• Optimal investment and asset pricing are often treated as separate problems (Markovitz vs. Black–Scholes).

• In practice, valuations have been largely disconnected from investment and risk management. This lead to large losses during 2008 e.g. with credit derivatives.

• Building on convex stochastic optimization, we describe a unified approach to optimal investment, valuation and risk management.

• The resulting valuations
  ○ are based on hedging costs,
  ○ extend and unify financial and actuarial valuations,
  ○ reduce to “risk neutral valuations” for replicable securities.
• Armstrong, Pennanen, Rakwongwan, Pricing and hedging of S&P500 options under illiquidity, manuscript.
• Nogueiras, Pennanen, Pricing and hedging EONIA swaps under illiquidity and credit risk, manuscript.
• Bonatto, Pennanen, Optimal hedging and valuation of oil refineries and supply contracts, manuscript.
• Pennanen, Perkkio, Convex duality in optimal investment and contingent claim valuation in illiquid markets, manuscript.
Let $\mathcal{M}$ be the linear space of adapted sequences of cash-flows on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$.

- The **financial market** is described by a convex set $\mathcal{C} \subset \mathcal{M}$ of claims that can be *superhedged* without cost (i.e. each $c \in \mathcal{C}$ is freely available in the financial market).
- In models with a *perfectly liquid cash-account*,

$$\mathcal{C} = \{ c \in \mathcal{M} \mid \sum_{t=0}^T c_t \in C \}$$

where $C \subset L^0(\Omega, \mathcal{F}_T, P)$ are the claims at $T$ that can be hedged without cost [Delbaen and Schachermayer, 2006].

- **Conical** $\mathcal{C}$: [Dermody and Rockafellar, 1991], [Jaschke and Küchler, 2001], [Jouini and Napp, 2001], [Madan, 2014].
Example 1 (The classical model) In the classical perfectly liquid market model with a cash-account

\[ C = \{ c \in \mathcal{M} \mid \exists x \in \mathcal{N} : \sum_{t=0}^{T} c_t \leq \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1} \} \]

which is a convex cone. This set has been extensively studied in the literature; see e.g. [Föllmer and Schied, 2004] or [Delbaen and Schachermayer, 2006] and their references.
Asset-Liability Management

• Consider an agent with liabilities $c \in \mathcal{M}$, access to $\mathcal{C}$ and a loss function $\mathcal{V} : \mathcal{M} \rightarrow \mathbb{R}$ that measures disutility/regret/risk/... of delivering $c \in \mathcal{M}$. For example,

$$\mathcal{V}(c) = \mathbb{E} \sum_{t=0}^{T} -u_t(-c_t).$$

• The agent’s ALM problem can be written as

$$\varphi(c) = \inf_{d \in \mathcal{C}} \mathcal{V}(c - d)$$

• We assume that $\mathcal{V}$ is convex and nondecreasing with $\mathcal{V}(0) = 0$. 
Example: Oil derivatives

- We study the problem of a derivatives trader who aims to optimize his end of the year derivatives book P/L.
- The market consists of 12 futures contracts with 12 maturities on five oil products:
  - Brent: North Sea Brent Crude,
  - WTI: West Texas Intermediate Crude Oil,
  - RBOB: Reformulated Gasoline Blendstock,
  - HO: NY Harbor Ultra Low Sulphur Diesel,
  - Gasoil: Low Sulphur Gasoil.
- This is joint work with Luciane Bonatto, Petrobras.
Example: Oil derivatives
Example: Oil derivatives

- Denote the **spot price** of underlying \( i \) at time \( t \) by \( S_{t,i} \).
- The payouts of long and short positions in a futures contract with maturity \( t \) are given by

\[
\begin{align*}
  c^l_t &= S_{t,i} - F^a_t, \\
  c^s_t &= F^b_t - S_{t,i}
\end{align*}
\]

for **outright contracts** and

\[
\begin{align*}
  c^l_t &= (S_{t,i} - S_{t,j}) - F^a_t \\
  c^s_t &= F^b_t - (S_{t,i} - S_{t,j})
\end{align*}
\]

for **spread contracts**.
- Here \( F^b_t, F^a_t \) are the bid and ask **futures prices**. Both bid and ask prices come with finite **quantities** \( q_a, q_b \).
Example: Oil derivatives

