On the minimum time control of the 2 and 3-body problem

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Dynamic of the CRTBP

$$\ddot{q}(t) + \nabla V_\mu(q(t)) + 2i\dot{q}(t) = u(t) \quad (1)$$

with $V_\mu(q) = \frac{1}{2}|q|^2 + \frac{1-\mu}{|q+\mu|} + \frac{1}{|q+1-\mu|}$ in the rotating frame.

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$$\Leftrightarrow \begin{cases} \dot{x} = F_0(x) + u_1 F_1(x) + u_2 F_2(x) \\ x = (q, \dot{q}). \end{cases}$$

M a 4-dimensionnal manifold

$$\dot{x} = F_0(x) + u_1 F_1(x) + u_2 F_2(x),\ x \in M,\ |u| \leq 1$$

Hypothesis : (A) $\forall x \in M$ $\mathrm{rank}(F_1(x), F_2(x), F_{01}(x), F_{02}(x)) = 4$, $F_{ij} = [F_i, F_j]$.

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Theorem

Let $(F_i)_{i=0\ldots m}$ be vector fields on M, if :

- *conv*(U) *contain a neighborhood of* $0 \in \mathbb{R}^m$
- $\forall x \in M,\ \boldsymbol{Lie}_x(F_0, \ldots, F_m) = T_x M$
- F_0 *is recurrent.*

then the system $\dot{x} = F_0(x) + \sum_i u_i F_i(x)$ *with* $x \in M$, $u \in U \subset \mathbb{R}^m$ *is controlable.*

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Definition

The hamiltonian associated to the dynamic $\dot{x} = f(x,u)$ and the cost $\int_0^{t_f} r(x(t),u(t))dt$ is defined by : $H(x,p,p_0,u) = \langle p, f(x,u)\rangle + p_0 r(x,u)$, $p \in TxM^*$, $p_0 \in \mathbb{R}$, $u \in U$.

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Here we have :

$$
H(x, p, p_0, u) = H_0(x,p) + u_1 H_1(x,p) + u_2 H_2(x,p) + p_0
$$

with $H_i(x,p) = \langle p, F_i(x)\rangle$.

Theorem (PMP)

If (x, u) *is an optimal pair, then* $\exists p : [0, t_f] \to T^*M$ *and a constant* $p_0 \leq 0$ *such as :*

- $H(x(t), p(t), u(t)) = \max_{u \in U} H(x(t), p(t), u)$
- $H(x(t), p(t)) \geq 0$
- $p(t) \neq 0$
- (x, p) *is solution of the maximized hamiltonian :*

$$H^*(x, p) = \max_u H(x, p, u) \ : \begin{cases} \dot{x}(t) = \frac{\partial H^*}{\partial p}(x(t), p(t)) \\ \dot{p}(t) = -\frac{\partial H^*}{\partial x}(x(t), p(t)) \end{cases}$$

Remark : We did not mention the dependance on p_0 since any $p_0 \leq 0$ can be taken.

In our case : $H^*(x,p) = H_1(x,p) + \sqrt{H_1^2(x,p) + H_2^2(x,p)}$, with $u = \frac{1}{||(H_1,H_2)||_2}(H_1, H_2)$.

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The singular locus :

$$\Sigma = \{(x,p) \in T^*M, H_1(x,p) = H_2(x,p) = 0\}$$

(A) implies that Σ is a well definied submanifold since $(x,p) \mapsto (H_1, H_2)$ is a submersion.
$\mathrm{Codim}(\Sigma) = 2 \Rightarrow$ most trajectory don't cross it.

Nilpotent approximation

The nilpotent approximation of an affine control system $\dot{x} = F_0(x) + \sum_{i=1}^{m} u_i F_i(x)$ is defined by considering a nilpotent (mean all lie bracket are finite) sub-algebra of $L = \mathbf{Lie}(F_0, \ldots, F_m)$. It is a polynomial approximation of the vector fields.

Idea.

- Find the right set of coordinates in which the F_i are easily expendables.
- Then cut the corresponding series at the desired lenght : a Lie algebra generated by vector fields with polynomial coeffecients is nilpotent.

Nilpotent approximation.

Here we come up with : $\begin{cases} \dot{x_1} = 1 + x_3 & \dot{x_3} = u_1 \\ \dot{x_2} = x_4 & \dot{x_4} = u_2. \end{cases}$

And end up with : $H(x,p) = p_1(1+x_3) + p_2 x_4 + \sqrt{p_3^2 + p_4^2}$.

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$p_1 = a$, $p_2 = c$, $p_3(t) = at + b$, $p_4(t) = ct + d$.
Codimension one submanifolds of initial conditions crossing Σ :

$$\Delta = \{(x, p), p_1 p_4 - p_2 p_3 = 0\}$$

Nilpotent approximation.

$$x_3(t, z_0) = \frac{a}{a^2+b^2}(\sqrt{(at+b)^2+(ct+d)^2} - \sqrt{b^2+d^2})$$
$$-c\frac{ad-bc}{(a^2+c^2)^{3/2}}(\operatorname{argsh}\left(\frac{(a^2+c^2)t+ab+cd}{ad-bc}\right) - \operatorname{argsh}\left(\frac{ab+cd}{ad-bc}\right)) + x_3(0)$$

The flow is continuous. Smooth out of Δ, and on this strata :
$x_3(t, a, c, \tau) = \frac{a}{\sqrt{a^2+c^2}}(|t-\tau| + |\tau|)$ with $\tau = \tau_1 = \tau_2 = -b/a$
The flow of the nilpotent approximation is piecewise smooth.

Initial dynamic.

