A completely positive representation of the cone of flow matrices

Guillaume Sagnol[1]

Joint work with Thibaut Sauvage[2]

[1]Zuse Institute Berlin, [2]École polytechnique

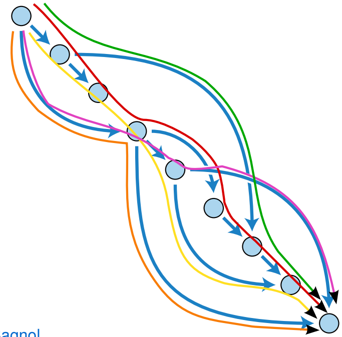
PGMO Days, November 9, 2016

Flows in DAG

Let $G = (V, A)$ be a Directed Acyclic Graph (DAG) with n arcs, two distinctive nodes $s, t \in V$ (called *source* and *sink*).

We assume that s has no incoming arc, t has no outgoing arc, and for all $v \in V$ there is an (s, t)-path that goes through v.

Let $\mathcal{P}$ denote the set of all paths from s to t. A *flow* is defined as a vector $\boldsymbol{f} \in (\mathbb{R}_+)^{|\mathcal{P}|}$.



Example: $\boldsymbol{f} = [3, 1, 2, 1, 2]$

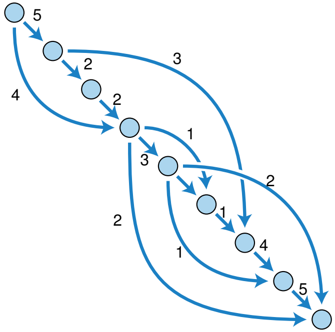
f_p is the amount of flow that goes through Path p.

$F = \sum_{p \in \mathcal{P}} f_P$ is the *value* of the flow

Arc-based flows

Specify a flow by its value on each arc: $\boldsymbol{x} \in \mathbb{R}^n_+$, such that

$$x_a = \sum_{p \ni a} f_p \qquad (\forall a \in A).$$



In vector notation,

$$\boldsymbol{x} = \sum_{p \in \mathcal{P}} f_p \mathbf{1}_p,$$

where $\mathbf{1}_p \in \{0,1\}^n$ is the incidence vector of path p.

The set of all unit-value (arc-based) flows is $\mathcal{F} = \textbf{conv}\{\mathbf{1}_p | p \in \mathcal{P}\}$

P-flows vs. A-flows

Pros and cons of arc-based flows

- Pros
  - A vector $\boldsymbol{x} \in \mathbb{R}_+^n$ is a flow of value F iff

$$\forall v \in V, \sum_{a \in \delta^-(v)} x_a - \sum_{a \in \delta^+(v)} x_a = \begin{cases} -F & \text{if } v = s; \\ F & \text{if } v = t; \\ 0 & \text{otherwise.} \end{cases}$$

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  - Given an arc-based flow $\boldsymbol{x}$, there is a simple algorithm which allows to find a path-based flow $\boldsymbol{f}$ such that $\boldsymbol{x} = \sum_{p \in \mathcal{P}} f_p \mathbf{1}_p$.

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- Cons
  - But arc-based flows have no “memory”, i.e., we do not know which other arc a particle of flow already visited.

Flow matrices

The memoryless property of arc-based flows make them inapropriate to handle path problems with coupling between arcs. For example:

- Shortest path with forbidden pairs
- Quadratic shortest path
- Optimal routing problems with fairness among users
- Maximum flow with paired arc-capacities

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$\rightarrow$ We are going to study a new object, called *flow matrix*, which captures interactions between any 2 arcs of the same path.

Outline

1. The cone of flow matrices
   - definition
   - The membership problem is NP-hard
2. Approximation Hierarchies
   - Tensor-based hierarchy
   - A completely positive representation
3. Application: Max-flow with paired arc-capacities

Flow matrices

- Let $(f_p)_{p\in\mathcal{P}}$ be a flow. We define the flow matrix of $\boldsymbol{f}$ by

$$M(\boldsymbol{f}) = \{M_{ij}\} \in \mathbb{S}^n, \text{ where } M_{ij} := \sum_{p\ni i, p\ni j} f_p.$$

