Optimization and Games in Congested Networks

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Models to predict flows under congestion
Outline

1. Traffic equilibrium
2. Transit equilibrium
3. Research opportunities
   - Adaptive dynamics
   - Dynamic equilibrium
   - TCP/IP multipath routing

Based on joint work with:

Jean-Bernard Baillon, Pablo Beltrán, Manuel Cepeda, Riccardo Colini-Baldeschi, José Correa, Miguel Dumett, Michael Florian, Cristóbal Guzmán, Emerson Melo, Panayotis Mertikopoulos, Marco Scarsini, Sylvain Sorin, Alfredo Torrico
Traffic equilibrium
Traffic flows under congestion

SANTIAGO

- 6,000,000 people
- 11,000,000 daily trips
- 1,750,000 car trips

Morning peak
- 500,000 car trips
- 29,000 OD pairs

2266 nodes / 7636 arcs / 409 centroids
Traffic flows under congestion

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Wardrop Equilibrium (Wardrop’52)

Given

\[
\begin{aligned}
\text{traffic network} & \quad (V, A) \\
\text{travel demands} & \quad g_i^d \geq 0 \\
\text{arc travel times} & \quad t_a = s_a(v_a)
\end{aligned}
\]
Wardrop Equilibrium (Wardrop’52)

Given

\[ (V, A) \]

traffic network

\[ g_i^d \geq 0 \]

travel demands

\[ t_a = s_a(v_a) \]

arc travel times

Split \( g_i^d = \sum_{r \in \mathcal{R}_i^d} x_r \) with \( x_r \geq 0 \) so that only shortest routes are used

\[ x_r > 0 \implies T_r = \tau_i^d \]
Wardrop Equilibrium (Wardrop’52)

Given \( (V, A) \) traffic network, travel demands \( g^d_i \geq 0 \), and arc travel times \( t_a = s_a(v_a) \),

Split \( g^d_i = \sum_{r \in R^d_i} x_r \) with \( x_r \geq 0 \) so that only shortest routes are used,

\[
x_r > 0 \implies T_r = \tau^d_i
\]

where

\[
T_r = \sum_{a \in r} s_a(v_a) \quad \text{(route times)}
\]

\[
v_a = \sum_{r \ni a} x_r \quad \text{(total arc flows)}
\]

\[
\tau^d_i = \min_{r \in R^d_i} T_r \quad \text{(minimal times)}
\]
Variational characterization (Beckman-McGuire-Winsten’56)

\[
(P) \begin{cases}
\text{Min} & \sum_a \int_0^{\nu_a} s_a(z) \, dz \\
\text{s.t.} & \text{flow conservation}
\end{cases}
\]

⇒ There exists a unique equilibrium \( \nu^* \)
Dual characterization \textit{(Fukushima'84)}

Change of variables: $\nu_a \leftrightarrow t_a$

\[
(D) \quad \text{Min}_t \quad \sum_a \int_0^{t_a} s_a^{-1}(z) \, dz - \sum_{i,d} g_i^d \tau_i^d(t) \\
\phi(t) \quad \text{strictly convex}
\]
Dual characterization (Fukushima'84)

Change of variables: $\nu_a \leftrightarrow t_a$

\[(D) \quad \min_t \sum_a \int_0^{t_a} s_a^{-1}(z) \, dz - \sum_{i,d} g_i^d \tau_i^d(t) \]

\[\phi(t) \quad \text{strictly convex} \]

$t \mapsto \tau_i^d(t) = \text{minimum travel time}$

concave, non-smooth, polyhedral

\[\tau_i^d = \min_{a \in A_i^+} [t_a + \tau_{ja}^d] \quad \text{Bellman's equations} \]
Quantifying inefficiency: Price-of-Anarchy

Social cost = Total travel time = \( \sum_{a \in A} x_a s_a(x_a) \)

PoA = \( \frac{\text{Cost of Equilibrium}}{\text{Minimal Cost}} \) \( \geq 1 \)

Theorem (Roughgarden-Tardos’2002, Roughgarden’2003)

- PoA \( \leq \frac{4}{3} \) for affine costs.
- PoA \( \sim O(d/\log d) \) for polynomials of degree \( d \).