Figure 1: Market bid and ask futures prices and volume
Example: Oil derivatives
Example: Oil derivatives
Example: Oil derivatives
Example: Oil derivatives
Example: Oil derivatives

Actual Prices and Simulated Prices Gasoil

- Market
- Simulation
Example: Oil derivatives

The set of superhedgeable claims becomes

\[ C = \{ c \in \mathcal{M} | \exists w \in \mathcal{M}, x \in (\mathbb{R}_{+}^{2K})^{12} : c_t + w_t \leq r_t w_{t-1} + \sum_{k \in K} (x_{t}^{l,k} c_{t}^{l,k} + x_{t}^{s,k} c_{t}^{s,k}) \} \]

where \( K \) is the set of traded contracts and

- \( w_t \) monthly cash position with \( w_{-1} := 0 \),
- \( r_t \) monthly return on cash,
- \( c_{t}^{l,k}, c_{t}^{s,k} \) net cash-flows of long/short position in the \( k \)th forward,
- \( x_{t}^{l,k}, x_{t}^{s,k} \) long/short position in the \( k \)th swap (to be optimized),
- \( c_t \) agent’s cash-flows to be hedged.
Example: Oil derivatives

- We describe risk preferences by

\[ \mathcal{V}(c) = \begin{cases} E \exp[\gamma c_T] & \text{if } c_t \leq 0 \text{ for } t < T, \\ +\infty & \text{otherwise.} \end{cases} \]

where \( \gamma > 0 \) describes the risk aversion of the agent.

- The ALM-problem can then be written

\[
\text{minimize } E \exp(-\gamma w_T) \text{ over } x \in [0, q],
\]

where \( w_T \) is given by the recursion

\[
c_t + w_t = r_t w_{t-1} + \sum_{k \in K} (x_t^{l,k} c_t^{l,k} + x_t^{s,k} c_t^{s,k}).
\]

- This is a 288-dimensional convex optimization problem.
Example: Oil derivatives

- We discretized the probability measure with 10,000 scenarios generated by antithetic sampling.
- The approximate problem was solved with the sequential quadratic programming algorithm of Matlab’s fmincon.
- The user can easily include other convex portfolio constraints coming e.g. from risk management.
Example: Oil derivatives
Example: Oil derivatives

![Optimal Portfolio in lots (1000 bbl)](image)

**Optimal Portfolio in lots (1000 bbl)**

- Data for Brent and WTI contracts from Jan to Dec
- Blue bars represent long positions
- Red bars represent short positions

**Valuations**
- Analysis of oil derivatives
- Valuation techniques

**Optionalities**
- Strategies for hedging
- Risk management

**Duality**
- Comparison of long and short positions
- Market dynamics
Example: EONIA swaps

- EONIA (Euro Over Night Index Average) is the average overnight interest rate on agreed interbank lending.
- We study indifference swap rates of EONIA swaps (Overnight Index Swaps).
- The hedging instruments consist of EONIA and other EONIA swaps.
- This is joint work with Maria Nogueiras, HSBC.
Example: EONIA swaps

![Graph showing historical and simulated rates for EONIA swaps](image-url)
Example: EONIA swaps

Table 1: Swap data:

<table>
<thead>
<tr>
<th>OIS Maturity</th>
<th>OIS Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1W</td>
<td>$-0.2730E-3$</td>
</tr>
<tr>
<td>2W</td>
<td>$-0.0500E-3$</td>
</tr>
<tr>
<td>3W</td>
<td>$-0.0300E-3$</td>
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<td>2M</td>
<td>$-0.0700E-3$</td>
</tr>
<tr>
<td>3M</td>
<td>$-0.1400E-3$</td>
</tr>
<tr>
<td>6M</td>
<td>$-0.1300E-3$</td>
</tr>
</tbody>
</table>
Example: EONIA swaps

We have

\[ C = \{ c \in \mathcal{M} \mid \exists x \in \mathcal{N}_0, z \in \mathbb{R}^K : x_t + c_t \leq (1 + r_t)x_{t-1} + \sum_{k \in K} z_k c_{k,t} \} \]

where

- \( x_t \): amount of overnight deposits,
- \( r_t \): EONIA rate,
- \( c_{k,t} \): net cash-flows of the \( k \)th swap,
- \( z_k \): position in the \( k \)th swap (to be optimized),
- \( c_t \): agent’s cash-flows to be hedged,
• We describe risk preferences by

\[
V(c) = \begin{cases} 
E \exp[\gamma c_T] & \text{if } c_t \leq 0 \text{ for } t < T, \\
+\infty & \text{otherwise.}
\end{cases}
\]

where \(\gamma > 0\) describes the risk aversion of the agent.

