Chennai, India

In-vehicle congestion
Sao Paulo Metro Station

Platform congestion
London Tube

Bus stop congestion
Problem

Given

\begin{align*}
\text{transit network} & \quad (V, A) \\
\text{travel demands} & \quad g^d_i \geq 0 \\
\text{arc travel times} & \quad t_a = s_a(v_a)
\end{align*}
Problem

Given

- transit network \((V, A)\)
- travel demands \(g_i^d \geq 0\)
- arc travel times \(t_a = s_a(v_a)\)
- bus frequencies \(\mu_a\)
- bus capacities \(c_a\)
Problem

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\end{align*}

Determine:

- passenger flows on line segments
- travel times for each OD pair
- waiting times at bus stops
Common lines – uncongested (Chriqui-Robillard’75)

Travel times $t_1 \leq \ldots \leq t_n$

Frequencies $\mu_1, \ldots, \mu_n$

Strategies $s \subseteq L$
**Common lines – uncongested** *(Chriqui-Robillard’75)*

Travel times \( t_1 \leq \ldots \leq t_n \)

Frequencies \( \mu_1, \ldots, \mu_n \)

Strategies \( s \subseteq L \)

\[
T_s = W_s + \sum_{a \in s} t_a \pi_s^a = \frac{1 + \sum_{a \in s} t_a \mu_a}{\sum_{a \in s} \mu_a}
\]

\[
\tau = \min_{s \subseteq L} T_s
\]
Common lines – uncongested (Chriqui-Robillard’75)

Theorem (Chriqui-Robillard’75)

Optimal strategy: $s^* = \{1, 2, \ldots, k^*\}$ take the $k^*$ fastest lines.

$\Rightarrow$ linear time algorithm: $s^* = \{a : t_a \leq T_{s^*}\}$
Extension to networks

Extension to networks


These models overlook capacity effects.

✓ On-board: increased discomfort
✓ Bus stops: increased waiting times
✓ Boardings & alightings: increased travel times
✗ Boarding probabilities: changes in flow distribution

Key point: estimate correctly the flows!
Denied boardings deferred to other services
“Sydney bus and train commuters say overcrowding is still their main public transport concern.”

“Buses in Sydney on the busiest routes are often overcrowded and do not stop for passengers, with an extraordinary 22% of people missing their service.”

Emma Freijinger, November 2013
Common lines with congestion

Split the demand \( g = \sum_{s \subseteq L} x_s \) so that only optimal strategies are used

\[
x_s > 0 \implies T_s \triangleq W_s + \sum_{a \in s} t_a \pi^a_s \text{ is minimal}
\]

\[
\begin{align*}
W_s &= W_s(x) \\
\pi^a_s &= \pi^a_s(x)
\end{align*}
\]

... queueing theory
Bulk queues $M|M|1$

\[ \nu \rightarrow \bigcirc \rightarrow \mu, c \quad (\lambda < \mu \lambda) \]

Little’s formula yields the waiting time

\[ W(\nu) = \frac{1}{\nu} \mathbb{E}(L) = \frac{1}{\nu} \frac{\rho}{1 - \rho} \]

where $\rho = \rho(\nu) \in [0, 1]$ is the solution of

\[ \mu(\rho + \rho^2 + \cdots + \rho^c) = \nu. \]
Bulk queues $M|M|1$

\[ \nu \xrightarrow{\bigcirc} \mu, c \quad (\lambda < \mu C) \]

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\[ \mu(\rho + \rho^2 + \cdots + \rho^c) = \nu. \]

**Definition**

**Effective frequency** is defined as

\[ f(\nu) \triangleq \frac{1}{W(\nu)} \]

**Remark:** for light demand \( \nu \sim 0 \) we have \( f(\nu) \sim \mu. \)
Strategies with congestion

Let \( s \subseteq L \) and denote \( f_a(v_a) \) the effective frequency of each line \( a \in L \).

**Lemma (C-Correa’2001)**

Assume Poisson arrivals with rate \( x_s \). Then the expected flows \( v_a \) on the lines \( a \in s \) are the unique solution of the system

\[
\begin{align*}
  v_a &= x_s \frac{f_a(v_a)}{\sum_{b \in s} f_b(v_b)} \\
  W_s &= \frac{1}{\sum_{b \in s} f_b} \\
  \pi^a_s &= f_a \frac{1}{\sum_{b \in s} f_b} \\
  T_s &= \frac{1 + \sum_{a \in s} t_a f_a}{\sum_{a \in s} f_a}
\end{align*}
\]

Moreover, setting \( f_a = f_a(v_a) \) we have

When there are several \( x_s’\)’s we have

\( v_a = v_a(x), f_a = f_a(x), ... \)
Strategies with congestion

Let $s \subseteq L$ and denote $f_a(v_a)$ the effective frequency of each line $a \in L$.

