Acyclic Gambling Games

Rida Laraki
CNRS, Université de Paris Dauphine, et École Polytechnique

With Jérôme Renault, Université de Toulouse Capitole
Bewlew-Kohlberg (resp. Mertens-Neyman) proved existence of the asymptotic (resp. the uniform value) in finite actions and state spaces stochastic games.
• Bewlew-Kohlberg (resp. Mertens-Neyman) proved existence of the asymptotic (resp. the uniform value) in finite actions and state spaces stochastic games.

• Recently, two main conjectures on existence without finiteness was proved to be wrong.
Bewlew-Kohlberg (resp. Mertens-Neyman) proved existence of the asymptotic (resp. the uniform value) in finite actions and state spaces stochastic games.

Recently, two main conjectures on existence without finiteness was proved to be wrong.

Vigeral (2013) proved that the limit of discounted values fail to exist in finite state space, compact actions sets and continuous payoffs and transitions.
Bewlew-Kohlberg (resp. Mertens-Neyman) proved existence of the asymptotic (resp. the uniform value) in finite actions and state spaces stochastic games.  
Recently, two main conjectures on existence without finiteness was proved to be wrong.  
Vigeral (2013) proved that the limit of discounted values fail to exist in finite state space, compact actions sets and continuous payoffs and transitions.  
Ziliotto (2013) proved divergence in many cases and in particular compact state space and finite actions sets, and continuous payoffs and transitions.
Bewlew-Kohlberg (resp. Mertens-Neyman) proved existence of the asymptotic (resp. the uniform value) in finite actions and state spaces stochastic games.

Recently, two main conjectures on existence without finiteness was proved to be wrong.

Vigeral (2013) proved that the limit of discounted values fail to exist in finite state space, compact actions sets and continuous payoffs and transitions.

Ziliotto (2013) proved divergence in many cases and in particular compact state space and finite actions sets, and continuous payoffs and transitions.

To avoid oscillations, Botle-Gaubert-Vigeral (2015) proved convergence in finite state space, definable compact action sets and definable transition functions.
Mertens-Zamir, Mertens-Neyman-Rosenberg, Sorin-Vigeral, Laraki, Renault, Li-Venel proved existence of the asymptotic or uniform value in many classes “irreversible” stochastic games, with general compact action and state spaces.
Mertens-Zamir, Mertens-Neyman-Rosenberg, Sorin-Vigeral, Laraki, Renault, Li-Venel proved existence of the asymptotic or uniform value in many classes “irreversible” stochastic games, with general compact action and state spaces.

Our paper provides a general definition of “irreversibility” in product stochastic games.
Mertens-Zamir, Mertens-Neyman-Rosenberg, Sorin-Vigeral, Laraki, Renault, Li-Venel proved existence of the asymptotic or uniform value in many classes “irreversible” stochastic games, with general compact action and state spaces.

Our paper provides a general definition of “irreversibility” in product stochastic games.

We show that “acyclicity” guarantees convergence and any weakening of it implies divergence.
Mertens-Zamir, Mertens-Neyman-Rosenberg, Sorin-Vigeral, Laraki, Renault, Li-Venel proved existence of the asymptotic or uniform value in many classes “irreversible” stochastic games, with general compact action and state spaces.

Our paper provides a general definition of “irreversibility” in product stochastic games.

We show that “acyclicity” guarantees convergence and any weakening of it implies divergence.

We also extend the Mertenz-Zamir characterization of the asymptotic value in repeated games with incomplete information on both sides to product stochastic games.
Given a compact metric space $S$, we denote by :

- $C(S)$: continuous, functions from $S \to \mathbb{R}$
Given a compact metric space $S$, we denote by:

- $C(S)$: continuous, functions from $S \rightarrow \mathbb{R}$
- $\Delta(S)$: Borel probabilities over $S$. 

$\delta_s \in \Delta(S)$: the Dirac measure on $s \in S$.

For $v \in C(S)$, $\tilde{v}$ is the affine extension to $\Delta(S)$: $\tilde{v}(p) = IE_p(v)$ for all $p$ in $\Delta(S)$.

$\Delta(S)$ is endowed with the weak* topology. A compatible distance being the Kantorovich-Rubinstein metric:

$$d_{KR}(p, p') = \sup_{v \in E_1} |\tilde{v}(p) - \tilde{v}(p')|$$

where $E_1$ is the set of 1-Lipschitz functions on $S$. 

Rida Laraki and Jérôme Renault
Given a compact metric space $S$, we denote by:

- $C(S)$: continuous, functions from $S \to \mathbb{R}$
- $\Delta(S)$: Borel probabilities over $S$.
- $\delta_s \in \Delta(S)$: the Dirac measure on $s \in S$. 

$\Delta(S)$ is endowed with the weak* topology. A compatible distance being the Kantorovich-Rubinstein metric:

$$d_{KR}(p, p') = \sup_{v \in E_1} |\tilde{v}(p) - \tilde{v}(p')|$$

where $E_1$ is the set of 1-Lipschitz functions on $S$. 

Rida Laraki and Jérôme Renault
General Notation

Given a compact metric space $S$, we denote by:

- $C(S)$: continuous, functions from $S \to \mathbb{R}$
- $\Delta(S)$: Borel probabilities over $S$.
- $\delta_s \in \Delta(S)$: the Dirac measure on $s \in S$.
- For $\nu \in C(S)$, $\tilde{\nu}$ is the affine extension to $\Delta(S)$:
  $$\tilde{\nu}(p) = E_p(\nu) \text{ for all } p \text{ in } \Delta(S).$$
Given a compact metric space $S$, we denote by:

- $C(S)$: continuous, functions from $S \rightarrow \mathbb{R}$
- $\Delta(S)$: Borel probabilities over $S$.
- $\delta_s \in \Delta(S)$: the Dirac measure on $s \in S$.
- For $\nu \in C(S)$, $\tilde{\nu}$ is the affine extension to $\Delta(S)$:
  $$\tilde{\nu}(p) = E_p(\nu) \text{ for all } p \in \Delta(S).$$
- $\Delta(S)$ is endowed with the weak$^*$ topology.
Given a compact metric space $S$, we denote by:

- $C(S)$: continuous, functions from $S \rightarrow \mathbb{R}$
- $\Delta(S)$: Borel probabilities over $S$.
- $\delta_s \in \Delta(S)$: the Dirac measure on $s \in S$.
- For $\nu \in C(S)$, $\tilde{\nu}$ is the affine extension to $\Delta(S)$:
  \[ \tilde{\nu}(p) = E_p(\nu) \text{ for all } p \in \Delta(S). \]
- $\Delta(S)$ is endowed with the weak* topology.
- A compatible distance being the Kantorovich-Rubinstein metric:
  \[ d_{KR}(p, p') = \sup_{\nu \in E_1} |\tilde{\nu}(p) - \tilde{\nu}(p')| \]
  where $E_1$ is the set of 1-Lipschitz functions on $S$. 
We consider a zero-sum product stochastic game where each player controls his gambling house.

- $X$ (resp. $Y$) is a non empty compact metric set of states controlled by player 1 (resp. player 2).
We consider a zero-sum \textbf{product stochastic game} where each player controls his gambling house.

- \( X \) (resp. \( Y \)) is a non empty compact metric set of states controlled by player 1 (resp. player 2).

- The transitions of player 1 (resp. player 2) are given by a continuous multifunction \( \Gamma : X \rightrightarrows \Delta(X) \) with non empty convex compact values, (resp. \( \Lambda : Y \rightrightarrows \Delta(Y) \)).
We consider a zero-sum product stochastic game where each player controls his gambling house.

