Douady’s Rabbit
= Julia set of the map $z \mapsto z^2 + c$, with $c$ s.t. $c^3 + 2c^2 + c + 1 = 0$
Dynamics in one variable

Some pictures of Julia sets

Julia set of the map $z \mapsto z^2 + 0,3$
Dynamics in one variable

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Fractal nature of the Julia set of the previous picture
Dynamics in one variable

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Julia set of the map $z \mapsto z^2 + c$ with $c = -0.74543 + 0.11301i$
Dynamics in one variable

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Julia set of the map $z \mapsto z^2 + c$ with $c = -0.75 + 0.11i$
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Julia set of the map $z \mapsto e^{2i\pi} + z^2$
Definition

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Consider the sphere

$$\mathbb{R}^3 \supset S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$
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$$\pi : (x, y, z) \mapsto \begin{cases} \infty & \text{if } (x, y, z) = (0, 0, 1) \\ \frac{x + iy}{1 - z} & \text{otherwise} \end{cases}$$

$$\pi^{-1} : z = x + iy \mapsto \begin{cases} (0, 0, 1) & \text{if } z = \infty \\ \frac{(2x, 2y, |z|^2 - 1)}{|z|^2 + 1} & \text{otherwise} \end{cases}$$
Definition (holomorphic)

If \( \mathcal{V} \subset \mathbb{C} \) is an open set of complex numbers, a function \( f : \mathcal{V} \to \mathbb{C} \) is called \textbf{holomorphic} if the first derivative

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is defined and continuous as a function from \( \mathcal{V} \) to \( \mathbb{C} \), or equivalently if \( f \) has a power series expansion about any point \( z_0 \in \mathcal{V} \) which converges to \( f \) in some neighborhood of \( z_0 \).
Definition (Normal family)

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$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ non constant holomorphic map

The domain of normality for the collection of iterates $\{f^n\}$ is called the **Fatou set** of $f$, and its complement is called the **Julia set** of $f$. Notations: $J_f = \text{Julia set}$, $\hat{\mathbb{C}} \setminus J_f = \text{Fatou set}$.
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Let us now assume that degree of \( f \geq 2 \).
Example

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\[ z \mapsto z^2 + (0, 99 + 0, 14i)z \]

a wild curve
Dynamics in one variable
Julia sets of polynomial maps

$z \mapsto z^2 + i$

a **dendrite**, that is, a compact, connected set without interior that does not separate the space
Dynamics in one variable
Julia sets of polynomial maps

\[ z \mapsto z^2 + (-0, 765 + 0, 12i) \]

A Cantor set: a rather thick totally disconnected set
Dynamics in one variable

Julia sets of polynomial maps

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Remark

in any of these pictures since the map is an even one the Julia set is centrally symmetric
Dynamics in one variable
Julia sets of polynomial maps

$z \mapsto z^2 + (-0, 122 + 0, 745i)$
the Douady rabbit

equation of Julia set whose complement has infinitely many connected components
non polynomial Julia sets can be even more diverse, as illustrated in the following pictures:
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\[ z \mapsto 1 - \frac{1}{z^2} \]
Dynamics in one variable

Julia sets of non polynomial sets

\[
z \mapsto \frac{1 + \frac{i \sqrt{3}}{2} + z^2}{1 - z^2}
\]
Dynamics in one variable

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\[ z \mapsto -0,138(z + \frac{1}{z}) - 0,303 \]
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Remark that if $z$ is a fixed point of $M$, then in a neighborhood of $z$, one has

$$M(z + h) = z + M(z)h + M^{(2)}(z)\frac{h^2}{2} + M^{(3)}(z)\frac{h^3}{3!} + \ldots$$

Hence
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Dynamics in one variable

How to draw Julia sets?

Newton's method

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**Definition**

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- $M(z) = z$,
- $\forall 1 \leq i \leq r$, $M^{(i)}(z) = 0$,
- $M^{(r+1)}(z) \neq 0$. 
Newton’s method:

Let $N: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the Newton's function of $z$, and $z_0 \in \mathbb{C}$.