• The ALM-problem can then be written as

\[
\text{minimize} \quad E \exp(-\gamma x_T) \quad \text{over} \quad z \in \mathbb{R}^K,
\]

where \(x_T\) is given by the recursion

\[
x_t = (1 + r_{t-1})x_{t-1} + \sum_{k \in K} z_k C_{k,t} - c_t.
\]
Example: EONIA swaps

Optimal terminal wealth distribution with $\gamma = 1$
Example: EONIA swaps

Optimal terminal wealth distribution with $\gamma = 5$
Example: EONIA swaps

Optimal terminal wealth distribution with $\gamma = 10$
Pre-crisis valuations

- **Risk neutral valuation** assumes that the payout of a claim can be replicated by trading and that the negative of the trading strategy replicates the negative claim (perfect liquidity).

- It follows that
  - there is only one sensible price for buying/selling the claim.
  - the price can be expressed as the expectation of the cash-flows under a “risk neutral measure”.
  - the price does not depend on our market expectations, risk preferences or financial position.

- The independence is peculiar to redundant securities whose cash-flows can be replicated by trading other assets.
Pre-crisis valuations

- Actuarial valuations come from the opposite direction where everything is invested on the “bank account” and nothing but fixed-income instruments can be replicated.

- Actuarial valuations can be divided roughly into
  - premium principles reminiscent of indifference valuations discussed below.
  - “best estimate” which is defined as the discounted expectation of future cash-flows.

- Such valuations are not market consistent: the “best estimate” of e.g. a European call tends to be too high.

- The “best estimate” is inherently procyclical: it increases when discount rates decrease during financial crises.

- A trick question: “What discount rate should be used?”
Pre-crisis valuations

- The flaws of pre-crisis valuations are well-known so it is common to adjust the incorrect valuations:
  - Credit valuation adjustment (CVA) tries to correct for credit risk that was ignored by a pricing model.
  - Funding valuation adjustment (FVA) tries to correct for incorrect lending/borrowing rates.
  - Risk margin in Solvency II tries to correct for the risk that is filtered out by the expectation in the “best estimate”.
- Instead of adjusting incorrect valuations, we will adjust the underlying model and derive values from hedging arguments à la Black–Scholes.
Valuation of contingent claims

• In incomplete markets, the hedging argument for valuation of contingent claims has two natural generalizations:
  ◦ accounting value: How much cash do we need to cover our liabilities at an acceptable level of risk?
  ◦ indifference price: What is the least price we can sell a financial product for without increasing our risk?

• The former is important in accounting, financial reporting and supervision (SII, IFRS) and in the BS-model.

• The latter is more relevant in trading.

• Classical math finance makes no distinction between the two.
Valuation of contingent claims

- In **incomplete markets**, the hedging argument for valuation of contingent claims has two natural generalizations:
  - **accounting value**: How much cash do we need to cover our liabilities at an acceptable level of risk?
  - **indifference price**: What is the least price we can sell a financial product for without increasing our risk?

- In general, such values depend on our **views**, **risk preferences** and **financial position**.

- Subjectivity is the driving force behind trading.

- Trying to avoid the subjectivity leads to inconsistencies and confusion.

- In **complete markets**, the two notions coincide and they are independent of the subjective factors.
Accounting values

- Let $\varphi : \mathcal{M} \to \overline{\mathbb{R}}$ be the optimum value function of (ALM).
- We define the accounting value for a liability $c \in \mathcal{M}$ by

$$\pi_s^0(c) = \inf \{ \alpha \in \mathbb{R} | \varphi(c - \alpha p^0) \leq 0 \}$$

where $p^0 = (1, 0, \ldots, 0)$.

- Similarly,

$$\pi_b^0(c) = \sup \{ \alpha \in \mathbb{R} | \varphi(\alpha p^0 - c) \leq 0 \}$$

gives the accounting value of an asset $c \in \mathcal{M}$.