- \( X \) (resp. \( Y \)) is a non empty compact metric set of states controlled by player 1 (resp. player 2).
- The transitions of player 1 (resp. player 2) are given by a continuous multifunction \( \Gamma : X \Rightarrow \Delta(X) \) with non empty convex compact values, (resp. \( \Lambda : Y \Rightarrow \Delta(Y) \)).
We consider a zero-sum product stochastic game where each player controls his gambling house.

- $X$ (resp. $Y$) is a non empty compact metric set of states controlled by player 1 (resp. player 2).
- The transitions of player 1 (resp. player 2) are given by a continuous multifunction $\Gamma : X \rightrightarrows \Delta(X)$ with non empty convex compact values, (resp. $\Lambda : Y \rightrightarrows \Delta(Y)$).
- The running payoff of $P_1$ is given by a continuous mapping $u : X \times Y \rightarrow \mathbb{R}$, the payoff to $P_2$ is given by $-u$. 
How the game is played?

Given a discount factor $\lambda \in (0, 1]$ and an initial state $(x_1, y_1) \in X \times Y$, the game is played as follows:

- At any stage $t \geq 1$, the payoff of P1 is $u(x_t, y_t)$. 

Rida Laraki and Jérôme Renault
Given a discount factor $\lambda \in (0, 1]$ and an initial state $(x_1, y_1) \in X \times Y$, the game is played as follows:

- At any stage $t \geq 1$, the payoff of P1 is $u(x_t, y_t)$.
- Independently P1 chooses $p_{t+1} \in \Gamma(x_t)$ and P2 chooses $q_{t+1} \in \Lambda(y_t)$. The new states $x_{t+1}$ and $y_{t+1}$ are selected according to $p_{t+1}$ and $q_{t+1}$.
- The stream of payoffs is evaluated according to $\sum_{t \geq 1} \lambda \left(1 - \lambda \right)^{t-1} u(x_t, y_t)$. 
How the game is played?

Given a discount factor \( \lambda \in (0, 1] \) and an initial state \((x_1, y_1) \in X \times Y\), the game is played as follows:

- At any stage \( t \geq 1 \), the payoff of P1 is \( u(x_t, y_t) \).
- Independently P1 chooses \( p_{t+1} \in \Gamma(x_t) \) and P2 chooses \( q_{t+1} \in \Lambda(y_t) \).
- New states \( x_{t+1} \) and \( y_{t+1} \) are selected according to \( p_{t+1} \) and \( q_{t+1} \).
How the game is played?

Given a discount factor $\lambda \in (0, 1]$ and an initial state $(x_1, y_1) \in X \times Y$, the game is played as follows:

- At any stage $t \geq 1$, the payoff of P1 is $u(x_t, y_t)$.
- Independently P1 chooses $p_{t+1} \in \Gamma(x_t)$ and P2 chooses $q_{t+1} \in \Lambda(y_t)$.
- New states $x_{t+1}$ and $y_{t+1}$ are selected according to $p_{t+1}$ and $q_{t+1}$.
- $x_{t+1}$ and $y_{t+1}$ are publicly announced, and the play goes to stage $t + 1$. 

The stream of payoffs is evaluated according to $\sum_{t=0}^{\infty} \lambda (1 - \lambda)^{t-1} u(x_t, y_t)$.
How the game is played?

Given a discount factor \( \lambda \in (0, 1] \) and an initial state \((x_1, y_1) \in X \times Y\), the game is played as follows:

- At any stage \( t \geq 1 \), the payoff of P1 is \( u(x_t, y_t) \).
- Independently P1 chooses \( p_{t+1} \in \Gamma(x_t) \) and P2 chooses \( q_{t+1} \in \Lambda(y_t) \).
- New states \( x_{t+1} \) and \( y_{t+1} \) are selected according to \( p_{t+1} \) and \( q_{t+1} \).
- \( x_{t+1} \) and \( y_{t+1} \) are publicly announced, and the play goes to stage \( t + 1 \).
- The stream of payoffs is evaluated according to

\[
\sum_{t} \lambda(1 - \lambda)^{t-1} u(x_t, y_t).
\]
Definition

Let \( v_\lambda \) be the unique element of \( C(X \times Y) \) s.t. \( \forall (x, y) \in X \times Y \),

\[
v_\lambda(x, y) = \max_{p \in \Gamma(x)} \min_{q \in \Lambda(y)} \left( \lambda u(x, y) + (1 - \lambda) \tilde{v}_\lambda(p, q) \right)
\]

\[
= \min_{q \in \Lambda(y)} \max_{p \in \Gamma(x)} \left( \lambda u(x, y) + (1 - \lambda) \tilde{v}_\lambda(p, q) \right).
\]
**Definition**

Let $v_\lambda$ be the unique element of $C(X \times Y)$ s.t. $\forall (x, y) \in X \times Y$, 

$$
v_\lambda(x, y) = \max_{p \in \Gamma(x)} \min_{q \in \Lambda(y)} \left( \lambda u(x, y) + (1 - \lambda) \tilde{v}_\lambda(p, q) \right)
$$

$$
= \min_{q \in \Lambda(y)} \max_{p \in \Gamma(x)} \left( \lambda u(x, y) + (1 - \lambda) \tilde{v}_\lambda(p, q) \right).
$$

- Existence and uniqueness of $v_\lambda$ follow from standard fixed-point arguments (Sion’s theorem).
Let $v_\lambda$ be the unique element of $C(X \times Y)$ s.t. $\forall (x, y) \in X \times Y$,

$$v_\lambda(x, y) = \max_{p \in \Gamma(x)} \min_{q \in \Lambda(y)} (\lambda u(x, y) + (1 - \lambda) \tilde{v}_\lambda(p, q))$$

$$= \min_{q \in \Lambda(y)} \max_{p \in \Gamma(x)} (\lambda u(x, y) + (1 - \lambda) \tilde{v}_\lambda(p, q)).$$

- Existence and uniqueness of $v_\lambda$ follow from standard fixed-point arguments (Sion’s theorem).
- $v_\lambda(x, y)$ is the value of the discounted gambling game.
Let \( v_\lambda \) be the unique element of \( C(X \times Y) \) s.t. \( \forall (x, y) \in X \times Y \),

\[
 v_\lambda(x, y) = \max_{p \in \Gamma(x)} \min_{q \in \Lambda(y)} (\lambda u(x, y) + (1 - \lambda) \tilde{v}_\lambda(p, q))
\]

\[
 = \min_{q \in \Lambda(y)} \max_{p \in \Gamma(x)} (\lambda u(x, y) + (1 - \lambda) \tilde{v}_\lambda(p, q)).
\]

- Existence and uniqueness of \( v_\lambda \) follow from standard fixed-point arguments (Sion’s theorem).
- \( v_\lambda(x, y) \) is the value of the discounted gambling game.
- Our goal is to establish tight conditions for convergence/divergence of \( v_\lambda \) (as \( \lambda \to 0 \)) and a characterization of the limit.
Example 1: red-and-black casino

\[ X \subset [0, \infty) \] is the space of fortune.
Example 1: red-and-black casino

- $X \subset [0, \infty)$ is the space of fortune.
- At each fortune $x \geq 0$, the gambler can stake any $s \in [0, x]$. 

Dubins & Savage (1965) introduced and solved the one player model.