Properties

1. Fix $N = \text{zeros of } f$
2. the convergence of a simple root is quadratic ($N' = f(f(z))/f'(z)$)
3. the convergence of a root of multiplicity $k$ is $1 - 1/k$
4. $\infty$ is fixed and repulsive
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Let $N: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by

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Example

Consider \( f(z) = z^2 + 1 \) roots of \( f \):
- \( i \), and
- \( -i \)

\( z_0 \in \mathbb{R} \Rightarrow \) the iterates of \( N(z) \) behave chaotically

\( z_0 \notin \mathbb{R} \Rightarrow \) Newton’s method converges

\( z_0 = 1 + 0.5i \),
\( z_1 = 0, 5, -0 + 0.0058i \)
\( z_2 = -0, 1853 + 1,2838i \)
\( z_3 = -0, 0058 - 1,0038i \)
\( z_4 = 0, 0009 + 0,9996i \),
\( z_5 = i \)
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\[
\begin{align*}
  z_0 &= 1 + 0.5i \\
  z_1 &= N(z_0) = 0, 1 + 0.4500i \\
  z_2 &= N^2(z_0) = -0.1853 + 1.2838i \\
  z_3 &= N^3(z_0) = -0.0376 - 1.0234i \\
  z_4 &= N^4(z_0) = -0.0009 + 0.9996i \\
  z_5 &= N^5(z_0) = i
\end{align*}
\]

\[
\begin{align*}
  z_0 &= 0.5 - i \\
  z_1 &= N(z_0) = 0.0500 - 0.9000i \\
  z_2 &= N^2(z_0) = -0.0058 - 1.0038i \\
  z_3 &= N^3(z_0) = -i
\end{align*}
\]
Basins of attraction for complex Newton’s method were first considered by Arthur Cayley.
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ζ root of \( f(z) \) \( \Rightarrow \)

basin of attraction of \( \zeta \)
\[ = \{ z_0 \in \mathbb{C} \mid \text{Newton’s method starting at } z_0 \text{ converges to } \zeta \} \]
To view the basin of attraction for complex polynomials of degree $\geq 2$ we make use of a computer.
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1. Compute $f'(z)$ and $N(z)$;
2. Compute the roots of $f$ (via factoring or numerical approximation on $f$);
3. Pick an initial point $z_0$ and compute the distance between $z_0$ and the roots of $f$. If the distance is less than some small $\varepsilon$, color the point the root color;
4. If not, iterate until the distance between the iterate and the roots of $f$ is less than small value $\varepsilon$. Color the original point the appropriate root color.
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How to draw Julia sets?

Basins of attractions for $f(z) = z^3 - z$

- Points which converge to 0 are colored blue.
- Points which converge to 1 are colored green.
- Points which converge to $-1$ are colored red.
Dynamics in one variable

How to draw Julia sets?

Pictures

basins of attractions for $f(z) = z^3 - z$

roots of $f$ are 0, 1 & $-1$
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Pictures

basins of attractions for $f(z) = z^3 - 1$
Dynamics in one variable

How to draw Julia sets?

Pictures

Basins of attractions for $f(z) = z^3 - 1$

Roots of $f$ are $1$, $\frac{-1}{2} + \frac{\sqrt{3}}{2}i$ & $\frac{-1}{2} - \frac{\sqrt{3}}{2}i$
Dynamics in one variable

How to draw Julia sets?

Pictures

basins of attractions for \( f(z) = z^3 - 1 \)

roots of \( f \) are 1, \( -\frac{1}{2} + \frac{\sqrt{3}}{2} \, i \) & \( -\frac{1}{2} - \frac{\sqrt{3}}{2} \, i \)
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points which converge to \( -\frac{1}{2} + \frac{\sqrt{3}}{2} \, i \) are colored red
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Dynamics in one variable

How to draw Julia sets?

Pictures

basins of attractions for \( f(z) = z^4 - 1 \)
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How to draw Julia sets?

- Pictures

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How to draw Julia sets?

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- Points which converge to $1$ are colored green.
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Lemma (Invariance Lemma)

\[ f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \text{ holomorphic map} \]
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\[ J_f \text{ is fully invariant under } f, \text{ that is, } z \in J_f \iff f(z) \in J_f. \]
Lemma (Invariance Lemma)

\( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) holomorphic map

\( J_f \) is fully invariant under \( f \), that is, \( z \in J_f \iff f(z) \in J_f \).

Idea of the proof.

Invariance Lemma \( \iff F_f \) fully invariant.
Lemma (Invariance Lemma)

$f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ holomorphic map

$J_f$ is fully invariant under $f$, that is, $z \in J_f \iff f(z) \in J_f$.

Idea of the proof.

Invariance Lemma $\iff F_f$ fully invariant.

In fact for any open set $\mathcal{U} \subset \hat{\mathbb{C}}$, some sequence of iterates $f^{n_j}$ converges uniformly on compact subsets of $\mathcal{U}$ $\iff$ the corresponding sequence of iterates $f^{n_j+1}$ converges uniformly on compact subsets of the open set $f^{-1}(\mathcal{U})$. 

