) + G(\mu_N)$$

Define

$$\mathrm{FP}_h(\mu, \nu) := \inf_{\rho, v: \partial_t \rho - \Delta\rho + \mathrm{div}(\rho v) = 0} \int_0^h \int_{\mathbf{T^d}} \frac{1}{2} \rho |v|^2 \ : \ \rho_{|_{t=0,h}} = \mu, \nu\}$$

And consider the time-discretization of MFG

$$\mathrm{MFG}_h := \inf_{\mu_0 = \rho_0, \mu_1, \cdots \mu_N} \sum_{i=0}^{N-1} \mathrm{FP}_h(\mu_{i+1}, \mu_i) + J_N(\mu_0, \cdots, \mu_N)$$

As well as the time-discretization of MFG:

$$\mathrm{SMFG}_h := \inf_{\mu_0=\rho_0,\mu_1,\cdots\mu_N} S_N(\mu_0,\cdots,\mu_N) + J_N(\mu_0,\cdots,\mu_N)$$

where S_N is given by the multi-marginal Schrödinger problem

$$S_N(\mu_0,\cdots,\mu_N) := \inf\{H(Q|R) \ \ e_{\frac{k}{N}\#}Q = \mu_k, k = 0,\cdots,N\}.$$

We first show that

$$S_N(\mu_0,\cdots,\mu_N) = \sum_{i=0}^{N-1} \mathrm{FP}_h(\mu_{i+1},\mu_i) + \int_{\mathbf{T^d}} \mu_0 \ln(\mu_0).$$

and then prove a Γ-convergence results for both approximations MFG_h and SMFG_h. Luca will explain how to solve numerically SMFG_h.