Lemma (C-Correa’2001)

Assume Poisson arrivals with rate $x_s$. Then the expected flows $v_a$ on the lines $a \in s$ are the unique solution of the system

\[(E) \quad v_a = x_s \frac{f_a(v_a)}{\sum_{b \in s} f_b(v_b)}\]

Moreover, setting $f_a = f_a(v_a)$ we have

\[
W_s = \frac{1}{\sum_{b \in s} f_b} \\
\pi^a_s = \frac{f_a}{\sum_{b \in s} f_b} \\
T_s = \frac{1 + \sum_{a \in s} t_a f_a}{\sum_{a \in s} f_a}
\]

When there are several $x_s$’s we have $v_a = v_a(x)$, $f_a = f_a(x)$,...

No analytic expressions available... yet!
Common lines – Equilibrium model

We postulate $f_a = f_a(v_a)$ as a decreasing function of the line’s load, and we denote $v = v(x)$ the unique solution of the system

$$v_a = \sum_s x_s \pi_s^a = \sum_s x_s \frac{f_a(v_a)}{\sum_{b \in s} f_b(v_b)}$$
Common lines – Equilibrium model

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$$(E) \quad v_a = \sum_s x_s \pi_s^a = \sum_{s \ni a} x_s \frac{f_a(v_a)}{\sum_{b \in s} f_b(v_b)}$$

**Equilibrium:** Split the demand $g = \sum_s x_s$ with $x_s \geq 0$ so that

$$(W) \quad x_s > 0 \Rightarrow T_s(v) = \tau(v)$$

where

$$T_s(v) = \frac{1 + \sum_{a \in s} t_a f_a(v_a)}{\sum_{a \in s} f_a(v_a)}$$

$$\tau(v) = \min_{s \subseteq L} T_s(v)$$
Characterization & Existence

Theorem (C-Correa’2001)

Let $\bar{v}_a(\alpha)$ be the inverse function of $v_a \mapsto \frac{v_a}{f_a(v_a)}$ and denote $\tau_\alpha \triangleq \tau(\bar{v}(\alpha))$. Then the equilibrium flows $v_a$ are the optimal solutions of

\[
\min_{(\alpha,v)} \sum_{a \in L} \left[ t_a v_a + \int_0^\alpha \left[ \tau_\xi - t_a \right]_+ \bar{v}_a'(\xi) \, d\xi \right]
\]

\[
\sum_{a \in L} v_a = g
\]

\[
0 \leq v_a \leq \bar{v}_a(\alpha).
\]

In particular, there exists an equilibrium.

Remark: Once $\alpha$ is known, the flows $v$ are easily obtained.
Characterization & Existence

**Theorem (C-Correa’2001)**

Let $\alpha$ be the unique solution of \( \sum_{t_a < \tau_\alpha} \bar{v}_a(\alpha) \leq g \leq \sum_{t_a \leq \tau_\alpha} \bar{v}_a(\alpha) \). Then $v$ is an equilibrium iff \( \sum_{a \in L} v_a = g \) with

\[
\frac{v_a}{f_a(v_a)} = \begin{cases} 
\alpha & \text{if } t_a < \tau_\alpha \\
\leq \alpha & \text{if } t_a = \tau_\alpha \\
0 & \text{if } t_a > \tau_\alpha
\end{cases}
\]
Comments

- Existence + characterization + conditions for uniqueness
- Constant time for some ranges of demand
- Co-existence of multiple equilibrium strategies
- Inefficient equilibria... Braess-type paradox
- Model consistent with simulations
Network Transit Equilibrium

Family of common line problems coupled by flow conservation.

\[ x_i \Rightarrow j_a \]

\[ t_a(v) \]

\[ \tau_{ja}^d \]

\[ \sum a_s \]

\[ f_a(v) \]

Flow Conservation:

\[ x_i \Rightarrow g_d i + \sum A_i v \Rightarrow d \]

\[ \sum A_i + i v \Rightarrow d \]

Existence of equilibria: Kakutani’s Fixed Point Theorem
Network Transit Equilibrium

Family of common line problems coupled by flow conservation.

\[ \tau_d^i = 0 ; \quad \tau_i^d = \min_{s \subseteq A_i^+} \frac{1 + \sum_{a \in s}[t_a(v) + \tau_{ja}^d]f_a(v)}{\sum_{a \in s} f_a(v)} \]

Flow Conservation:

\[ x_i^d \triangleq g_i^d + \sum_{A_i^-} v_a^d = \sum_{A_i^+} v_a^d \]

Existence of equilibria: Kakutani’s Fixed Point Theorem
Characterization

Theorem (Cepeda-C-Florian’2006)