What if there are two players competing $u(x, y) = x - y$?
Example 1: red-and-black casino

- $X \subset [0, \infty)$ is the space of fortune.
- At each fortune $x \geq 0$, the gambler can stake any $s \in [0, x]$.
- The gambler wins back the stake $s$ and an equal amount more with probability $w \in ]0, 1[$, but looses with probability $1 - w$:

$$\Gamma_w(x) = \{ w\delta(x+s) + (1-w)\delta(x-s) : 0 \leq s \leq x \}, \quad x \in X,$$
Example 1: red-and-black casino

- $X \subset [0, \infty)$ is the space of fortune.
- At each fortune $x \geq 0$, the gambler can stake any $s \in [0, x]$.
- The gambler wins back the stake $s$ and an equal amount more with probability $w \in ]0, 1[$, but looses with probability $1 - w$:

$$\Gamma_w(x) = \{ w\delta(x+s) + (1-w)\delta(x-s) : 0 \leq s \leq x \}, \quad x \in X,$$

- Dubins & Savage (1965) introduced and solved the one player model.
Example 1: red-and-black casino

- $X \subset [0, \infty)$ is the space of fortune.
- At each fortune $x \geq 0$, the gambler can stake any $s \in [0, x]$.
- The gambler wins back the stake $s$ and an equal amount more with probability $w \in ]0, 1[$, but loses with probability $1 - w$:
  \[
  \Gamma_w(x) = \{ w\delta(x+s)+(1-w)\delta(x-s) : 0 \leq s \leq x \}, \quad x \in X,
  \]
- Dubins & Savage (1965) introduced and solved the one player model.
- What if there are two players competing $u(x, y) = x - y$?
$X$ and $Y$ are two convex compact metric sets.
$X$ and $Y$ are two convex compact metric sets.

$\Gamma(x)$ is the set of probability distributions on $X$ that are centered at $x$. 

\( \mathcal{X} \) and \( \mathcal{Y} \) are two convex compact metric sets.

\( \Gamma(x) \) is the set of probability distributions on \( \mathcal{X} \) that are centered at \( x \).

\( \Gamma(y) \) is the set of probability distributions on \( \mathcal{Y} \) that are centered at \( y \).
\( X \) and \( Y \) are two convex compact metric sets.

\( \Gamma(x) \) is the set of probability distributions on \( X \) that are centered at \( x \).

\( \Gamma(y) \) is the set of probability distributions on \( Y \) that are centered at \( y \).

Recall that $\Gamma : X \Rightarrow \Delta(X)$. \[\text{THE GRAPH OF } \tilde{\Gamma} \text{ IS DEFINED AS THE CLOSURE OF THE CONVEX HULL OF THE GRAPH OF } \Gamma, \text{ VIEWED AS A SUBSET OF } \Delta(X) \times \Delta(X) \{ (\delta x, p), x \in X, p \in \Gamma(x) \}. \]
Recall that $\Gamma : X \Rightarrow \Delta(X)$.

\(\tilde{\Gamma} : \Delta(X) \Rightarrow \Delta(X)\) extends linearly $\Gamma$. 

Recall that $\Gamma : X \Rightarrow \Delta(X)$.

$\tilde{\Gamma} : \Delta(X) \Rightarrow \Delta(X)$ extends linearly $\Gamma$.

The graph of $\tilde{\Gamma}$ is defined as the closure of the convex hull of the graph of $\Gamma$, viewed as a subset of $\Delta(X) \times \Delta(X)$

$$\{(\delta_x, p), x \in X, p \in \Gamma(x)\}$$
Define inductively $(\tilde{\Gamma}^n)_n : \Delta(X) \Rightarrow \Delta(X)$ by :

- $\tilde{\Gamma}^0(p) = \{p\}$ for every state $p$ in $\Delta(X)$,
Define inductively \((\tilde{\Gamma}^n)_n : \Delta(X) \Rightarrow \Delta(X)\) by:

- \(\tilde{\Gamma}^0(p) = \{p\}\) for every state \(p\) in \(\Delta(X)\),
- For each \(n \geq 0\), \(\tilde{\Gamma}^{n+1} = \tilde{\Gamma}^n \circ \tilde{\Gamma}\).
Define inductively $(\tilde{\Gamma}^n)_n : \Delta(X) \Rightarrow \Delta(X)$ by:

- $\tilde{\Gamma}^0(p) = \{p\}$ for every state $p$ in $\Delta(X)$,
- For each $n \geq 0$, $\tilde{\Gamma}^{n+1} = \tilde{\Gamma}^n \circ \tilde{\Gamma}$.
- $\Gamma^\infty(x)$ is the reachable set of Player 1 from state $x$ in $X$ and is defined as the closure of $\bigcup_{n \geq 0} \tilde{\Gamma}^n(\delta_x)$.
Define inductively \((\tilde{\Gamma}^n)_n : \Delta(X) \Rightarrow \Delta(X)\) by:

- \(\tilde{\Gamma}^0(p) = \{p\}\) for every state \(p\) in \(\Delta(X)\),
- For each \(n \geq 0\), \(\tilde{\Gamma}^{n+1} = \tilde{\Gamma}^n \circ \tilde{\Gamma}\).
- \(\Gamma^\infty(x)\) is the reachable set of Player 1 from state \(x\) in \(X\) and is defined as the closure of \(\bigcup_{n \geq 0} \tilde{\Gamma}^n(\delta_x)\).
- Similarly we define \(\Lambda^\infty(x)\).
The game is called non-expansive if:

\[ \forall x \in X, \forall x' \in X, \forall p \in \Gamma(x), \exists p' \in \Gamma(x'), \text{ s.t. } d_{KR}(p, p') \leq d(x, x'), \]

and similarly for \( \Lambda \).

Example 1: When \( X \) and \( Y \) are finite, the game is non-expansive.

Example 2: Splitting games is non-expansive and leavable.

Example 3: Red-black casinos are leavable. They are non-expansive if and only if \( w \leq \frac{1}{2} \).
The game is called **non-expansive** if:

\[
\forall x \in X, \forall x' \in X, \forall p \in \Gamma(x), \exists p' \in \Gamma(x'), \text{ s.t. } d_{KR}(p, p') \leq d(x, x'),
\]

and similarly for \(\Lambda\).

The game is called **leavable** if:

\[
\forall x \in X, \delta_x \in \Gamma(x) \text{ and } \forall y \in Y, \delta_y \in \Lambda(y)
\]
The game is called **non-expansive** if:

\[ \forall x \in X, \forall x' \in X, \forall p \in \Gamma(x), \exists p' \in \Gamma(x'), \text{ s.t. } d_{KR}(p, p') \leq d(x, x'), \]

and similarly for \( \Lambda \)

The game is called **leavable** if:

\[ \forall x \in X, \delta_x \in \Gamma(x) \text{ and } \forall y \in Y, \delta_y \in \Lambda(y) \]

**Example 1**: When \( X \) and \( Y \) are finite, the game is non-expansive.
The game is called **non-expansive** if:

\[ \forall x \in X, \forall x' \in X, \forall p \in \Gamma(x), \exists p' \in \Gamma(x'), \text{ s.t. } d_{KR}(p, p') \leq d(x, x'), \]

and similarly for \( \Lambda \).