$u = \frac{1}{||(H_1,H_2)||}(H_1, H_2) = (\cos\theta, \sin\theta)$.
$H^* = H_1 + \sqrt{H_1^2 + H_2^2} = H_0 + \cos\theta H_1 + \sin\theta H_2 = H_0 + H_\theta$.
Let $\bar{\lambda} \in \Sigma$ and $O_{\bar{\lambda}}$ a neighborhood in local coordinates ($\subset \mathbb{R}^4 \times (\mathbb{R}^4)^*$).
Since we have the rank hypothesis (A), the map $(x, p) \mapsto (x, H_1, H_2, H_{01}, H_{02}) \in O_{\bar{\lambda}} \subset \mathbb{R}^4 \times \mathbb{R}^4$ is a smooth change of variables.

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Remark :

The singular locus becomes
$\Sigma = \{(x, H_1, H_2, H_{01}, H_{02}),\ H_1 = H_2 = 0\}$

Polar blow-up.

Now consider $(H_1, H_2) = \rho(\cos(\theta), \sin(\theta))$
$H(x, \rho, \theta, H_{01}, H_{02}) = H_0 + \rho$
The dynamic becomes :

$$\begin{cases} \dot{x} = F_0(x) + \cos\theta F_1(x) + \sin\theta F_2(x) \\ \dot{\rho} = H_{0\theta}(\lambda) \\ \dot{\theta} = \frac{1}{\rho}(H_{12}(\lambda) + \cos\theta H_{02}(\lambda) - \sin\theta H_{01}(\lambda)) \\ \dot{H}_{01} = H_{001}(\lambda) + H_{\theta 01}(\lambda) \\ \dot{H}_{02} = H_{002}(\lambda) + H_{\theta 02}(\lambda) \end{cases}$$

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$\rho \geq 0$ is preserved by this dynamic.

Equilibria.

$h_{ij} := H_{ij}(\bar{\lambda})$

Lemma

$\theta \mapsto h_{12} + \cos\theta h_{02} - \sin\theta h_{01}$ has two zero if $h_{12}^2 < h_{01}^2 + h_{02}^2$, one zero if $h_{12}^2 = h_{01}^2 + h_{02}^2$ and doesn't vanish if $h_{12}^2 > h_{01}^2 + h_{02}^2$.

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- The zeros of this function are precisly the equilibrium points for the system $(*)$.
- In the first case, we have a six-dimensional submanifold of equilibrium points $\lambda_\pm = (\bar{x}, 0, \theta_\pm, \bar{H}_{01}, \bar{H}_{02})$.

Central manifold.

Theorem

$\dot{z} = Bz + r(z,\varepsilon)$, $z \in \mathbb{R}^n$, $\varepsilon \in \mathbb{R}^l$, with B a diagonalizable $n \times n$ matrix and r smooth, $r(0,0) = \partial_z r(0,0) = 0$. Assume that B has eigenvalues of positive, negative and null real parts, and note X the subspace of eigenvectors which correspond to eigenvalues on the imaginary axes. Then the dynamic around the equilibrium is conjugate (topologicaly) to :

$$\begin{cases} \dot{y}^+ = y^+ \\ \dot{y}^- = y^- \\ \dot{x} = B_1 x + r_1(x,\varepsilon) \end{cases} \qquad \text{with } B_1 = B_{|X} \text{ and } r_1 = r_{|X}.$$

The jacobian of

$$f:\begin{pmatrix} x\\ \rho\\ \theta\\ H_{01}\\ H_{02}\end{pmatrix}\mapsto\begin{pmatrix}\rho(F_0(x)+cos(\theta)F_1(x)+\sin(\theta)F_2(x))\\ \rho F_{0\theta}\\ cos(\theta)H_{02}-\sin(\theta)H_{01}\\ \rho(H_{001}+H_{\theta01})\\ \rho(H_{002}+H_{\theta02})\end{pmatrix}$$

at $\lambda_\pm$ has for eigenvalues $H_{0\theta}(\bar{\lambda}_\pm)$, $-H_{0\theta}(\bar{\lambda}_\pm)$ and 6 dimensional kernel. Stable and unstable manifolds of dimension 1 $\Rightarrow$ only one extremal goes in $\{\rho=0\}$ at λ_-, and one goes out it at λ_+.

Theorem.

Extremals in $O_{\bar{\lambda}}$ exists and are unique.

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Corollary.

The extremal flow is smooth in $O_{\bar{\lambda}}$.

Application to the controlled CRTBP

- $F_0(x) = \nabla V_\mu(x)$
- $F_1(q, v) = \frac{q}{\|q\|} \frac{\partial}{\partial v}$
- $F_2(q, v) = \frac{q \wedge v}{\|q \wedge v\|} \wedge \frac{q}{\|q\|} \frac{\partial}{\partial v}$

The rank hypothesis is verified, besides, we get
$H_{12}(x, p) = \langle p, F_{12}(x) \rangle = 0$ Since $[F_1, F_2] = 0$.

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Conclusion :

Singularites correspond to switching, or so called π-singularites : instant rotation of angle π of the control feedback.

Expectations

Studying the set of initial conditions leading to the singular locus $\Sigma : \bigsqcup_{(\bar{x},\bar{H}_{01},\bar{H}_{02})} W^s(\bar{\lambda})$ which is of codimension 1 submanifold, and the behavior of the flow on it.

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Studying the set of initial conditions leading to the singular locus $\Sigma : \bigsqcup_{(\bar{x},\bar{H}_{01},\bar{H}_{02})} W^s(\bar{\lambda})$ which is of codimension 1 submanifold, and the behavior of the flow on it. We expect it to be, as in the nilpotent case, piecewise smooth.