It follows that $J_f$ possesses a great deal of self-similarity:
It follows that $J_f$ possesses a great deal of self-similarity: whenever $f(z_1) = z_2$ in $J_f$, with $f'(z_1) \neq 0$, there is an induced conformal isomorphism from a neighborhood $N_1$ of $z_1$ to a neighborhood $N_2$ of $z_2$, which takes $N_1 \cap J_f$ precisely onto $N_2 \cap J_f$. 
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For any $k \geq 0$, $J_{f^k} = J_f$. 

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Again we can equally well work with $F_f$. Suppose, for example, that $z \in F_{f^2}$. This means that, for some neighborhood $\mathcal{U}$ of $z$, the collection of $f_{|\mathcal{U}}^{2n}$ is contained in a compact subset $K \subset \text{Hol}(\mathcal{U}, \hat{\mathbb{C}})$. 
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Suppose, for example, that \( z \in F_{f^2} \). This means that, for some neighborhood \( \mathcal{U} \) of \( z \), the collection of \( f_{|\mathcal{U}}^{2n} \) is contained in a compact subset \( K \subset \text{Hol}(\mathcal{U}, \hat{\mathbb{C}}) \). It follows that every iterate of \( f \), restricted to \( \mathcal{U} \), belongs to the compact set \( K \cup f \circ K \subset \text{Hol}(\mathcal{U}, \hat{\mathbb{C}}) \Rightarrow z \in F_f \).
Definition (critical point)

\[ z = \text{critical point of } f \iff f'(z) = 0 \]
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Definition (multiplier)

\[ f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \text{ rational map} \]
Definition (critical point)

\[ z = \text{critical point of } f \iff f'(z) = 0 \]

Definition (multiplier)

\( f : \hat{C} \to \hat{C} \) rational map
\( z_0 \) periodic point of period \( n \)
Definition (critical point)

\[ z = \text{critical point of } f \Leftrightarrow f'(z) = 0 \]

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\[ f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \text{ rational map} \]
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\[ \lambda_{z_0} = (f^n)'(z_0) = \text{multiplier of the periodic orbit} \]
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**Definition (forward orbit of a point)**

\[ f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \text{ rational map} \]
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\[ \text{The sequence } \{z_n\} \text{ inductively defined by} \]
\[ z_{n+1} = f(z_n) \]
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Definition (forward orbit of a point)

\( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) rational map
The sequence \( \{z_n\} \) inductively defined by
\[ z_{n+1} = f(z_n) \]
is called the \textbf{forward orbit of} \( z_0 \), and is denoted \( \mathcal{O}^+(z_0) \).
**Remark**

chain rule $\Rightarrow \lambda_{z_0} = \text{product of the derivatives of } f \text{ along the orbit}$
Remark

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$\Rightarrow \lambda_{z_0}$ is an invariant of $\mathcal{O}^+(z_0)$ rather than the particular point $z_0$. 
Remark

chain rule \( \Rightarrow \lambda z_0 = \text{product of the derivatives of } f \text{ along the orbit} \)
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Whenever we discuss just one periodic orbit we will simply denote the multiplier by \( \lambda \).
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Definition (attracting, superattracting, repelling, neutral orbit)

A periodic orbit $O^+(z_0)$ is
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Definition (attracting, superattracting, repelling, neutral orbit)

A periodic orbit $O^+(z_0)$ is

- **attracting** if $0 < |\lambda| < 1$,
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chain rule ⇒ $\lambda_{z_0} = \text{product of the derivatives of } f \text{ along the orbit}$
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Definition (attracting, superattracting, repelling, neutral orbit)

A periodic orbit $\mathcal{O}^+(z_0)$ is
- **attracting** if $0 < |\lambda| < 1$,
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### Remark

Chain rule ⇒ $\lambda z_0 = \text{product of the derivatives of } f \text{ along the orbit}$  
⇒ $\lambda z_0$ is an invariant of $\mathcal{O}^+(z_0)$ rather than the particular point $z_0$. Whenever we discuss just one periodic orbit we will simply denote the multiplier by $\lambda$.

### Definition (attracting, superattracting, repelling, neutral orbit)

A periodic orbit $\mathcal{O}^+(z_0)$ is
- **attracting** if $0 < |\lambda| < 1$,
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Definition (attracting, superattracting, repelling, neutral orbit)

A periodic orbit $O^+(z_0)$ is

- **attracting** if $0 < |\lambda| < 1$,
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- **neutral** if $|\lambda| = 1$. 
Remark

In the special case where the point at infinity is periodic under a rational map, \( f^m(\infty) = \infty \), this definition may be confusing.
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In the special case where the point at infinity is periodic under a rational map, i.e. $f^m(\infty) = \infty$, this definition may be confusing. The multiplier $\lambda$ is not equal to

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but rather turns out to be equal to the reciprocal of this number.
Remark