The game is called **leavable** if:

\[ \forall x \in X, \delta_x \in \Gamma(x) \text{ and } \forall y \in Y, \delta_y \in \Lambda(y) \]

**Example 1**: When \( X \) and \( Y \) are finite, the game is non-expansive.

**Example 2**: Splitting games is non-expansive and leavable,

**Example 3**: Red-black casinos are leavable. They are non-expansive if and only if \( w \leq \frac{1}{2} \).
The game is called **non-expansive** if:

\[ \forall x \in X, \forall x' \in X, \forall p \in \Gamma(x), \exists p' \in \Gamma(x'), \text{ s.t. } d_{KR}(p, p') \leq d(x, x'), \]

and similarly for \( \Lambda \)

The game is called **leavable** if:

\[ \forall x \in X, \exists \delta_x \in \Gamma(x) \text{ and } \forall y \in Y, \exists \delta_y \in \Lambda(y) \]

- **Example 1**: When \( X \) and \( Y \) are finite, the game is non-expansive.
- **Example 2**: Splitting games is non-expansive and leavable,
- **Example 3**: Red-black casinos are leavable. They are non-expansive if and only if \( w \leq \frac{1}{2} \).
Excessive, Depressive, Balanced

Definition

1) \( v \) is excessive if: \( \forall (x, y), \ v(x, y) = \max_{p \in \Gamma(x)} \tilde{v}(p, y) \).

Observe that under non-expansivity and compactness, a uniform limit \( v \) of \( (v_\lambda) \) is necessarily continuous, and balanced.
Excessive, Depressive, Balanced

Definition

1) $v$ is excessive if: $\forall (x, y), v(x, y) = \max_{p \in \Gamma(x)} \tilde{v}(p, y)$.
2) $v$ is depressive if: $\forall (x, y), v(x, y) = \min_{q \in \Gamma(y)} \tilde{v}(x, q)$.
Definition

1) \(v\) is excessive if: \(\forall (x, y), \ v(x, y) = \max_{p \in \Gamma(x)} \tilde{v}(p, y).\)

2) \(v\) is depressive if: \(\forall (x, y), \ v(x, y) = \min_{q \in \Gamma(y)} \tilde{v}(x, q).\)

3) \(v\) is balanced if \(\forall (x, y),\)
\[
\begin{align*}
v(x, y) &= \max_{p \in \Gamma(x)} \min_{q \in \Gamma(y)} \tilde{v}(p, q) = \min_{q \in \Gamma(y)} \max_{p \in \Gamma(x)} \tilde{v}(p, q).
\end{align*}
\]
Definition

1) \( v \) is **excessive** if : \( \forall (x, y), \ v(x, y) = \max_{p \in \Gamma(x)} \tilde{v}(p, y). \)

2) \( v \) is **depressive** if : \( \forall (x, y), \ v(x, y) = \min_{q \in \Gamma(y)} \tilde{v}(x, q). \)

3) \( v \) is **balanced** if \( \forall (x, y), \)
\[ v(x, y) = \max_{p \in \Gamma(x)} \min_{q \in \Gamma(y)} \tilde{v}(p, q) = \min_{q \in \Gamma(y)} \max_{p \in \Gamma(x)} \tilde{v}(p, q). \]

Observe that under non-expansivity and compactness, a uniform limit \( \nu \) of \((\nu_\lambda)\) is necessarily continuous, and balanced.
Main result for 1-player games

**Theorem**

Consider a one player gambling game non-expansive and leavable. Then \((v_\lambda)\) uniformly converges to \(v\) (called the “réduite”)

\(v\) is the unique excessive function in \(C(X)\) satisfying:

\[ \forall x \in X, \exists p \in \Gamma^\infty(x), \; v(x) \leq \tilde{v}(p) \leq \tilde{u}(p). \]

\[ \forall x \in X, \; v(x) \geq u(x) \]
Main result for 1-player games

**Theorem**

Consider a one player gambling game non-expansive and leavable. Then \((v_\lambda)\) uniformly converges to \(v\) (called the “réduite”).

\(v\) is the unique excessive function in \(C(X)\) satisfying:

\[
\forall x \in X, \exists p \in \Gamma^\infty(x), \quad v(x) \leq \tilde{v}(p) \leq \tilde{u}(p).
\]

\[
\forall x \in X, \quad v(x) \geq u(x)
\]

This is the “fundamental theorem” of gambling (Dubins-Savage).
Main result for 1-player games

Theorem
Consider a one player gambling game non-expansive and leavable. Then \((v_\lambda)\) uniformly converges to \(v\) (called the “réduite”) \(v\) is the unique excessive function in \(C(X)\) satisfying:

\[
\forall x \in X, \exists p \in \Gamma^\infty(x), \ v(x) \leq \tilde{v}(p) \leq \tilde{u}(p).
\]

\[
\forall x \in X, \ v(x) \geq u(x)
\]

This is the “fundamental theorem” of gambling (Dubins-Savage). Our theorem is a new characterization.
\( \Gamma \) is weakly acyclic if there exists \( \varphi \in C(X) \) such that:

\[
\forall x \in X, \ \text{Argmax}_{p \in \Gamma(x)} \tilde{\varphi}(p) = \{\delta_x\}
\]
• $\Gamma$ is weakly acyclic if there exists $\varphi \in C(X)$ such that:

$$\forall x \in X, \text{Argmax}_{p \in \Gamma(x)} \tilde{\varphi}(p) = \{\delta_x\}$$

• $\Gamma$ is strongly acyclic if there exists $\varphi \in C(X)$ such that:

$$\forall x \in X, \text{Argmax}_{p \in \Gamma^\infty(x)} \tilde{\varphi}(p) = \{\delta_x\}$$
• $\Gamma$ is weakly acyclic if there exists $\varphi \in C(X)$ such that:

$$\forall x \in X, \text{Argmax}_{p \in \Gamma(x)} \tilde{\varphi}(p) = \{\delta_x\}$$

• $\Gamma$ is strongly acyclic if there exists $\varphi \in C(X)$ such that:

$$\forall x \in X, \text{Argmax}_{p \in \Gamma(X)} \tilde{\varphi}(p) = \{\delta_x\}$$

Observe that our definition of acyclicity implies leavable.
Main results for 2-player games

Theorem

Assume the gambling game is non-expansive and strongly acyclic. Then $(\nu_\lambda)$ uniformly converges to the unique continuous function $\nu : X \times Y \to \mathbb{R}$ satisfying:

$\nu$ is excessive and depressive,
Main results for 2-player games

Theorem

Assume the gambling game is non-expansive and strongly acyclic. Then \((v, \lambda)\) uniformly converges to the unique continuous function \(v : X \times Y \rightarrow \mathbb{R}\) satisfying:

- \(v\) is excessive and depressive,
- \(P1: \forall (x, y) \in X \times Y, \exists p \in \Gamma(x), v(x, y) \leq \tilde{\nu}(p, y) \leq \tilde{u}(p, y)\).
Main results for 2-player games

**Theorem**

Assume the gambling game is non-expansive and strongly acyclic. Then \((v_\lambda)\) uniformly converges to the unique continuous function \(v : X \times Y \rightarrow \mathbb{R}\) satisfying:

1. \(v\) is excessive and depressive, \(P1: \forall (x, y) \in X \times Y, \exists p \in \Gamma_\infty(x), v(x, y) \leq \widetilde{v}(p, y) \leq \check{u}(p, y)\),
2. \(P2: \forall (x, y) \in X \times Y, \exists q \in \Lambda_\infty(y), v(x, y) \geq \widetilde{v}(x, q) \geq \check{u}(x, q)\).

It applies to red-black casino when \(w \leq \frac{1}{2}\) and to splitting games (with a new characterization of the so-called Mertens Zamir system).
Main results for 2-player games

Theorem

Assume the gambling game is non-expansive and strongly acyclic. Then \((v_{\lambda})\) uniformly converges to the unique continuous function \(v : X \times Y \longrightarrow \mathbb{R}\) satisfying:

- \(v\) is excessive and depressive,
- \(P1: \forall (x, y) \in X \times Y, \exists p \in \Gamma_\infty(x), v(x, y) \leq \tilde{v}(p, y) \leq \tilde{u}(p, y),\)
- \(P2: \forall (x, y) \in X \times Y, \exists q \in \Lambda_\infty(y), v(x, y) \geq \tilde{v}(x, q) \geq \tilde{u}(x, q).\)

It applies to red-black casino when \(w \leq \frac{1}{2}\) and to splitting games (with a new characterization of the so called Mertens Zamir system).
Proposition (A)

Assume the transitions are non-expansive. The family $(v_\lambda)_{\lambda \in (0,1]}$ is equicontinuous.
Steps of the Proof

**Proposition (A)**

Assume the transitions are **non-expansive**. The family \((v_{\lambda})_{\lambda \in (0,1]}\) is **equicontinuous**.