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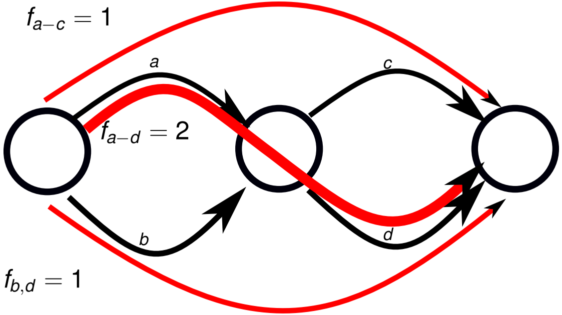
- In particular, note that $\mathbf{diag}\, M(\boldsymbol{f}) = \sum_{p\in\mathcal{P}} f_p \mathbf{1}_p$ is the arc-based flow associated with $\boldsymbol{f}$.
- The set of all flow-matrices of value 1 is

$$\mathcal{M} := \mathbf{conv}\{\mathbf{1}_p \mathbf{1}_p^T | \, p \in \mathcal{P}\}.$$

- The set of all flow-matrices (of any nonnegative value)

$$\mathcal{K} := \mathbf{cone}\{\mathbf{1}_p \mathbf{1}_p^T | \, p \in \mathcal{P}\}.$$

Example



$$\boldsymbol{x} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \qquad M(\boldsymbol{f}) = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 3 \end{bmatrix}.$$

Membership problem

Question: Given a symmetric matrix $X \in \mathbb{S}^n$, how hard is it to decide whether $X \in \mathcal{K}$? And, if should be the case, find a decomposition $X = \sum_p f_p \mathbf{1}_p \mathbf{1}_p^T$?

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Proposition: If $X \in \mathcal{K}$, there always exists a decomposition of the form

$$X = \sum_{i=1}^{N} f_i \mathbf{1}_{p_i} \mathbf{1}_{p_i}^T, \text{ for some } p_1, \ldots, p_N \in \mathcal{P},$$

where $N \leq \frac{n(n+1)}{2} + 1$.

Proof: Caratheodory Theorem.

The quadratic shortest path problem

Given a cost vector $\boldsymbol{c} \in \mathbb{R}^n$, where c_a is the cost of arc a, the shortest path problem is to find the path $p \in \mathcal{P}$ minimizing $\sum_{a \in P} c_a = \boldsymbol{c}^T \mathbf{1}_P$. We write

$$\mathbf{spl}(\boldsymbol{c}) = \min_{p \in \mathcal{P}} \boldsymbol{c}^T \mathbf{1}_P.$$

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Analagously, assume there is a cost $Q_{i,j}$ if you choose a path going through both i and j. This is the quadratic shortest path problem (QSPP):

$$\mathbf{qspl}(Q) = \min_{p \in \mathcal{P}} \mathbf{1}_P Q \mathbf{1}_P.$$

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Theorem [Rostami et. al. 2015]

The QSPP is NP-hard, even when $Q \succeq 0$.

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$$\mathbf{qspl}(Q) = \min_{p \in \mathcal{P}} \mathbf{1}_P Q \mathbf{1}_P = \min_{M \in \mathcal{M}} \langle M, Q \rangle.$$

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The dual of the cone of flow matrices

$$\mathcal{K} = \{\sum_p f_p \mathbf{1}_p \mathbf{1}_p^T \,|\, f_1, \ldots, f_{|\mathcal{P}|} \geq 0\}$$

Recall that $\mathbb{S}^n$ is equipped with inner product $\langle A, B \rangle = \textbf{trace}\, AB$.

$$\begin{aligned}
\mathcal{K}^* =& \{Y \in \mathbb{S}^n \,|\, \forall M \in \mathcal{K},\ \langle M, Y \rangle \geq 0\} \\
=& \{Y \in \mathbb{S}^n \,|\, \forall p \in \mathcal{P},\ \langle \mathbf{1}_P \mathbf{1}_P^T, Y \rangle \geq 0\} \\
=& \{Y \in \mathbb{S}^n \,|\, \forall p \in \mathcal{P},\ \mathbf{1}_P^T Y \mathbf{1}_P \geq 0\} \\
=& \{Y \in \mathbb{S}^n \,|\, \mathbf{qspl}(Y) \geq 0\}
\end{aligned}$$

Duality membership

Weak membership problem (WMEM): Given a set $S \subseteq \mathbb{R}^n$, a point $\boldsymbol{x} \in \mathbb{R}^n$ and $\epsilon > 0$, decide whether

(1) $\boldsymbol{x}$ is $\epsilon-$close to S

(2) S containts the $\epsilon-$ball centered at $\boldsymbol{x}$

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Theorem [Friedland & Lim, 2016]

Let K be a cone with nonempty interior. Then, the WMEM problem for K is reducible to the WMEM for K^* in polynomial time.