In the special case where the point at infinity is periodic under a rational map, i.e. $f^m(\infty) = \infty$, this definition may be confusing. The multiplier $\lambda$ is not equal to

$$\lim_{z \to \infty} (f^m)'(z)$$

but rather turns out to be equal to the reciprocal of this number. As examples, if $f(z) = 2z$, then $\infty$ is an attracting fixed point with multiplier $\lambda = \frac{1}{2}$, while if $f$ is a polynomial of degree $d \geq 2$, then $\infty$ is a superattracting fixed point, with $\lambda = 0$. 
Definition (basin of attraction)

\( \mathcal{O} \) attracting periodic orbit of period \( m \)
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basin of attraction

\[ = \text{open set } \mathcal{A} \]
\[ = \{ z \in \mathbb{C} \mid f^m(z), f^{2m}(z), \ldots \text{ converge towards some point of } \mathcal{O} \} \]
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= open set \( \mathcal{A} \)

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\( \mathbb{C} \) compact \( \Rightarrow \)
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Lemma (Basins & repelling points)

*Every attracting periodic orbit is contained in \( F_f \)*
Definition (basin of attraction)

\( O \) attracting periodic orbit of period \( m \)

**basin of attraction**

= open set \( A \)

= \{ z \in \hat{\mathbb{C}} | f^m(z), f^{2m}(z), \ldots \text{ converge towards some point of } O \} \)

\( \hat{\mathbb{C}} \) compact \( \Rightarrow \)

Lemma (Basins & repelling points)

*Every attracting periodic orbit is contained in \( F_f \)*

*In fact the entire basin of attraction for an attracting periodic orbit is contained in \( F_f \)**
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Lemma (Basins & repelling points)

Every attracting periodic orbit is contained in \( F_f \)

In fact the entire basin of attraction for an attracting periodic orbit is contained in \( F_f \)

However, every repelling periodic orbit is contained in \( J_f \)
Idea of the proof.

First consider a fixed point $z_0 = f(z_0)$ with multiplier $\lambda$. 

If $|\lambda| > 1$, then no sequence of iterates of $f$ can converge uniformly near $z_0$, for the first derivative of $f$ at $z_0$ is $\lambda$, which diverges to infinity as $n \to \infty$.

On the other hand, if $|\lambda| < 1$, then choosing $|\lambda| < c < 1$ it follows from Taylor's Theorem that $|f(z) - z_0| \leq c|z - z_0|$ for $z$ sufficiently close to $z_0 \Rightarrow$ the successive iterates of $f$, restricted to a small neighborhood, converge uniformly to the constant function $z \mapsto z_0$.

These statements for fixed points generalize immediately to periodic points, using Iteration Lemma which says $\forall k, Jf^k = Jf$ since a periodic point of $f$ is just a fixed point of some iterate $f^m$. 
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On the other hand, if \( |\lambda| < 1 \), then choosing \( |\lambda| < c < 1 \) it follows from Taylor’s Theorem that
\[
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for \( z \) sufficiently close to \( z_0 \) ⇒ the successive iterates of \( f \), restricted to a small neighborhood, converge uniformly to the constant function \( z \mapsto z_0 \). The corresponding statement for any compact subset of the basin \( \mathcal{A} \) then follows easily. These statements for fixed points generalize immediately to periodic points, using Iteration Lemma which says
\[
\forall k, \ J_{f^k} = J_f
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since a periodic point of \( f \) is just a fixed point of some iterate \( f^m \).
The case of a neutral periodic point is much more difficult. One particularly important case is the following:

Definition (parabolic periodic point)

A periodic point $z_0 = f^n(z_0)$ is called parabolic if the multiplier $\lambda$ at $z_0$ is equal to 1, yet $f^n$ is not the identity map, or more generally if $\lambda$ is a root of unity, yet no iterate of $f$ is the identity.

Example

$f(z) = z^2 - 1$ has two fixed points, both with multiplier equal to $-1$.

However these do not count as parabolic points since $f \circ f = \text{id}$.

We must exclude such cases so that the following will be true.

Lemma (Parabolic Points)

Every parabolic periodic point belongs to $J_f$. 