- $\pi_s^0$ can be interpreted like a risk measure in [Artzner, Delbaen, Eber and Heath, 1999]. However, we have not assumed the existence of a cash-account so $\pi_s^0$ is defined on sequences of cash-flows.
Accounting values

Define the super- and subhedging costs

\[ \pi^0_{\text{sup}}(c) := \inf\{\alpha \mid c - \alpha p^0 \in C\}, \quad \pi^0_{\text{inf}}(c) := \inf\{\alpha \mid \alpha p^0 - c \in C\} \]

**Theorem 2** The accounting value \( \pi^0_s \) is convex and nondecreasing with respect to \( C^\infty \). We have \( \pi^0_s \leq \pi^0_{\text{sup}} \) and if \( \pi^0_s(0) \geq 0 \), then

\[ \pi^0_{\text{inf}}(c) \leq \pi^0_b(c) \leq \pi^0_s(c) \leq \pi^0_{\text{sup}}(c) \]

with equalities throughout if \( c - \alpha p^0 \in C \cap (-C) \) for \( \alpha \in \mathbb{R} \).

- \( \pi^0_s \) is “translation invariant”: if \( c' - \alpha p^0 \in C^\infty \cap (-C^\infty) \) (i.e. \( c' \in \mathcal{M} \) is replicable with initial cash \( \alpha \)), then
  \[ \pi^0(c + c') = \pi^0(c) + \alpha. \]

- In complete markets, \( c - \alpha p^0 \in C^\infty \cap (-C^\infty) \) always for some \( \alpha \in \mathbb{R} \), so \( \pi^0_s(c) \) is independent of preferences and views.
In a swap contract, an agent receives a sequence $p \in \mathcal{M}$ of premiums and delivers a sequence $c \in \mathcal{M}$ of claims.

Examples:
- Swaps with a “fixed leg”: $p = (1, \ldots, 1)$, random $c$.
- In credit derivatives (CDS, CDO, . . .) and other insurance contracts, both $p$ and $c$ are random.
- Traditionally in mathematical finance,
  \[
  p = (1, 0, \ldots, 0) \quad \text{and} \quad c = (0, \ldots, 0, c_T).
  \]
- Futures contracts:
  \[
  p = (0, \ldots, 0, 1) \quad \text{and} \quad c = (0, \ldots, 0, c_T).
  \]

Claims and premiums live in the same space.
Swap contracts

- Let $\varphi : M \rightarrow \mathbb{R}$ be the optimum value function of (ALM).
- If we already have liabilities $\bar{c} \in M$, then

$$
\pi(\bar{c}, p; c) := \inf \{ \alpha \in \mathbb{R} \mid \varphi(\bar{c} + c - \alpha p) \leq \varphi(\bar{c}) \}
$$

gives the least swap rate that would allow us to enter a swap contract without worsening our financial position.

- Similarly,

$$
\pi^b(\bar{c}, p; c) := \sup \{ \alpha \in \mathbb{R} \mid \varphi(\bar{c} - c + \alpha p) \leq \varphi(\bar{c}) \} = -\pi(\bar{c}, p; -c)
$$

gives the greatest swap rate we would need on the opposite side of the trade.

- When $p = (1, 0, \ldots, 0)$ and $c = (0, \ldots, 0, c_T)$, we get an extension of the indifference price of [Hodges and Neuberger, 1989] to nonproportional transactions costs.
Swap contracts

Define the super- and subhedging swap rates,
\[ \pi_{\text{sup}}(p; c) = \inf \{ \alpha \mid c - \alpha p \in C^\infty \}, \quad \pi_{\text{inf}}(p; c) = \sup \{ \alpha \mid \alpha p - c \in C^\infty \}. \]

If \( C \) is a cone and \( p = (1, 0, \ldots, 0) \), we recover the super- and subhedging costs \( \pi^0_{\text{sup}} \) and \( \pi^0_{\text{inf}} \).

**Theorem 3** If \( \pi(\bar{c}, p; 0) \geq 0 \), then
\[ \pi_{\text{inf}}(p; c) \leq \pi_b(\bar{c}, p; c) \leq \pi(\bar{c}, p; c) \leq \pi_{\text{sup}}(p; c) \]

*with equalities if* \( c - \alpha p \in C^\infty \cap (-C^\infty) \) *for some* \( \alpha \in \mathbb{R} \).