NP-hardness

Theorem

The membership problem for $\mathcal{K}$ is NP-hard.

Sketch of proof

- Reduction from an instance $\mathcal{I}$ of SAT to QSPP: Construct Q such that

$$\begin{cases} \mathbf{qspl}(Q) \leq -1 & \text{if } I \text{ is a yes-instance;} \\ \mathbf{qspl}(Q) \geq 1 & \text{if } I \text{ is a no- instance;} \end{cases}$$

- This implies that the WMEM problem for $\mathcal{K}^* = \{Y \in \mathbb{S}^n \,|\, \mathbf{qspl}(Y) \geq 0\}$ is NP-hard
- So WMEM for $\mathcal{K}$ is NP-hard

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Polyhedral approximation

Let $X \in \mathcal{K}$. We already observed that $\mathbf{diag} X$ is a flow-vector.

It is easy to see that columns of X are flows, too:
$X\boldsymbol{e}_a = [X_{1a}, X_{2a}, \ldots, X_{na}]^T$ is the "subflow of all particles that go through arc a".

Proposition

Define

$$\mathcal{K}_2 := \left\{ X \in \mathbb{S}^n \;\middle|\; \begin{array}{l} \mathbf{diag} X \text{ is a flow vector (of any value)} \\ X\boldsymbol{e}_i \text{ is a flow vector of value } X_{ii} \; (i = 1, \ldots, n) \end{array} \right\}$$

We have $\mathcal{K} \subseteq \mathcal{K}_2$.

A tensor-based hierarchy

More generally, denote by $\mathcal{I}_d$ the set of all nonempty subsets of $\{1,\ldots,n\}$ with at most d elements.
We denote by $\mathbf{Sym}^d$ the set of all functions that map $\mathcal{I}_d \to \mathbb{R}$.
For $T \in \mathbf{Sym}^d$, we define

$$\mathbf{diag}\, T = [T_{\{1\}}, \ldots, T_{\{n\}}] \in \mathbb{R}^n$$

$$\mathbf{mat}\ T = \{T_{\{ij\}}\}_{1\leq i,j\leq n} \in \mathbb{S}^n$$

$$T_{i,:} = \big(T_{\{i\}\cup S}\big)_{S\in\mathcal{I}_{d-1}} \in \mathbf{Sym}^{d-1}$$

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We define recursively the following sets:

- $\mathcal{T}_1(u) := u\mathcal{F}$ (set of all flow-vectors of value u)
- $\mathcal{T}_d(u) := \left\{ X \in \mathbf{Sym}^d \;\middle|\; \begin{array}{l} \mathbf{diag} X \text{ is a flow vector of value } u \\ X_{i:} \in \mathcal{K}_{d-1}(X_{\{i\}})) \ (i = 1, \ldots, n) \end{array} \right\}$

A tensor-based hierarchy (continued)

Finally, let $\mathcal{K}_d$ be the set of symmetric matrices obtained by taking any combinations of 2 elements in order-k tensors of $\mathcal{K}_d(u)$:

$$\mathcal{K}_d := \{\mathbf{mat}\, X | \, X \in \mathcal{T}_d(u) \text{ for some } u \geq 0\}$$

By construction, we have:

Proposition

$$\mathcal{K} = \mathcal{K}_N \subseteq \cdots \subseteq \mathcal{K}_3 \subseteq \mathcal{K}_2,$$

where N is the length of the longest $(s,t)-$path in G.

A completely positive representation

Let $X \in \mathcal{K}$. By definition, $X = \sum_P f_P \mathbf{1}_P \mathbf{1}_P^T$. So X is positive semidefinite: $X \in \mathbb{S}_n^+$.

So we have: $\mathcal{K} \subseteq \underbrace{\mathcal{K}_2^+}_{\mathcal{K}_2 \cap \mathbb{S}_+^n} \subseteq \mathcal{K}_2$.