The case of a neutral periodic point is much more difficult. One particularly important case is the following:

**Definition (parabolic periodic point)**

A periodic point $z_0 = f^n(z_0)$ is called **parabolic** if the multiplier $\lambda$ at $z_0$ is equal to 1, yet $f^n$ is not the identity map, or more generally if $\lambda$ is a root of unity, yet no iterate of $f$ is the identity.
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**Example**

$f(z) = \frac{z}{z-1}$ has two fixed points, both with multiplier equal to $-1$. However these do not count as parabolic points since $f \circ f = \text{id}$. 
The case of a neutral periodic point is much more difficult. One particularly important case is the following:

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Before proving it let us recall the following statement:

**Theorem (Weierstrass Uniform Convergence Theorem)**

\[ \mathcal{U} \text{ open subset of } \mathbb{C} \]
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**Theorem (Weierstrass Uniform Convergence Theorem)**

$\mathcal{U}$ open subset of $\mathbb{C}$

If a sequence of holomorphic functions $\{f_n: \mathcal{U} \to \mathbb{C}\}$ converges uniformly to the limit function $f$, then $f$ itself is holomorphic.
Before proving it let us recall the following statement:

**Theorem (Weierstrass Uniform Convergence Theorem)**

Let \( U \) be an open subset of \( \mathbb{C} \).

If a sequence of holomorphic functions \( \{ f_n : U \to \mathbb{C} \} \) converges uniformly to the limit function \( f \), then \( f \) itself is holomorphic.

Furthermore, the sequence of derivatives \( \{ f'_n \} \) converges uniformly on any compact subset of \( U \) to the derivative \( f' \).
Before proving it let us recall the following statement:

**Theorem (Weierstrass Uniform Convergence Theorem)**

\( U \) open subset of \( \mathbb{C} \)

If a sequence of holomorphic functions \( \{f_n : U \to \mathbb{C}\} \) converges uniformly to the limit function \( f \), then \( f \) itself is holomorphic.

Furthermore, the sequence of derivatives \( \{f'_n\} \) converges uniformly on any compact subset of \( U \) to the derivative \( f' \).
Proof of Parabolic Points Lemma.
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Let \( w \) be a local uniformizing parameter, with \( w = 0 \) corresponding to the periodic point.
Proof of Parabolic Points Lemma.

Let $w$ be a local uniformizing parameter, with $w = 0$ corresponding to the periodic point. Then some iterate $f^m$ corresponds to a local mapping of the $w$-plane with power series expansion of the form

$$ w \mapsto w + a_q w^q + a_{q+1} w^{q+1} + \ldots \quad q \geq 2, a_q \neq 0 $$
Proof of Parabolic Points Lemma.

Let \( w \) be a local uniformizing parameter, with \( w = 0 \) corresponding to the periodic point. Then some iterate \( f^m \) corresponds to a local mapping of the \( w \)-plane with power series expansion of the form

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$\Rightarrow$ (qth derivative of $f^{mk}$ at 0) = $q ! \, k a_q$ which diverges to infinity as $k \to \infty$.  


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$\Rightarrow$ $(q$th derivative of $f^{mk}$ at $0) = q! \cdot ka_q$ which diverges to infinity as $k \to \infty$. no subsequence $\{f^{mk_j}\}$ can converge locally uniformly as $k_j \to \infty$ (Weierstrass Uniform Convergence Theorem). \qed
Lemma ($J_f$ is not empty)

$f : \mathbb{C} \to \mathbb{C}$ rational map of degree $\geq 2$
Lemma \((J_f \text{ is not empty})\)

\[ f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \text{ rational map of degree } \geq 2 \]

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$J_f$ were vacuous $\Rightarrow$ some sequence of iterates $\{f^{n_i}\}$ would converge, uniformly over the entire sphere $\hat{\mathbb{C}}$, to a holomorphic limit $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Here we are using the fact that normality is a local property.
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\(J_f\) were vacuous \(\Rightarrow\) some sequence of iterates \(\{f^{n_j}\}\) would converge, uniformly over the entire sphere \(\hat{\mathbb{C}}\), to a holomorphic limit \(g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\). Here we are using the fact that normality is a local property. A standard topological argument would then show that \(\deg f^{n_j} :\!\!=\!\!= \deg g\) for large \(j\) (in fact if two maps \(f_j\) and \(g\) are sufficiently close that the spherical distance \(\sigma(f_j(z), g(z))\) is uniformly less than the distance between antipodal points, then we can deform \(f_j(z)\) to \(g(z)\) along the unique shortest geodesic ; hence these two maps are homotopic and have the same degree). But \(\deg f^n \neq \deg g\) for large \(n\), since

\[
\lim_{n \to \infty} \deg f^n = \lim_{n \to \infty} d^n = \infty.
\]
Definition (grand orbit)

Let $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, 
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**grand orbit** of $z$ under $f = \text{GO}(z, f) = \{ p \in \hat{\mathbb{C}} | \mathcal{O}^+(p) \cap \mathcal{O}^+(z) \neq \emptyset \}$
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By a **Riemann surface** \( S \) we mean a connected complex analytic manifold of complex dimension 1.
Let us recall the following statement:

**Theorem (Montel Theorem)**

Let $S$ be a Riemann surface, $F$ a collection of holomorphic maps from $S$ to $\hat{\mathbb{C}}$, which omit three different values (i.e., assume that there are three distinct points $a, b, c$ in $\hat{\mathbb{C}}$ so that $f(S) \subset \hat{\mathbb{C}} \setminus \{a, b, c\}$ for every $f \in F$). Then $F$ is a normal family, i.e., $F \subset \text{Hol}(S, \hat{\mathbb{C}})$ is a compact set.