- Agents with identical views, preferences and financial position have no reason to trade with each other.
- Prices are independent of such subjective factors when \( c - \alpha p \in C^\infty \cap (-C^\infty) \) *for some* \( \alpha \in \mathbb{R} \). If in addition, \( p = p^0 \), then swap rates coincide with accounting values.
Example 4 (The classical model)  Consider the classical perfectly liquid market model where

\[ C = \{ c \in M \mid \exists x \in N : \sum_{t=0}^{T} c_t \leq \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1} \} \]

and \( C^\infty = C \). The condition \( c - \alpha p \in C^\infty \cap (-C^\infty) \) holds if there exist \( x \in N \) such that

\[ \sum_{t=0}^{T} c_t = \alpha \sum_{t=0}^{T} p_t + \sum_{t=0}^{T-1} x_t \cdot \Delta s_{t+1}. \]

The converse holds under the no-arbitrage condition.
The ALM-problem again

\[
\text{minimize} \quad E \exp(-\gamma w_T) \quad \text{over} \quad x \in [0, q],
\]

where \( w_t = r_tw_{t-1} + \sum_{k \in K} (x_t^{l,k} c_t^{l,k} + x_t^{s,k} c_t^{s,k}) - c_t. \)

- Consider a forward contract where the trader receives the \( L \) barrels of Brent in December in exchange of a \( L\alpha \) units of cash in December.
- That is, \( c = (0, \ldots, 0, Ls_{12}^1) \) and \( p = (0, \ldots, 0, L) \).
- The indifference swap rate

\[
\pi(\bar{c}, p; c) = \inf \{ \alpha \in \mathbb{R} \mid \varphi(\bar{c} + c - \alpha p) \leq \varphi(\bar{c}) \}
\]

can be found by line search and numerical optimization.
Example: Oil derivatives

- The quoted bid forward rate for December Brent is 44.72 up to 44 barrels.
- Line search and numerical optimization finds that the indifference price for going long
  - $L = 10$ barrels is 44.72 (this position can be perfectly hedged by the available December futures).
  - $L = 1000$ barrels is 41.80 (the optimal hedging strategy involves futures contracts with several maturities).
The ALM-problem again:

\[
\text{minimize } E \exp(-\gamma x_T) \text{ over } z \in \mathbb{R}^K,
\]

where \( x_t = (1 + r_{t-1})x_{t-1} + \sum_{k \in K} z_k c_{k,t} - c_t \).

- Consider a swap where the agent delivers the floating leg \( c \) of an EONIA swap and receives a multiple \( p \equiv 1 \).
- The indifference swap rate

\[
\pi(\bar{c}, p; c) = \inf \{ \alpha \in \mathbb{R} \mid \varphi(\bar{c} + c - \alpha p) \leq \varphi(\bar{c}) \}
\]

can be found by a simple line search with respect to \( \alpha \) by computing the optimum value \( \varphi(\bar{c} + c - \alpha p) \) at each iteration.
**Example: EONIA swaps**

**Reality check:** The indifference rate of a quoted 6M swap equals the quoted rate $-1.300 \times 10^{-4}$. This is independent of views and risk preferences just like the Black–Scholes formula.

**Table 2: Optimal portfolios before and after the trade**

<table>
<thead>
<tr>
<th>OIS Maturity</th>
<th>before</th>
<th>after</th>
</tr>
</thead>
<tbody>
<tr>
<td>1W</td>
<td>9.3882</td>
<td>9.3882</td>
</tr>
<tr>
<td>2W</td>
<td>-9.7979</td>
<td>-9.7979</td>
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<tr>
<td>3W</td>
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<td>0.1242</td>
</tr>
<tr>
<td>6M</td>
<td>-0.0345</td>
<td>0.9655</td>
</tr>
</tbody>
</table>
Example: EONIA swaps

Indifference rate of an unquoted 100 day swap:
\[-1.4184 \times 10^{-4}\]

Table 3: Optimal portfolios before and after the trade

<table>
<thead>
<tr>
<th>OIS Maturity</th>
<th>before</th>
<th>after</th>
</tr>
</thead>
<tbody>
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<tr>
<td>6M</td>
<td>-0.0345</td>
<td>0.1849</td>
</tr>
</tbody>
</table>
Example: EONIA swaps

Table 4: Dependence of indifference rate on the initial cash position

<table>
<thead>
<tr>
<th>units of cash</th>
<th>ID rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td>$4.2938 \times 10^{-5}$</td>
</tr>
<tr>
<td>0</td>
<td>$-1.4184 \times 10^{-4}$</td>
</tr>
<tr>
<td>5</td>
<td>$-3.1705 \times 10^{-4}$</td>
</tr>
</tbody>
</table>
In practice, many contracts and products involve optionalities: American options, Callable bonds, Swing options, Delivery contracts, Storages, ... 