Worst-case bounds overly pessimistic on real instances.
In practice PoA \( \approx 1 \) under low or high congestion.
True for a large class of costs but false in general!
Quantifying inefficiency: Price-of-Anarchy

PoA may oscillate and stay away from 1 even for simple networks with smooth strongly convex costs:

\[
s_1(x) = \left[1 + \frac{1}{2} \sin(\log x)\right] x^2
\]

\[
s_2(x) = x^2
\]

\[
s_3(x) = \left[1 + \frac{1}{2} \cos(\log x)\right] x^2
\]

Theorem (Colini-C-Mertikopoulos-Scarsini, 2016, 2017)

- Regularly varying costs: PoA \( \rightarrow 1 \) in the high congestion regime.
- Polynomial costs: PoA \( \rightarrow 1 \) plus sharp convergence rates.
What if travel times are uncertain?

Copenhagen – Source: DTU Transport (www.transport.dtu.dk)

Figure 2: Example of real time illustration of congestion (Source: Vejdirektoratet, www.trafikken.dk)

Figure 7: Observations of travel time by time of day. Frederikssundsvej, inward direction
Stochastic User Equilibrium (Dial’71, Fisk’80)

Drivers have different assessments for route travel times

\[
\begin{align*}
\tilde{t}_a &= t_a + \epsilon_a \\
\tilde{T}_r &= \sum_{a \in r} \tilde{t}_a \\
\tau_{i}^d &= \min_{r \in R_i^d} \tilde{T}_r
\end{align*}
\]

\[
\begin{aligned}
\text{random variables} \\
\text{with } t_a &= s_a(v_a) \text{ and } v_a = \sum_{r \supset a} x_r \text{ as before, and } \mathbb{E}(\epsilon_a) = 0.
\end{aligned}
\]
Stochastic User Equilibrium (Dial’71, Fisk’80)

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\begin{align*}
\tilde{t}_a &= t_a + \epsilon_a \\
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\]

\[
\begin{aligned}
\text{random} \\
\text{variables}
\end{aligned}
\]

with \( t_a = s_a(v_a) \) and \( v_a = \sum_{r \ni a} x_r \) as before, and \( \mathbb{E}(\epsilon_a) = 0 \).

Demand splits according to the pbb of each route being optimal

\[
x_r = g_i^d \mathbb{P}(\tilde{T}_r = \tilde{\tau}_i^d)
\]
Logit model (Dial’71, Fisk’80)

\( \epsilon_r \) i.i.d. Gumbel noise (supported by Gnedenko’s theorem)

\[
x_r = g_i^d \frac{\exp(-\beta T_r)}{\sum_{s \in R_i^d} \exp(-\beta T_s)}
\]

Drawback: independence is unlikely

Probit model (Daganzo’82)

\( \epsilon_r \) correlated Normal noise

No closed form equations \( \Rightarrow \) Montecarlo

Drawback: tractable only for very small networks
Markovian Traffic Equilibrium (Akamatsu’00, Baillon-C’06)

Routing as a stochastic dynamic programming process

\[
\begin{align*}
\tilde{t}_a &= t_a + \epsilon_a \\
\tilde{T}_r &= \sum_{a \in r} \tilde{t}_a \\
\tilde{\tau}_i^d &= \min_{r \in R_i^d} \tilde{T}_r
\end{align*}
\]

\{ random variables \}

At every intermediate node \(i\), users select a *random optimal arc*

\[\arg\min_{a \in A_i^+} \tilde{t}_a + \tilde{\tau}_j^d\]

\(\Rightarrow\) Markov chain for each destination \(d\)
MTE equations

Expected in-flow

\[ x_i^d = g_i^d + \sum_{a \in A_i^-} \nu_a^d \]

leaves node \( i \) according to

\[ \nu_a^d = x_i^d \mathbb{P}(\tilde{t}_a + \tilde{\tau}_a^d \leq \tilde{t}_b + \tilde{\tau}_b^d \ \forall \ b \in A_i^+) \]

where \( t_a = s_a(\nu_a) \) and \( \nu_a = \sum_d \nu_a^d \)
**Variational formulation**

\[ \tilde{\tau}_i^d = \min_{a \in A_i^+} \{ \tilde{t}_a + \tilde{\tau}_{ja}^d \} \]

**Theorem (Baillon-C’06)**

\[ \tau_i^d = \mathbb{E}(\tilde{\tau}_i^d) \] is the unique solution of the stochastic Bellman equations

\[ \tau_d^d = 0 \quad ; \quad \tau_i^d = \mathbb{E} \left( \min_{a \in A_i^+} \{ t_a + \tau_{ja}^d + \varepsilon_a^d \} \right) . \]