Using it one can prove the following statement:

**Lemma (Exceptional points)**

If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is rational of degree $\geq 2$, then the set $E_f$ of exceptional points can have at most two elements. These exceptional points, if they exist, must always be superattracting periodic points of $f$ and hence must belong to the $F_f$. 


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If $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is rational of degree $\geq 2$, then the set $E_f$ of exceptional points can have at most two elements. These exceptional points, if they exist, must always be superattracting periodic points of $f$ and hence must belong to the $F_f$. 

Theorem (Transitivity)

Let $f : \mathbb{C} \to \mathbb{C}$ be a rational fraction of degree $\geq 2$, 

$U$ contains the entire Julia set, i.e. $J_f \subset U$

$U$ contains all but at most two points of $\mathbb{C}$, i.e. $\mathbb{C} \setminus \Delta \subset U$ with $\# \Delta \leq 2$

more precisely if $V$ is sufficiently small, then $U$ is the complement $\mathbb{C} \setminus E_f$ of the set of exceptional points.
Theorem (Transitivity)

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Theorem (Transitivity)

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Dynamics in one variable

Grand orbit and topology

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Proof of Transitivity Theorem.

First remark that the complementary set $\hat{\mathbb{C}} \setminus \mathcal{U}$ can contain at most two points. For otherwise since $f(\mathcal{U}) \subset \mathcal{U}$ it would follow from Montel’s Theorem that $\mathcal{U}$ must be contained in $F_f$ which is impossible since $z \in \mathcal{U} \cap J_f$. 
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Proof of Transitivity Theorem.

First remark that the complementary set \( \hat{C} \setminus \mathcal{U} \) can contain at most two points. For otherwise since \( f(\mathcal{U}) \subset \mathcal{U} \) it would follow from Montel’s Theorem that \( \mathcal{U} \) must be contained in \( F_f \) which is impossible since \( z \in \mathcal{U} \cap J_f \). Again making use of the fact that \( f(\mathcal{U}) \subset \mathcal{U} \) we see that any preimage of a point \( z_1 \in \hat{C} \setminus \mathcal{U} \) must itself belongs to the finite set \( \hat{C} \setminus \mathcal{U} \). It follows by a counting argument that some iterated preimage of \( z_1 \) is periodic; hence \( z_1 \) itself is periodic and exceptional.
Proof of Transitivity Theorem.

First remark that the complementary set \( \hat{C} \setminus U \) can contain at most two points. For otherwise since \( f(U) \subset U \) it would follow from Montel’s Theorem that \( U \) must be contained in \( F_f \) which is impossible since \( z \in U \cap J_f \). Again making use of the fact that \( f(U) \subset U \) we see that any preimage of a point \( z_1 \in \hat{C} \setminus U \) must itself belongs to the finite set \( \hat{C} \setminus U \). It follows by a counting argument that some iterated preimage of \( z_1 \) is periodic; hence \( z_1 \) itself is periodic and exceptional. Since \( E_f \) is disjoint from \( J_f \) it follows that \( J_f \subset U \).
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**Corollary (Julia set with interior)**

\[ f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \text{ rational fraction of degree } \geq 2 \]
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\[ J_f \neq \emptyset \Rightarrow J_f = \hat{\mathbb{C}} \]
As a consequence one has:

**Corollary (Julia set with interior)**

\[ f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \text{ rational fraction of degree } \geq 2 \]

\[ \mathcal{J}_f \neq \emptyset \Rightarrow \mathcal{J}_f = \hat{\mathbb{C}} \]

**Idea of the proof.**

For if \( \mathcal{J}_f \) has an interior point \( z \) then choosing a neighborhood \( \mathcal{V} \subset \mathcal{J}_f \) of \( z \) the union \( \mathcal{U} \subset \mathcal{J}_f \) of forward images of \( \mathcal{V} \) is everywhere dense, \( \overline{\mathcal{U}} = \hat{\mathbb{C}} \).
Corollary (Basin Boundary = Julia set)

Let $A \subset \hat{\mathbb{C}}$ be the basin of attraction for some attracting periodic orbit.
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Let $A \subset \hat{\mathbb{C}}$ be the basin of attraction for some attracting periodic orbit $\Rightarrow \partial A = \overline{A} \setminus A$ is equal to $J_f$
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Every connected component of \( F_f \) either coincides with some connected component of this basin \( A \) or else disjoint from \( A \).
Corollary (Basin Boundary=Julia set)

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Proof.