In general, a long position in a contract with optionalities allows an agent to choose a sequence of cash-flows within a given (possibly random) set \( C \subset \mathbb{R}^{T+1} \).

Example: an American option on an underlying \( X = (X_t)_{t=0}^T \):

\[
C = \{c \mid c \leq e X, \ e \geq 0, \ \sum_{t=0}^T e_t \leq 1, \ e_t \in \{0, 1\}\}.
\]

The indifference swap rate for a long position in \( C \) becomes

\[
\pi_l(\bar{c}, p; C) = \sup\{\alpha \mid \inf_{c \in \mathcal{M}(C)} \varphi(\bar{c} - c + \alpha p) \leq \varphi(\bar{c})\},
\]

where \( \mathcal{M}(C) \) denotes the \((\mathcal{F}_t)_{t=0}^T\)-adapted selectors of \( C \).
Long positions in a contract with optionalities can be treated mathematically and numerically much like nonoptional contracts.

Quantitative analysis of a short position requires a model for the behaviour (exercise strategy) of the counterparty.

The mathematical literature on American options concentrates on superhedging against the worst case.

In practice, superhedging and the worst case assumption often lead to excessively high prices.

Indifference pricing of both long and short positions is studied in [Koch Medona, Munari, Pennanen, 2016].
• Embedding optimal investment problems in the general Conjugate duality framework of [Rockafellar, 1974] yields extensions of many classical duality results from financial mathematics to more general problems with e.g. illiquidity effects, transaction costs, portfolio constraints, . . .

• Combined with certain measure theoretic results from financial mathematics, convex analysis allows for closing the duality gap in quite a general class of models.

• In particular, in the classical perfectly liquid market model, we obtain a simple derivation of the sharpest form of the “fundamental theorem of asset pricing” due to [Dalang, Morton and Willinger, 1992].
Duality

- Let $\mathcal{M}^p = \{ c \in \mathcal{M} \mid c_t \in L^p(\Omega, \mathcal{F}_t, P; \mathbb{R}) \}$.
- The bilinear form
  $$\langle c, y \rangle := E \sum_{t=0}^{T} c_t y_t$$
  puts $\mathcal{M}^1$ and $\mathcal{M}^\infty$ in separating duality.
- The conjugate of a function $f$ on $\mathcal{M}^1$ is defined by
  $$f^*(y) = \sup_{c \in \mathcal{M}^1} \{ \langle c, y \rangle - f(c) \}.$$  
- If $f$ is proper, convex and lower semicontinuous, then
  $$f(y) = \sup_{y \in \mathcal{M}^\infty} \{ \langle c, y \rangle - f^*(y) \}.$$
We assume from now on that

\[ \mathcal{V}(c) = E \sum_{t=0}^{T} V_t(c_t) \]

for convex random functions \( V_t : \mathbb{R} \times \Omega \rightarrow \mathbb{R} \) with \( V_t(0) = 0 \).

**Theorem 5**  
If \( S_t(x, \cdot) \in L^1 \) for all \( x \in \mathbb{R}^J \), then

\[ \varphi^*(y) = \mathcal{V}^*(y) + \sigma_C(y) \]

where \( \mathcal{V}^*(y) = E \sum_{t=0}^{T} V_t^*(y_t) \) and \( \sigma_C(y) = \sup_{c \in C} \langle c, y \rangle \).