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But we also have $X \in \mathcal{C}_n^*$, where $X \in \mathcal{C}_n^* = \textbf{conv}\{\boldsymbol{x}\boldsymbol{x}^T : \boldsymbol{x} \in \mathbb{R}_+^n\}$ is the cone of completely positive matrices.

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Max-flow with paired arc-capacities

MFPAC Problem

Given a nonnegative capacity matrix $C \in \mathbb{S}^n$, find a flow $(f_P)_{P \in \mathcal{P}}$ of maximum value $F = \sum_P f_P$ that satisfies the paired arc-capacity constraints:

$$\forall i, j \in \{1, \ldots, n\}, \quad \sum_{P \ni i, P \ni j} f_P \leq C_{ij}.$$

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$$\forall i, j \in \{1, \ldots, n\}, \qquad \sum_{P \ni i, P \ni j} f_P \leq C_{ij}.$$

This problem can be formulated as a linear problem over $\mathcal{K}$:

$$\max_X \quad \sum_{a\in\delta^+(S)} X_{aa}$$

$$X \leq C$$

$$X \in \mathcal{K}.$$

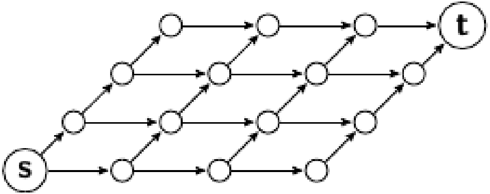
Exact solution: Column Generation

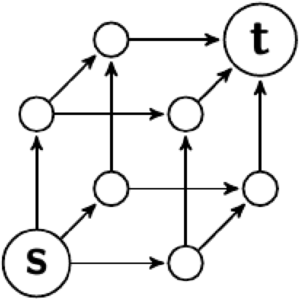
1. Initialize subset of paths $\hat{\mathcal{P}} \subset \mathcal{P}$
2. Solve the *Restricted Master Problem*:

$$\begin{aligned}\max_{\boldsymbol{f}} \quad & \sum_{P \in \hat{\mathcal{P}}} f_P \\ & X \leq C \\ & X = \sum_{P \in \hat{\mathcal{P}}} f_P \mathbf{1}_P \mathbf{1}_P^T.\end{aligned}$$

3. *pricing problem:* Find the Quadratic Shortest Path $P^* \in \mathcal{P}$ for the cost matrix Y, where $Y \in \mathbb{S}^n$ is the optimal dual variable of constraints $X \leq C$.
4.
   - **If qspl**$(Y) < 1$: $\hat{\mathcal{P}} \leftarrow \hat{\mathcal{P}} \cup \{P^*\}$ and **goto** 2.
   - **Else**: **return** $(f_P)_{P \in \hat{\mathcal{P}}}$

Tests on grid graphs





Results on 2D-Grids

	size of the grid							
Relaxation	1	2	3	4	5	6	7	8
$\mathcal{K}_2$ (average)	1.00	1.00	1.01	1.04	1.04	1.08	1.08	1.08
$\mathcal{K}_2$ (worst case)	1.00	1.00	1.06	1.17	1.12	1.29	1.17	1.16
$\mathcal{K}_2^+$ (average)	1.00	1.00	1.00	1.03	1.04	1.06	1.06	?
$\mathcal{K}_2^+$ (worst case)	1.00	1.00	1.03	1.11	1.11	1.22	1.17	?
$\mathcal{K}_3$ (average)	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$\mathcal{K}_3$ (worst case)	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00

Ratio of the optimal value of the relaxation to the true optimal value over $\mathcal{K}$, for 10 randomly generated capacity matrices C.

Results on 3D-Grids

	size of the grid		
Relaxation	1	2	3
$\mathcal{K}_2$ (average)	1.00	1.06	1.03
$\mathcal{K}_2$ (worst case)	1.00	1.16	1.10
$\mathcal{K}_2^+$ (average)	1.00	1.04	?
$\mathcal{K}_2^+$ (worst case)	1.00	1.12	?
$\mathcal{K}_3$ (average)	1.00	1.00	1.00
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Conclusions & Perspectives

Conclusions

- Some combinatorial problems can be written as linear optimization over the cone $\mathcal{K}$
- $\mathcal{K}$ is intractable, but we can use approximation hierarchies

Perspectives

- Decomposition methods to optimize over $\mathcal{K}_2$,$\mathcal{K}_3$,...
- Real-world application ?