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Proof.

If \( \mathcal{V} \) is any neighborhood of a point of \( J_f \) then Transitivity Theorem implies that some \( f^k(\mathcal{V}) \) intersects \( A \), hence \( \mathcal{V} \) itself intersects \( A \). This proves that \( J_f \subset \overline{A} \).
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**Corollary (Basin Boundary=Julia set)**

Let \( A \subset \hat{\mathbb{C}} \) be the basin of attraction for some attracting periodic orbit

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Every connected component of \( F_f \) either coincides with some connected component of this basin \( A \) or else disjoint from \( A \)

**Proof.**

If \( V \) is any neighborhood of a point of \( J_f \) then Transitivity Theorem implies that some \( f^k(V) \) intersects \( A \), hence \( V \) itself intersects \( A \). This proves that \( J_f \subset \overline{A} \).

But \( J_f \) is disjoint from \( A \) so \( J_f \subset \partial A \).
Corollary (Basin Boundary = Julia set)

Let $A \subset \hat{\mathbb{C}}$ be the basin of attraction for some attracting periodic orbit
⇒ $\partial A = \overline{A} \setminus A$ is equal to $J_f$

Every connected component of $F_f$ either coincides with some connected component of this basin $A$ or else disjoint from $A$

Proof.

If $\mathcal{V}$ is any neighborhood of a point of $J_f$ then Transitivity Theorem implies that some $f^k(\mathcal{V})$ intersects $A$, hence $\mathcal{V}$ itself intersects $A$. This proves that $J_f \subset \overline{A}$.

But $J_f$ is disjoint from $A$ so $J_f \subset \partial A$.

If $\mathcal{V}$ is a neighborhood of a point of $\partial A$, then any limit of iterates $f^k|_\mathcal{V}$ must have a jump discontinuity between $A$ and $\partial A$, ...
Corollary (Basin Boundary=Julia set)

Let $A \subset \hat{\mathbb{C}}$ be the basin of attraction for some attracting periodic orbit
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Proof.

If $V$ is any neighborhood of a point of $J_f$ then Transitivity Theorem implies that some $f^k(V)$ intersects $A$, hence $V$ itself intersects $A$. This proves that $J_f \subset \overline{A}$.

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If $V$ is a neighborhood of a point of $\partial A$, then any limit of iterates $f^k|_V$ must have a jump discontinuity between $A$ and $\partial A$, hence $\partial A \subset J_f$. 
Corollary (Basin Boundary=Julia set)

Let \( A \subset \hat{\mathbb{C}} \) be the basin of attraction for some attracting periodic orbit
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Every connected component of \( F_f \) either coincides with some connected component of this basin \( A \) or else disjoint from \( A \)

Proof.

If \( \mathcal{V} \) is any neighborhood of a point of \( J_f \) then Transitivity Theorem implies that some \( f^k(\mathcal{V}) \) intersects \( A \), hence \( \mathcal{V} \) itself intersects \( A \). This proves that \( J_f \subset \overline{A} \).
But \( J_f \) is disjoint from \( A \) so \( J_f \subset \partial A \).
If \( \mathcal{V} \) is a neighborhood of a point of \( \partial A \), then any limit of iterates \( f^k|_\mathcal{V} \) must have a jump discontinuity between \( A \) and \( \partial A \), hence \( \partial A \subset J_f \).
Any connected Fatou component which intersects \( A \) must coincide with some component of \( A \) since it cannot intersect \( \partial A \).

\qed
Corollary (Iterated preimages are dense)

Let $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ rational fraction of degree $\geq 2$, 

Corollary (Iterated preimages are dense)

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ rational fraction of degree $\geq 2$, let $z_0$ be any point of $J_f$
Corollary (Iterated preimages are dense)

Let \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) rational fraction of degree \( \geq 2 \), let \( z_0 \) be any point of \( J_f \) \( \Rightarrow \) the set of all iterated preimages of \( z_0 \)

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\{ z \in \hat{\mathbb{C}} | \exists n \geq 0 \text{ s.t. } f^n(z) = z_0 \}
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is everywhere dense in \( J_f \)
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Proof.

\( z_0 \not\in E_f \) so Transitivity Theorem implies that every point \( z_1 \in J_f \) can be approximated arbitrarily closely by points \( z \) whose forward orbits contain \( z_0 \).
Corollary (No isolated points)

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Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ rational fraction of degree $\geq 2$,
⇒ $J_f$ has no isolated point.