Moreover \( t \mapsto \tau_i^d(t) \) is concave & smooth, and MTE is characterized by

\[
(D) \quad \min_t \phi(t) \triangleq \sum_a \int_0^{t_a} s_a^{-1}(x) \, dx - \sum_{i,d} g_{id}^d \tau_i^d(t).
\]

...same formulation as Wardrop equilibrium!
Method of Successive Averages

- Compute current arc travel times $t^k_a = s_a(v^k_a)$
- Solve stochastic Bellman equations
- Compute invariant measures of Markov chains $\tilde{\nu}^d_a$
- Aggregate flows $\tilde{\nu}_a = \sum \tilde{\nu}^d_a$
- Update $\nu^{k+1} = (1 - \alpha_k)\nu^k + \alpha_k\tilde{\nu}$
Method of Successive Averages

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$$\frac{\nu^{k+1} - \nu^k}{\alpha_k} = -\nabla \phi(t^k)$$
Method of Successive Averages

- Compute current arc travel times \( t^k_a = s_a(v^k_a) \)
- Solve stochastic Bellman equations
- Compute invariant measures of Markov chains \( \tilde{v}^d_a \)
- Aggregate flows \( \tilde{v}_a = \sum \tilde{v}^d_a \)
- Update \( v^{k+1} = (1 - \alpha_k) v^k + \alpha_k \tilde{v} \)

\[
\frac{v^{k+1} - v^k}{\alpha_k} = -\nabla \phi(t^k) = -D(v^k)^{-1} \nabla \tilde{\phi}(v^k)
\]
Method of Successive Averages

- Compute current arc travel times $t_a^k = s_a(v_a^k)$
- Solve stochastic Bellman equations
- Compute invariant measures of Markov chains $\tilde{v}_a^d$
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$$\frac{v^{k+1} - v^k}{\alpha_k} = -\nabla \phi(t^k) = -D(v^k)^{-1}\nabla \tilde{\phi}(v^k)$$

Theorem (Baillon-C’06)

$$\sum \alpha_k = \infty \text{ and } \sum \alpha_k^2 < \infty \Rightarrow \text{convergence to MTE}$$
Stochastic MSA iterations

Absolute precision: \( \log(||\hat{w}^k - w^k||) \)

Iterations ~
Stochastic MSA-Newton iterations

Absolute precision: $\log(||\tilde{w}^k - w^k||)$

Iterations
Risk-averse routing

So far we focused on expected travel times ≡ risk-neutral approach. What is the risk of a route with random travel time $\tilde{T}_r = \sum_{a \in r} \tilde{t}_a$?
Two natural axioms for a risk map $X \mapsto \phi(X)$

Standard risk maps ? Markowitz, $\text{VaR}(\alpha)$, $\text{CVaR}(\alpha)$
Two natural axioms for a risk map $X \mapsto \phi(X)$

Standard risk maps? Markowitz, VaR($\alpha$), CVaR($\alpha$)

**Monotonicity:** $X \leq Y$ *almost surely* $\Rightarrow \phi(X) \leq \phi(Y)$
Two natural axioms for a risk map $X \mapsto \phi(X)$

Standard risk maps? $\text{VaR}(\alpha), \text{CVaR}(\alpha)$

**Monotonicity:** $X \leq Y$ almost surely $\Rightarrow \phi(X) \leq \phi(Y)$
Two natural axioms for a risk map $X \mapsto \phi(X)$

Standard risk maps? \(\text{VaR}(\alpha), \text{CVaR}(\alpha)\)

**Monotonicity:** \(X \leq Y \text{ almost surely } \Rightarrow \phi(X) \leq \phi(Y)\)

**Add-consistency:** \(\text{If } \phi(X) \leq \phi(Y) \text{ and } Z \text{ independent then}\)
\[
\phi(X + Z) \leq \phi(Y + Z)
\]
Two natural axioms for a risk map $X \mapsto \phi(X)$

Standard risk maps?

**Monotonicity:** $X \leq Y$ almost surely $\Rightarrow \phi(X) \leq \phi(Y)$

**Add-consistency:** If $\phi(X) \leq \phi(Y)$ and $Z$ independent then

$$\phi(X + Z) \leq \phi(Y + Z)$$

![Diagram](attachment:image.png)

Remark: Independent link travel times

$$\phi(T_r) = \sum a^2 \phi(\sim t_a)$$

Risk-optimal path

Shortest path with lengths $\ell_a = \phi(\sim t_a)$.