**Proposition (B)**

Assume the game is **non-expansive** and **leavable**. Let \(v\) be a limit point of \((v_{\lambda})_{\lambda \in (0,1]}\) for the uniform convergence. Then \(v\) is **balanced** and satisfies \(P1\) and \(P2\).
Proof proposition B

Let \((\lambda_n)_n\) such that \(\|v_{\lambda_n} - v\| \to_{n \to \infty} 0\).
Proof proposition B

Let \((\lambda_n)\) such that \(\|v_{\lambda_n} - v\| \to_{n \to \infty} 0\).
Assume \(v_{\lambda_n}(x, y) > u(x, y) + \lambda_n\) for \(n\) large.
Proof proposition B

Let \((\lambda_n)_n\) such that \(\|v_{\lambda_n} - v\| \to_{n \to \infty} 0\).
Assume \(v_{\lambda_n}(x, y) > u(x, y) + \lambda_n\) for \(n\) large.
Define inductively \((p^n_t)_{t=0, \ldots, T_n}\) in \(\Delta(X)\) by:

\[
\text{Define inductively } (p^n_t)_{t=0, \ldots, T_n} \text{ in } \Delta(X) \text{ by :}
\]
Proof proposition B

Let \((\lambda_n)_n\) such that \(\|v_{\lambda_n} - v\| \to_{n \to \infty} 0\).

Assume \(v_{\lambda_n}(x, y) > u(x, y) + \lambda_n\) for \(n\) large.

Define inductively \((p^n_t)_{t=0,\ldots,T_n}\) in \(\Delta(X)\) by:

- \(p^n_0 = \delta_x\),
Proof proposition B

Let \((\lambda_n)_n\) such that \(\|v_{\lambda_n} - v\| \to_{n \to \infty} 0\).

Assume \(v_{\lambda_n}(x, y) > u(x, y) + \lambda_n\) for \(n\) large.

Define inductively \((p^n_t)_{t=0, \ldots, T_n}\) in \(\Delta(X)\) by:

- \(p^n_0 = \delta_x\),
- If (*): \(\tilde{v}_{\lambda_n}(p^n_t, y) > \tilde{u}(p^n_t, y) + \lambda_n\), let \(p^n_{t+1}\) be s.t.

\[
\lambda_n \tilde{u}(p^n_t, y) + (1 - \lambda_n) \tilde{v}_{\lambda_n}(p^n_{t+1}, y) \geq \tilde{v}_{\lambda_n}(p^n_t, y).
\]
Let \((\lambda_n)_n\) such that \(\|v_{\lambda_n} - v\| \to_{n \to \infty} 0\).
Assume \(v_{\lambda_n}(x, y) > u(x, y) + \lambda_n\) for \(n\) large.
Define inductively \((p^n_t)_{t=0,\ldots,T_n}\) in \(\Delta(X)\) by:

- \(p^n_0 = \delta_x\),
- If \((*)\): \(\tilde{v}_{\lambda_n}(p^n_t, y) > \tilde{u}(p^n_t, y) + \lambda_n\), let \(p^n_{t+1}\) be s.t.

\[\lambda_n \tilde{u}(p^n_t, y) + (1 - \lambda_n) \tilde{v}_{\lambda_n}(p^n_{t+1}, y) \geq \tilde{v}_{\lambda_n}(p^n_t, y)\.

Consequently, \(\exists T_n\) such that \(\tilde{v}_{\lambda_n}(p^n_{T_n}, y) \leq \tilde{u}(p^n_{T_n}, y) + \lambda_n\).

Define \(p^n = p^n_{T_n}\) and consider a limit point \(p^* \in \Gamma^\infty(x)\). Then:

\(v(x, y) \leq \tilde{v}_{\lambda_n}(p^*, y) \leq \tilde{u}(p^*, y)\).
Proof proposition B

Let \((\lambda_n)_n\) such that \(\|v_{\lambda_n} - v\| \to_{n \to \infty} 0\).
Assume \(v_{\lambda_n}(x, y) > u(x, y) + \lambda_n\) for \(n\) large.
Define inductively \((p^n_t)_{t=0, \ldots, \tau_n}\) in \(\Delta(X)\) by:

- \(p^n_0 = \delta_x\),
- If (*) : \(\tilde{v}_{\lambda_n}(p^n_t, y) > \tilde{u}(p^n_t, y) + \lambda_n\), let \(p^n_{t+1}\) be s.t.
  \[\lambda_n \tilde{u}(p^n_t, y) + (1 - \lambda_n) \tilde{v}_{\lambda_n}(p^n_{t+1}, y) \geq \tilde{v}_{\lambda_n}(p^n_t, y)\].
- If (*) then :
  \[\tilde{v}_{\lambda_n}(p^n_{t+1}, y) \geq \tilde{v}_{\lambda_n}(p^n_t, y) + \frac{\lambda_n^2}{1 - \lambda_n} > \tilde{v}_{\lambda_n}(p^n_t, y)\].
Let \((\lambda_n)_n\) such that \(\|v_{\lambda_n} - v\| \to_{n \to \infty} 0\).
Assume \(v_{\lambda_n}(x, y) > u(x, y) + \lambda_n\) for \(n\) large.
Define inductively \((p^n_t)_{t=0, \ldots, T_n}\) in \(\Delta(X)\) by:

- \(p^n_0 = \delta_x\),
- If (*) : \(\tilde{v}_{\lambda_n}(p^n_t, y) > \tilde{u}(p^n_t, y) + \lambda_n\), let \(p^n_{t+1}\) be s.t.
  \[\lambda_n \tilde{u}(p^n_t, y) + (1 - \lambda_n) \tilde{v}_{\lambda_n}(p^n_{t+1}, y) \geq \tilde{v}_{\lambda_n}(p^n_t, y).\]
- If (*) then :
  \[\tilde{v}_{\lambda_n}(p^n_{t+1}, y) \geq \tilde{v}_{\lambda_n}(p^n_t, y) + \frac{\lambda_n^2}{1 - \lambda_n} > \tilde{v}_{\lambda_n}(p^n_t, y).\]
- Consequently, \(\exists T_n\) such that \(\tilde{v}_{\lambda_n}(p^n_{T_n}, y) \leq \tilde{u}(p^n_{T_n}, y) + \lambda_n\).
Proof proposition B

Let $(\lambda_n)_n$ such that $\|v_{\lambda_n} - v\| \to_{n\to\infty} 0$.

Assume $v_{\lambda_n}(x, y) > u(x, y) + \lambda_n$ for $n$ large.