Proof.

First remark that $J_f$ must be an infinite set.
Corollary (No isolated points)

Let \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) rational fraction of degree \( \geq 2 \),
\( \Rightarrow J_f \) has no isolated point.

Proof.

First remark that \( J_f \) must be an infinite set. For if \( J_f \) were finite it would consist of grand orbit finite points: contradiction with Exceptional points Lemma.
**Corollary (No isolated points)**

*Let* $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *rational fraction of degree* $\geq 2$, 
*⇒* $J_f$ *has no isolated point.*

**Proof.**

First remark that $J_f$ must be an infinite set. For if $J_f$ were finite it would consist of grand orbit finite points: contradiction with Exceptional points Lemma. 
*⇒* $J_f$ contains at least one limit point $z_0$. 
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\( \Rightarrow \) \( J_f \) contains at least one limit point \( z_0 \). The iterated preimages of \( z_0 \) form a dense set of nonisolated points in \( J_f \).
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Corollary (Julia components)

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ rational fraction of degree $\geq 2$, 
$J_f$ is either connected or else has uncountably many connected components
**Corollary (Topological Transitivity)**

*For a generic choice of the point \( z \in J_f \) the forward orbit

\[
O^+(z) = \{ z, f(z), f^2(z), \ldots \}
\]

is everywhere dense in \( J_f \).
Corollary (Topological Transitivity)

For a generic choice of the point $z \in J_f$ the forward orbit

$$\mathcal{O}^+(z) = \{z, f(z), f^2(z), \ldots\}$$

is everywhere dense in $J_f$

Proof.

For any integer $j > 0$ one can cover $J_f$ by finitely many open sets $\mathcal{V}_{jk}$ of diameter less than $1/j$. 
**Corollary (Topological Transitivity)**

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\mathcal{O}^+(z) = \{ z, f(z), f^2(z), \ldots \}
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**Proof.**

For any integer \( j > 0 \) one can cover \( J_f \) by finitely many open sets \( \mathcal{V}_{jk} \) of diameter less than \( 1/j \).

Let \( \mathcal{U}_{jk} = \bigcup_n f^{-n}(\mathcal{V}_{jk}) \).
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Let \( U_{jk} = \bigcup_n f^{-n}(V_{jk}) \).
From Corollary "Iterated preimages are dense" one gets \( \overline{U_{jk}} \cap J_f = J_f \),
Corollary (Topological Transitivity)

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For any integer $j > 0$ one can cover $J_f$ by finitely many open sets $V_{jk}$ of diameter less than $1/j$.

Let $U_{jk} = \bigcup_n f^{-n}(V_{jk})$.

From Corollary ”Iterated preimages are dense” one gets $\overline{U_{jk} \cap J_f} = J_f$, i.e. $U_{jk} \cap J_f$ is a dense open subset of $J_f$. 
Corollary (Topological Transitivity)

*For a generic choice of the point $z \in J_f$ the forward orbit*

$$\mathcal{O}^+(z) = \{z, f(z), f^2(z), \ldots\}$$

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Proof.

For any integer $j > 0$ one can cover $J_f$ by finitely many open sets $\mathcal{V}_{jk}$ of diameter less than $1/j$.

Let $\mathcal{U}_{jk} = \bigcup_n f^{-n}(\mathcal{V}_{jk})$.

From Corollary ”Iterated preimages are dense” one gets $\mathcal{U}_{jk} \cap J_f = J_f$, i.e. $\mathcal{U}_{jk} \cap J_f$ is a dense open subset of $J_f$. If $z$ belongs to the intersection of these dense open sets, then $\mathcal{O}^+(z)$ intersects every one of the $\mathcal{V}_{jk}$
Corollary (Topological Transitivity)

*For a generic choice of the point* \( z \in J_f \) *the forward orbit*

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\mathcal{O}^+(z) = \{ z, f(z), f^2(z), \ldots \}
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**Proof.**

For any integer \( j > 0 \) one can cover \( J_f \) by finitely many open sets \( \mathcal{V}_{jk} \) of diameter less than \( 1/j \).

Let \( \mathcal{U}_{jk} = \bigcup_n f^{-n}(\mathcal{V}_{jk}) \).

From Corollary "Iterated preimages are dense" one gets \( \mathcal{U}_{jk} \cap J_f = J_f \), i.e. \( \mathcal{U}_{jk} \cap J_f \) is a dense open subset of \( J_f \). If \( z \) belongs to the intersection of these dense open sets, then \( \mathcal{O}^+(z) \) intersects every one of the \( \mathcal{V}_{jk} \) and hence is everywhere dense in \( J_f \). \( \square \)
References:

- Wikipedia.