\( (v^d_a) \) are equilibrium flows iff there exist \( (\alpha^d_i) \) such that

\[
\begin{align*}
\frac{v_a^d}{f_a(v)} &= \alpha^d_i \quad \text{if } t_a(v) + \tau^d_{j(a)}(v) < \tau^d_i(v) \\
&\leq \alpha^d_i \quad \text{if } t_a(v) + \tau^d_{j(a)}(v) = \tau^d_i(v) \\
&= 0 \quad \text{if } t_a(v) + \tau^d_{j(a)}(v) > \tau^d_i(v)
\end{align*}
\]

These are precisely the optimal solutions of

\[
\min_{(v^d_a)} \sum_d \left[ \sum_{a \in A} t_a(v) v^d_a + \sum_{i \in V} \max_{a \in A^+_i} \frac{v^d_a}{f_a(v)} - \sum_{i \in V} g^d_i \tau^d_i(v) \right]
\]

s.t. \( g^d_i + \sum_{A^-_i} v^d_a = \sum_{A^+_i} v^d_a \)
Applications

Model implemented as a macro within EMME software (INRO).

Has been used in

- 2005: Sao Paulo
- 2010: San Francisco, Bangkok
- 2011: Sidney, Mexico City, Los Angeles,
- 2012: Santiago
- 2013: Brisbane, Rio de Janeiro
- 2014: Stockholm
- 2016: Toronto

Source: Michael Florian (INRO)
Application: Santiago (developed by SECTRA, 2012)
Emme Model – Model Construction

Trunk Routes (2011 coverage)

Regular nodes: 20,093
Stops and stations: 12,500
Centroids: 2,916
Regular links: 60,528
Connectors: 24,714

(Meta data size and license)

Metro Lines (2011 coverage)

Trunk routes: 390 service-direction / 3,700 vehicles / 7,800 km

Feeder routes: 430 service-direction / 2,200 vehicles / 5,100 km

Source: SECTRA

Gobierno de Chile | Ministerio de Transportes y Telecomunicaciones
Flows on all transit modes
Emme Model – Results with Capacity Considerations

Source: SECTRA
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Source: SECTRA
Emme Model – Results with Capacity Considerations

Source: SECTRA
Assigned vs. metro counts segment volume

\[ R^2 = 0.99, \quad STD = 438.14 \]

Emma Freijinger, November 2013
Research opportunities

- Adaptive dynamics
- Dynamic equilibrium
- TCP/IP multipath routing
Adaptive dynamics

- Are drivers fully rational? Do they have full information?
- Do myopic adaptive dynamics support equilibrium?
- Recent results by C-Melo-Sorin’2010 and Bravo’2011 provide partial answers and lead to other notions of equilibria
Adaptive dynamics

- Are drivers fully rational? Do they have full information?
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- Recent results by C-Melo-Sorin’2010 and Bravo’2011 provide partial answers and lead to other notions of equilibria

- Many open questions remain!
  - almost sure convergence under small noise
  - speeds of convergence of stochastic adaptive dynamics
  - multiplicity and bifurcations of equilibria
  - large population asymptotics
  - more realistic adaptive dynamics
  - robustness under model specification
Dynamic equilibrium

- Traffic equilibrium under automated guidance software tools
- Dynamic equilibrium more appropriate than static equilibrium
- Recent progress for fluid queuing networks:
  "derivatives of equilibria \equiv normalized \ thin \ flows \ with \ resetting"
  - characterization & computation of equilibria: Koch-Skutella’2010
  - characterization & existence of equilibria: C-Correa-Larré’2015
  - long-term behavior of equilibria: C-Correa-Olver’2017

Many open questions remain!
- finite convergence of reconstruction algorithm
- efficient computation of NTFR’s and equilibria
- multiple origin-destination networks
- other link dynamics: LWR, spillbacks
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TCP/IP multipath routing

- Each source $s \in S$ transmits packets from $o_s$ to $d_s$
- TCP = At which rate? / IP = Along which route?
- Random delays $\tilde{\tau}_a = \text{queuing} + \text{transmission} + \text{propagation}$
- Finite queuing buffers $\Rightarrow$ packet loss probabilities $p_a$

Congestion control – TCP Reno/Tahoe/Vegas
Sources adjust transmission rates in response to congestion.
Feedback mechanism: higher congestion $\Leftrightarrow$ smaller rates.
Kelly *et al.* 1998: Steady states of TCP protocols are
- Equilibria for an associated potential game
- TCP reverse-engineered as a decentralized asynchronous algorithm that solves a network optimization problem
Congestion control and multipath routing

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  - Equilibria for an associated potential game
  - TCP reverse-engineered as a decentralized asynchronous algorithm that solves a network optimization problem

- Many open questions remain!
  - Convergence analysis accounting for stochastics & delays
  - Increase transmission rates: multi-path routing
  - Combine NUM with MTE (C-Guzmán’2014)
  - Design stable packet-level protocols
Thanks!

Reprints available at

https://sites.google.com/site/cominettiroberto/