Define inductively $(p^n_t)_{t=0,\ldots,T_n}$ in $\Delta(X)$ by:

- $p^n_0 = \delta_x$,
- If $(\ast)$ : $\tilde{v}_{\lambda_n}(p^n_t, y) > \tilde{u}(p^n_t, y) + \lambda_n$, let $p^n_{t+1}$ be s.t.
  
  \[ \lambda_n \tilde{u}(p^n_t, y) + (1 - \lambda_n) \tilde{v}_{\lambda_n}(p^n_{t+1}, y) \geq \tilde{v}_{\lambda_n}(p^n_t, y). \]

- If $(\ast)$ then :
  
  \[ \tilde{v}_{\lambda_n}(p^n_{t+1}, y) \geq \tilde{v}_{\lambda_n}(p^n_t, y) + \frac{\lambda_n^2}{1 - \lambda_n} > \tilde{v}_{\lambda_n}(p^n_t, y). \]

- Consequently, $\exists T_n$ such that $\tilde{v}_{\lambda_n}(p^n_{T_n}, y) \leq \tilde{u}(p^n_{T_n}, y) + \lambda_n$.
- Define $p^n = p^n_{T_n}$ and consider a limit point $p^* \in \Gamma_{\infty}(x)$. Then :
  
  \[ v(x, y) \leq \tilde{v}(p^*, y) \leq \tilde{u}(p^*, y). \]
Proposition (C)

Assume the gambling game is weakly acyclic and non-expansive. If \( \nu \) in \( C(X \times Y) \) is balanced, then \( \nu \) is excessive-depressive.
Steps of the Proof

Proposition (C)

Assume the gambling game is weakly acyclic and non-expansive. If \( \nu \in C(X \times Y) \) is balanced, then \( \nu \) is excessive-depressive.

False without weak acyclicity.
Proposition (C)

Assume the gambling game is weakly acyclic and non-expansive. If \( v \in C(X \times Y) \) is balanced, then \( v \) is excessive-depressive.

False without weak acyclicity.

Proposition (D)

Assume the gambling game is strongly-acyclic. There is at most one excessive-depressive function \( v \) in \( C(X \times Y) \) satisfying \( P1 \) and \( P2 \).
Proposition (C)

Assume the gambling game is weakly acyclic and non-expansive. If \( v \) in \( C(X \times Y) \) is balanced, then \( v \) is excessive-depressive.

False without weak acyclicity.

Proposition (D)

Assume the gambling game is strongly-acyclic. There is at most one excessive-depressive function \( v \) in \( C(X \times Y) \) satisfying P1 and P2.

Uniqueness is false with only weak-acyclicity.
Proof proposition D: a maximum principle

Assume \( v_1 \) and \( v_2 \) two excessive-depressive continuous functions satisfying \( P1 \) and \( P2 \) respectively. Show that \( v_1 \leq v_2 \).
Proof proposition D : a maximum principle

Assume \( v_1 \) and \( v_2 \) two excessive-depressive continuous functions satisfying \( P1 \) and \( P2 \) respectively. Show that \( v_1 \leq v_2 \).

- \( v_1 - v_2 \) being continuous on \( X \times Y \), define the compact set :

\[
Z = \text{Argmax}_{(x,y) \in X \times Y} v_1(x, y) - v_2(x, y).
\]
Proof proposition D : a maximum principle

Assume \( v_1 \) and \( v_2 \) two excessive-depressive continuous functions satisfying \( P1 \) and \( P2 \) respectively. Show that \( v_1 \leq v_2 \).

- \( v_1 - v_2 \) being continuous on \( X \times Y \), define the compact set:

\[
Z = \text{Argmax}_{(x,y) \in X \times Y} v_1(x,y) - v_2(x,y).
\]

- Let \( (x_0, y_0) \in \text{Argmin}_{(x,y) \in Z} \varphi(x) - \psi(y) \).

Rida Laraki and Jérôme Renault
Proof proposition D : a maximum principle

Assume $v_1$ and $v_2$ two excessive-depressive continuous functions satisfying $P1$ and $P2$ respectively. Show that $v_1 \leq v_2$.

- $v_1 - v_2$ being continuous on $X \times Y$, define the compact set:

$$Z = \text{Argmax}_{(x,y) \in X \times Y} v_1(x,y) - v_2(x,y).$$

- Let $(x_0, y_0) \in \text{Argmin}_{(x,y) \in Z} \varphi(x) - \psi(y)$.

- By $P1$ and excessivity in $x$, there is $p$ in $\Gamma^\infty(x_0)$ such that:

$$v_1(x_0, y_0) = \tilde{v}_1(p, y_0) \leq \tilde{u}(p, y_0).$$
Proof proposition D: a maximum principle

Assume \( v_1 \) and \( v_2 \) two excessive-depressive continuous functions satisfying \( P1 \) and \( P2 \) respectively. Show that \( v_1 \leq v_2 \).

- \( v_1 - v_2 \) being continuous on \( X \times Y \), define the compact set:
  \[
  Z = \text{Argmax}_{(x, y) \in X \times Y} v_1(x, y) - v_2(x, y).
  \]

- Let \( (x_0, y_0) \in \text{Argmin}_{(x, y) \in Z} \varphi(x) - \psi(y) \).
- By \( P1 \) and excessivity in \( x \), there is \( p \) in \( \Gamma^\infty(x_0) \) such that:
  \[
  v_1(x_0, y_0) = \tilde{v}_1(p, y_0) \leq \tilde{u}(p, y_0).
  \]

- Because \( v_2 \) is excessive in \( x \), \( v_2(p, y_0) \leq v_2(x_0, y_0) \).
Proof proposition D : a maximum principle

Assume \( v_1 \) and \( v_2 \) two excessive-depressive continuous functions satisfying \( P1 \) and \( P2 \) respectively. Show that \( v_1 \leq v_2 \).

- \( v_1 - v_2 \) being continuous on \( X \times Y \), define the compact set :
  \[
  Z = \text{Argmax}_{(x,y) \in X \times Y} v_1(x, y) - v_2(x, y).
  \]

- Let \((x_0, y_0) \in \text{Argmin}_{(x,y) \in Z} \varphi(x) - \psi(y)\).

- By \( P1 \) and excessivity in \( x \), there is \( p \) in \( \Gamma_\infty(x_0) \) such that :
  \[
  v_1(x_0, y_0) = \tilde{v}_1(p, y_0) \leq \tilde{u}(p, y_0).
  \]

- Because \( v_2 \) is excessive in \( x \), \( v_2(p, y_0) \leq v_2(x_0, y_0) \).

- Thus, \( \tilde{v}_1(p, y_0) - \tilde{v}_2(p, y_0) \geq v_1(x_0, y_0) - v_2(x_0, y_0) \).
Proof proposition D: a maximum principle

Assume \( v_1 \) and \( v_2 \) two excessive-depressive continuous functions satisfying \( P1 \) and \( P2 \) respectively. Show that \( v_1 \leq v_2 \).

- \( v_1 - v_2 \) being continuous on \( X \times Y \), define the compact set:

\[
Z = \text{Argmax}_{(x,y) \in X \times Y} v_1(x, y) - v_2(x, y).
\]

- Let \( (x_0, y_0) \in \text{Argmin}_{(x,y) \in Z} \varphi(x) - \psi(y) \).

- By \( P1 \) and excessivity in \( x \), there is \( p \) in \( \Gamma^\infty(x_0) \) such that:

\[
v_1(x_0, y_0) = \tilde{v}_1(p, y_0) \leq \tilde{u}(p, y_0)
\]

- Because \( v_2 \) is excessive in \( x \), \( v_2(p, y_0) \leq v_2(x_0, y_0) \).

- Thus, \( \tilde{v}_1(p, y_0) - \tilde{v}_2(p, y_0) \geq v_1(x_0, y_0) - v_2(x_0, y_0) \).

- Since \( (x_0, y_0) \in Z \), \( \text{Supp}(p) \times \{y_0\} \subset Z \).
Proof proposition D: a maximum principle

Assume \( v_1 \) and \( v_2 \) two excessive-depressive continuous functions satisfying \( P1 \) and \( P2 \) respectively. Show that \( v_1 \leq v_2 \).