Risk-equilibrium

Wardrop model for $s_a(v_a) = \phi(\sim t_a)$ with $\sim t_a F(v_a)$. 
Two natural axioms for a risk map $X \mapsto \phi(X)$

Standard risk maps? Any alternatives?

Monotonicity: $X \leq Y$ almost surely $\Rightarrow \phi(X) \leq \phi(Y)$

Add-consistency: If $\phi(X) \leq \phi(Y)$ and $Z$ independent then $\phi(X + Z) \leq \phi(Y + Z)$

![Diagram showing relationships between X, Y, and Z.]
Two natural axioms for a risk map \( X \mapsto \phi(X) \)

Standard risk maps? Any alternatives?

Monotonicity: \( X \leq Y \) almost surely \( \Rightarrow \phi(X) \leq \phi(Y) \)

Add-consistency: If \( \phi(X) \leq \phi(Y) \) and \( Z \) independent then
\[
\phi(X + Z) \leq \phi(Y + Z)
\]

Remark: Independent link travel times \( \Rightarrow \phi(\tilde{T}_r) = \sum_{a \in r} \phi(\tilde{t}_a) \)

- Risk-optimal path \( \equiv \) shortest path with lengths \( \ell_a = \phi(\tilde{t}_a) \).
- Risk-equilibrium \( \equiv \) Wardrop model for \( s_a(v_a) = \phi(\tilde{t}_a) \) with \( \tilde{t}_a \sim F(v_a) \).
Theories of choice & Risk aversion

**Expected utility:** \( \phi(X) = u^{-1}(\mathbb{E}[u(X)]) \)

Exaggerate the **cost** of bad outcomes through a convex increasing \( u : \mathbb{R} \to \mathbb{R} \)
(VonNeuman-Morgenstern’1947)

**Distorted risk measure:** \( \phi(X) = \mathbb{E}(X^h) \) with \( \mathbb{P}(X^h \leq x) = h(\mathbb{P}(X \leq x)) \)

Exaggerate the **probability** of bad outcomes through a distortion map \( h : [0, 1] \to [0, 1] \) increasing from 0 to 1 with \( h(s) \leq s \) (Yaari’1987)

**Rank-dependent utility:** \( \phi(X) = u^{-1}(\mathbb{E}[u(X^h)]) \)
(Schmeidler’1989; Quiggin’1993)
One additional axiom

Each of these theories of choice are characterized by monotonicity, weak continuity, and some form of independence axiom:

\[ \phi(X) \leq \phi(Y) \Rightarrow \phi(\mathcal{L}(p, X, Z)) \leq \phi(\mathcal{L}(p, Y, Z)). \]
Entropic risks

Theorem (C-Torrico’2015)

Among expected utilities, distorted risk measures, and rank dependent utilities, the only add-consistent risk measures are

$$\phi_{\beta}(X) = \frac{1}{\beta} \ln(\mathbb{E}[e^{\beta X}])$$.
### Entropic risks

**Theorem (C-Torrico’2015)**

Among expected utilities, distorted risk measures, and rank dependent utilities, the only add-consistent risk measures are

\[
\phi_\beta(X) = \frac{1}{\beta} \ln(\mathbb{E}[\exp^{\beta X}]).
\]

- Mixtures of entropic risks \( \int_{\mathbb{R}} \phi_\beta(X) dH(\beta) \) are also add-consistent.
- In atomic spaces there are other exotic add-consistent measures.
Entropic risks

Theorem (C-Torrico’2015)

Among expected utilities, distorted risk measures, and rank dependent utilities, the only add-consistent risk measures are

\[ \phi_\beta(X) = \frac{1}{\beta} \ln(\mathbb{E}[e^{\beta X}]). \]

- Mixtures of entropic risks \( \int_\mathbb{R} \phi_\beta(X) dH(\beta) \) are also add-consistent.
- In atomic spaces there are other exotic add-consistent measures.

Open questions:
- Are mixtures of entropic risks the only add-consistent measures in non-atomic probability spaces?
- Risk-optimal paths without independence?
- Populations with heterogeneous risk-aversion?
- Price-of-anarchy for risk averse equilibrium?
Transit equilibrium
Many large cities suffer from overcrowded transit systems