Moreover,

\[ \sigma_C(y) = \inf_{v \in \mathcal{N}^1} E \sum_{t=0}^{T} [(y_t S_t)^*(v_t) + \sigma_{D_t} (E[\Delta v_{t+1} | \mathcal{F}_t])] \]

where the infimum is attained for all \( y \in \mathcal{M}^\infty \).
Duality

Example 6 If $S_t(\omega, x) = s_t(\omega) \cdot x$ and $D_t(\omega)$ is a cone,
$$C^* = \{ y \in \mathcal{M}^\infty \mid E[\Delta(y_{t+1}s_{t+1}) \mid \mathcal{F}_t] \in D_t^* \}.$$  

Example 7 If $S_t(\omega, x) = \sup \{ s \cdot x \mid s \in [s^b_t(\omega), s^a_t(\omega)] \}$ and $D_t(\omega) = \mathbb{R}^J$, then
$$C^* = \{ y \in \mathcal{M}^\infty \mid ys \text{ is a martingale for some } s \in [s^b, s^a] \}.$$  

Example 8 In the classical model, $C^*$ consists of positive multiples of martingale densities.
Duality

**Theorem 9** Assume the linearity condition, the Inada condition $V_t^\infty = \delta_{\mathbb{R}_-}$ and that $p^0 \notin C^\infty$ and $\inf \varphi < 0$. Then

$$\pi^0(c) = \sup_{y \in M^\infty} \{ \langle c, y \rangle - \sigma_C(y) - \sigma_B(y) \mid y_0 = 1 \},$$

where $B = \{ c \in M^1 \mid V(c) \leq 0 \}$. In particular, when $C$ is conical and $V$ is positively homogeneous,

$$\pi^0(c) = \sup_{y \in M^\infty} \{ \langle c, y \rangle \mid y \in C^* \cap B^*, \ y_0 = 1 \}.$$

- Extends good deal bounds to sequences of cash-flows.
Theorem 10  Assume the linearity condition, the Inada condition and that \( p \notin C^\infty \) and \( \inf \varphi < \varphi(\bar{c}) \). Then

\[
\pi(\bar{c}, p; c) = \sup_{y \in M^\infty} \{ \langle c, y \rangle - \sigma_C(y) - \sigma_{B(\bar{c})}(y) \mid \langle p, y \rangle = 1 \},
\]

where \( B(\bar{c}) = \{ c \in M^1 \mid V(\bar{c} + c) \leq \varphi(\bar{c}) \} \). In particular, if \( C \) is conical,

\[
\pi(\bar{c}, p; c) = \sup_{y \in M^\infty} \{ \langle c, y \rangle - \sigma_{B(\bar{c})}(y) \mid u \in C^*, \langle p, y \rangle = 1 \}.
\]
Example 11 In the classical model, with \( p = (1, 0, \ldots, 0) \) and \( V_t = \delta_{\mathbb{R}_-} \) for \( t < T \), we get

\[
\pi(\bar{c}, p; c) = \sup_{Q \in \mathcal{Q}} \sup_{\alpha > 0} E^Q \left\{ \sum_{t=0}^{T} (\bar{c}_t + c_t) - \alpha \left[ V^*_T \left( \frac{dQ}{dP} / \alpha \right) - \varphi(\bar{c}) \right] \right\}
\]

where \( \mathcal{Q} \) is the set of absolutely continuous martingale measures; see [Biagini, Frittelli, Grasselli, 2011] for a continuous-time version.
Theorem 12 (FTAP) Assume that $S^\infty$ is finite-valued and that $D \equiv \mathbb{R}^J$. Then the following are equivalent

1. $S$ satisfies the robust no-arbitrage condition.

2. There is a strictly consistent price system: adapted processes $y$ and $s$ such that $y > 0$, $s_t \in \text{ri dom } S_t^*$ and $ys$ is a martingale.

- In the classical linear market model, $\text{ri dom } S_t^* = \{1, \tilde{s}_t\}$ so we recover the Dalang–Morton–Willinger theorem.

- The robust no-arbitrage condition means that there exists a sublinear arbitrage-free cost process $\tilde{S}$ with $\text{dom } \tilde{S}_t^* \subseteq \text{ri dom } S_t^*$. 
Summary

- Post-crisis FM is **subjective**: optimal investment and valuations depend on *views*, *risk preferences*, *financial position* and *trading expertise*.
- ALM brings pricing, accounting and risk management under a single consistent framework.
- Not a quick solution but a coherent and universal approach based on risk management.
- Requires techniques from statistics, optimization, and computer science.
- With some convex analysis, classical “fundamental theorems” can be extended to illiquid market models.