- \( v_1 - v_2 \) being continuous on \( X \times Y \), define the compact set:
  \[
  Z = \text{Argmax}_{(x,y) \in X \times Y} v_1(x, y) - v_2(x, y).
  \]

- Let \( (x_0, y_0) \in \text{Argmin}_{(x,y) \in Z} \varphi(x) - \psi(y) \).
- By \( P1 \) and excessivity in \( x \), there is \( p \) in \( \Gamma^\infty(x_0) \) such that:
  \[
  v_1(x_0, y_0) = \tilde{v}_1(p, y_0) \leq \tilde{u}(p, y_0).
  \]

- Because \( v_2 \) is excessive in \( x \), \( v_2(p, y_0) \leq v_2(x_0, y_0) \).
- Thus, \( \tilde{v}_1(p, y_0) - \tilde{v}_2(p, y_0) \geq v_1(x_0, y_0) - v_2(x_0, y_0) \).
- Since \( (x_0, y_0) \in Z \), \( \text{Supp}(p) \times \{ y_0 \} \subset Z \).
- By definition of \( (x_0, y_0) : \tilde{\varphi}(p) - \psi(y_0) \geq \varphi(x_0) - \psi(y_0) \).
Assume \( v_1 \) and \( v_2 \) two excessive-depressive continuous functions satisfying \( P1 \) and \( P2 \) respectively. Show that \( v_1 \leq v_2 \).

- \( v_1 - v_2 \) being continuous on \( X \times Y \), define the compact set:

\[
Z = \text{Argmax}_{(x,y) \in X \times Y} v_1(x, y) - v_2(x, y).
\]

- Let \((x_0, y_0) \in \text{Argmin}_{(x,y) \in Z} \varphi(x) - \psi(y)\).
- By \( P1 \) and excessivity in \( x \), there is \( p \) in \( \Gamma^\infty(x_0) \) such that:

\[
v_1(x_0, y_0) = \tilde{v}_1(p, y_0) \leq \tilde{u}(p, y_0)
\].

- Because \( v_2 \) is excessive in \( x \), \( v_2(p, y_0) \leq v_2(x_0, y_0) \).
- Thus, \( \tilde{v}_1(p, y_0) - \tilde{v}_2(p, y_0) \geq v_1(x_0, y_0) - v_2(x_0, y_0) \).
- Since \((x_0, y_0) \in Z\), \( \text{Supp}(p) \times \{y_0\} \subset Z \).
- By definition of \((x_0, y_0)\): \( \varphi(p) - \psi(y_0) \geq \varphi(x_0) - \psi(y_0) \).
- Acyclicity implies that \( p = \delta_{x_0} \).
- Thus, \( v_1(x_0, y_0) \leq u(x_0, y_0) \). Similarly, \( v_2(x_0, y_0) \geq u(x_0, y_0) \).
Proof proposition D: a maximum principle

Assume $v_1$ and $v_2$ two excessive-depressive continuous functions satisfying $P1$ and $P2$ respectively. Show that $v_1 \leq v_2$.

- $v_1 - v_2$ being continuous on $X \times Y$, define the compact set:
  $$Z = \text{Argmax}_{(x,y) \in X \times Y} v_1(x,y) - v_2(x,y).$$

- Let $(x_0, y_0) \in \text{Argmin}_{(x,y) \in Z} \varphi(x) - \psi(y)$.
- By $P1$ and excessivity in $x$, there is $p$ in $\Gamma^\infty(x_0)$ such that:
  $$v_1(x_0, y_0) = \tilde{v}_1(p, y_0) \leq \tilde{u}(p, y_0).$$

- Because $v_2$ is excessive in $x$, $v_2(p, y_0) \leq v_2(x_0, y_0)$.
- Thus, $\tilde{v}_1(p, y_0) - \tilde{v}_2(p, y_0) \geq v_1(x_0, y_0) - v_2(x_0, y_0)$.
- Since $(x_0, y_0) \in Z$, $\text{Supp}(p) \times \{y_0\} \subset Z$.
- By definition of $(x_0, y_0)$: $\varphi(p) - \psi(y_0) \geq \varphi(x_0) - \psi(y_0)$.
- Acyclicity implies that $p = \delta_{x_0}$.
- Thus, $v_1(x_0, y_0) \leq u(x_0, y_0)$. Similarly, $v_2(x_0, y_0) \geq u(x_0, y_0)$.
- Consequently, $v_1(x_0, y_0) - v_2(x_0, y_0) \leq 0$. 

Rida Laraki and Jérôme Renault
Theorem

*With only weak acyclicity in the main theorem and everything the same: the limit of* \( v_\lambda \) *may not exist.*
Theorem

*With only weak acyclicity in the main theorem and everything the same: the limit of $v_\lambda$ may not exist.*

For a counter example, we adapt a counter example from Ziliotto.
A weakly acyclic game without convergence

We have $X = \{a, b, c\}$, $c$ is absorbing, $\alpha$ and $\beta$ belong to a compact set $I \subset [0, 1/4]$ such that 0 is in the closure of $I \setminus \{0\}$.

\[ 1 - \alpha - \alpha^2 \]
\[ \beta \]
\[ \alpha \]
\[ \alpha^2 \]
\[ 1 - \beta - \beta^2 \]
\[ \beta^2 \]

The gambling house of player 2 is a copy, with $Y = \{a', b', c'\}$ and a compact set of choices $I' \subset [0, 1/4]$ but $I$ and $I'$ may be different.
The utility function $u$ is supposed to be:

\[
\begin{array}{ccc}
  a' & b' & c' \\
  a & 0 & 1 & 1 \\
b & 1 & 0 & 1 \\
c & 1 & 1 & 0 \\
\end{array}
\]

Player 2 wants to be at the same location as player 1, and player 1 the opposite.
Uniqueness in proposition D fails

The gambling game is weakly-acyclic

thus excessive-depressive = balanced.
Uniqueness in proposition D fails

The gambling game is weakly-acyclic

thus excessive-depressive = balanced.

However we can show that:

A function $v : X \times Y \rightarrow \mathbb{R}$ is excessive-depressive and satisfies $P1$ and $P2$ if and only if there exists a number $x \in [0, 1]$ s. t.:

$$v = \begin{array}{ccc}
    a' & b' & c' \\
    a & x & x & 1 \\
    b & x & x & 1 \\
    c & 0 & 0 & 0 \\
\end{array}$$

There are infinitely many such functions.

Consequently: $P1$ and $P2 +$ excessive-depressive are not sufficient to characterize a unique limiting value (if such a limit exists!!!).

Rida Laraki and Jérôme Renault
Uniqueness in proposition D fails

The gambling game is weakly-acyclic

thus excessive-depressive $= \text{balanced}$.

However we can show that:

A function $v : X \times Y \rightarrow \mathbb{R}$ is excessive-depressive and satisfies $P1$ and $P2$ if and only if there exists a number $x \in [0, 1]$ s. t.:

$$v = \begin{array}{ccc}
    a' & b' & c' \\
    a & x & x & 1 \\
    b & x & x & 1 \\
    c & 0 & 0 & 0 \\
\end{array}$$

There are infinitely many such functions.

Consequently: $P1$ and $P2 +$ excessive-depressive are not sufficient to characterize a unique limiting value (if such a limit exists!!!).
Uniqueness in proposition D fails

The gambling game is weakly-acyclic

thus excessive-depressive = balanced.

However we can show that:

A function $v : X \times Y \rightarrow \mathbb{R}$ is excessive-depressive and satisfies $P1$ and $P2$ if and only if there exists a number $x \in [0, 1]$ s. t.:

$$v = \begin{array}{ccc}
a' & b' & c' \\
a & x & x & 1 \\
b & x & x & 1 \\
c & 0 & 0 & 0
d\end{array}$$

There are infinitely many such functions.

Consequently: $P1$ and $P2 +$ excessive-depressive are not sufficient to characterize a unique limiting value (if such a limit exists!!!).
Existence of the limit?

Proposition

Assume \( I' = [0, 1/4] \)

A) If \( I = [0, 1/4] \), the limit value exists with \( x = \frac{1}{2} \).
Proposition

Assume $I' = [0, 1/4]$

A) If $I = [0, 1/4]$, the limit value exists with $x = \frac{1}{2}$.

B) If $I = \{0, \frac{1}{4}\}$, the limit value exists with $x = 0$. 

Rida Laraki and Jérôme Renault
Existence of the limit?

**Proposition**

Assume $I' = [0, 1/4]$

A) If $I = [0, 1/4]$, the limit value exists with $x = \frac{1}{2}$.

B) If $I = \{0, \frac{1}{4}\}$, the limit value exists with $x = 0$.

C) If $I = \left\{\frac{1}{2^n}, n \in \mathbb{N}^*\right\} \cup \{0\}$, $(v_\lambda)$ may diverge.
Convergence in A) and B) is implied by Bolte-Gaubert-Vigeral 2015.
• Convergence in A) and B) is implied by Bolte-Gaubert-Vigeral 2015.

• They show that any stochastic game with finitely many states, separable-definable transitions and definable compact action sets has a uniform value.
Convergence in A) and B) is implied by Bolte-Gaubert-Vigeral 2015.

They show that any stochastic game with finitely many states, separable-definable transitions and definable compact action sets has a uniform value.

Example in C is weakly acyclic, has finitely many states, separable-definable transitions but the compact action set of player 1 is not definable.
Remarks

- Convergence in A) and B) is implied by Bolte-Gaubert-Vigeral 2015.
- They show that any stochastic game with finitely many states, separable-definable transitions and definable compact action sets has a uniform value.
- Example in C is weakly acyclic, has finitely many states, separable-definable transitions but the compact action set of player 1 is not definable.
- Example C) may be related to a counter-example of Ziliotto 2013.
The recent counter examples rise the question: under which conditions does $\nu_\lambda$ converges for compact state spaces?
The recent counter examples rise the question: under which conditions does $\nu_\lambda$ converge for compact state spaces?

It was in the “air” that a sufficient (and perhaps necessary) condition is “irreversibility”.

Moreover we offer a characterization of the asymptotic value that extends the Mertens-Zamir system of equations.

We can prove existence of the uniform value under a bounded variation assumption (splitting games by Oliu-Barton).

Under progress:
- Non existence of the uniform value without bounded variation;
- Convergence and characterization without leavablity;
- Convergence when only one house is strongly acyclic;
- Relaxing separability (the dependent case).

Rida Laraki and Jérôme Renault
The recent counter examples rise the question: under which conditions does \( \nu_\lambda \) converge for compact state spaces?

It was in the “air” that a sufficient (and perhaps necessary) condition is “irreversibility”.

Our paper made this statement precise in the class of product stochastic games: if the game is strongly acyclique \( \nu_\lambda \) converges and if it is weakly acyclique, it may diverge.
Conclusion, Extensions

- The recent counter examples rise the question: under which conditions does $\nu_\lambda$ converges for compact state spaces?
- It was in the “air” that a sufficient (and perhaps necessary) condition is “irreversibility”.
- Our paper made this statement precise in the class of product stochastic games: if the game is strongly acyclique $\nu_\lambda$ converges and if it is weakly acyclique, it may diverge.
- Moreover we offer a characterization of the asymptotic value that extends the Mertens-Zamir system of equations.
The recent counter examples rise the question: under which conditions does $\nu_{\lambda}$ converge for compact state spaces?

It was in the “air” that a sufficient (and perhaps necessary) condition is “irreversibility”.

Our paper made this statement precise in the class of product stochastic games: if the game is strongly acyclic, $\nu_{\lambda}$ converges and if it is weakly acyclic, it may diverge.

Moreover we offer a characterization of the asymptotic value that extends the Mertens-Zamir system of equations.

We can prove existence of the uniform value under a bounded variation assumption (splitting games by Oliu-Barton).
The recent counter examples rise the question: under which conditions does $\nu_\lambda$ converges for compact state spaces?

It was in the “air” that a sufficient (and perhaps necessary) condition is “irreversibility”.

Our paper made this statement precise in the class of product stochastic games: if the game is strongly acyclique $\nu_\lambda$ converges and if it is weakly acyclique, it may diverge.

Moreover we offer a characterization of the asymptotic value that extends the Mertens-Zamir system of equations.

We can prove existence of the uniform value under a bounded variation assumption (splitting games by Oliu-Barton).

Under progress:
The recent counter examples raise the question: under which conditions does $\nu_\lambda$ converge for compact state spaces?

It was in the “air” that a sufficient (and perhaps necessary) condition is “irreversibility”.

Our paper made this statement precise in the class of product stochastic games: if the game is strongly acyclique $\nu_\lambda$ converges and if it is weakly acyclique, it may diverge.

Moreover we offer a characterization of the asymptotic value that extends the Mertens-Zamir system of equations.

We can prove existence of the uniform value under a bounded variation assumption (splitting games by Oliu-Barton).

Under progress:
- Non existence of the uniform value without bounded variation;
The recent counter examples rise the question: under which conditions does \( \nu_\lambda \) converges for compact state spaces?

It was in the “air” that a sufficient (and perhaps necessary) condition is “irreversibility”.

Our paper made this statement precise in the class of product stochastic games: if the game is strongly acyclique \( \nu_\lambda \) converges and if it is weakly acyclique, it may diverge.

Moreover we offer a characterization of the asymptotic value that extends the Mertens-Zamir system of equations.

We can prove existence of the uniform value under a bounded variation assumption (splitting games by Oliu-Barton).

Under progress:
- Non existence of the uniform value without bounded variation;
- Convergence and characterization without leavablity;
The recent counter examples rise the question: under which conditions does $v_\lambda$ converges for compact state spaces?.

It was in the “air” that a sufficient (and perhaps necessary) condition is “irreversibility”.

Our paper made this statement precise in the class of product stochastic games: if the game is strongly acyclic $v_\lambda$ converges and if it is weakly acyclic, it may diverge.

Moreover we offer a characterization of the asymptotic value that extends the Mertens-Zamir system of equations.

We can prove existence of the uniform value under a bounded variation assumption (splitting games by Oliu-Barton).

Under progress:
- Non existence of the uniform value without bounded variation;
- Convergence and characterization without leavablity;
- Convergence when only one house is strongly acyclic;
The recent counter examples rise the question: under which conditions does $v_\lambda$ converges for compact state spaces?

It was in the “air” that a sufficient (and perhaps necessary) condition is “irreversibility”.

Our paper made this statement precise in the class of product stochastic games: if the game is strongly acylic $v_\lambda$ converges and if it is weakly acyclic, it may diverge.

Moreover we offer a characterization of the asymptotic value that extends the Mertens-Zamir system of equations.

We can prove existence of the uniform value under a bounded variation assumption (splitting games by Oliu-Barton).

Under progress:

- Non existence of the uniform value without bounded variation;
- Convergence and characterization without leavablity;
- Convergence when only one house is strongly acyclic;
- Relaxing separability (the dependent case).