Weierstrass Institute for
Applied Analysis and Stochastics

Aspects of Nonsmoothness for Gaussian Probability Functions

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Probability functions

We consider probability functions of the type

$$\varphi(x) := \mathbb{P}(g(x,\xi) \leq 0),$$

where

- $x \in X$ is a decision variable in a separable and reflexive Banach space X
- ξ is an m-dimensional random vector defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$
- $g : X \times \mathbb{R}^m \to \mathbb{R}$ is a mapping defining the random inequality constraint $g(x,\xi) \leq 0$

Our basic assumptions:

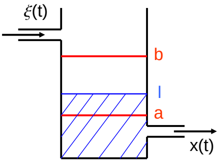
- g locally Lipschitzian
- $g(x,\cdot)$ convex for all $x \in X$
- ξ is a Gaussian random vector

Probability functions occur in many optimization problems from engineering, e.g.

$$\max\{\varphi(x) \mid x \in X\} \quad \text{reliability maximization}$$
$$\min\{f(x) \mid \varphi(x) \geq p\} \quad \text{probabilistic constraints}$$

Reservoir control problem

Consider a reservoir with random inflow ξ and controlled release x:



Assume a finitely parameterized inflow process

$\xi(t) = \langle \xi, a(t) \rangle, \quad \xi \sim \mathcal{N}(\mu, \Sigma)$ (e.g., K-L expansion)

Water level at time t:

$$l(\xi, x, t) = l_0 + \int_0^t \langle \xi, a(\tau) \rangle d\tau - \int_0^t x(\tau) d\tau$$

Probability of satisfying a critical lower level profile l_* given a release profile x:

$$\varphi(x) := \mathbb{P}(l(\xi, x, t) \geq l_*(t) \quad \forall t \in [0, T]) = \mathbb{P}\left(\underbrace{\max_{t \in [0,T]} \{l_*(t) - l(\xi, x, t)\}}_{g(x,\xi)} \leq 0 \right)$$

g locally Lipschitz and convex in $\xi \implies$ basic assumptions satisfied.

Slater point assumption

Let $\bar{x} \in X$ be a point of interest for our probability function $\varphi(x) := \mathbb{P}(g(x, \xi) \leq 0)$.

In addition to our basic assumptions

g locally Lipschitz, $g(x, \cdot)$ convex, $\xi \sim \mathcal{N}(\mu, \Sigma)$

suppose that: $g(\bar{x}, \mu) < 0$ (mean is a Slater point).

Slater point assumption

- is satisfied whenever $\varphi(\bar{x}) \geq 0.5$ $\Longrightarrow$ no restriction of generality
- implies continuity of φ at $\bar{x}$.

Question: Does the Slater point assumption for the mean along with $g \in \mathcal{C}^1$ imply that $\varphi \in \mathcal{C}^1$?

Answer: **No** in general, **Yes** for g linear in ξ.

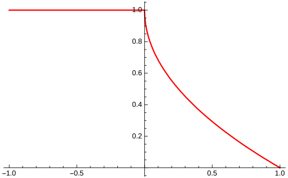
Let $\xi \sim \mathcal{N}(\mu, \Sigma)$ and

$$g(x,z) := \langle a(x), z\rangle - b(x), \quad a \in \mathcal{C}^1(X, \mathbb{R}^m),\ b \in \mathcal{C}^1(X, \mathbb{R}),\ X \text{ - Banach space}$$

Slater point assumption at point of interest: $\langle a(\bar{x}), \mu\rangle < b(\bar{x})$. Then, with Φ = CDF of $\mathcal{N}(0,1)$:

$$\varphi(\bar{x}) = \Phi\left(\frac{b(\bar{x}) - \langle a(\bar{x}), \mu\rangle}{\langle a(\bar{x}), \Sigma a(\bar{x})\rangle}\right) \in \mathcal{C}^1$$

Let $g(x, z_1, z_2) := x^2 \cdot 1_{[0,\infty)}(x) \cdot \exp(-1 - 4\log(1 - \Phi(z_1))) + z_2 - 1 \in \mathcal{C}^1$.



Then, g is convex in (z_1, z_2) for every $x \in \mathbb{R}$.

Let $\xi = (\xi_1, \xi_2) \sim \mathcal{N}(0, I_2)$. Then, $g(\bar{x} := 0, \mu = 0) < 0$

(Slater point assumption) and

$$\varphi(x) = \begin{cases} \frac{1}{\sqrt{2\pi}} \int e^{-s^2/2}\Phi(1)ds & x \le 0 \\ \frac{1}{\sqrt{2\pi}} \int e^{-s^2/2}\Phi(1 - x^2\exp(-1 - 4\log(1 - \Phi(s))))ds & x > 0 \end{cases}$$

φ is continuous (by Slater point assumption) but **not even locally Lipschitz**.

The cone of nice directions

Definition

According to our basic assumptions, let $g : X \times \mathbb{R}^m \to \mathbb{R}$ be locally Lipschitz.
For $L > 0$, we define the L- cone of nice directions at $\bar{x} \in \mathbb{R}^n$, as

$$C_L := \{h \in X \mid d^C g(\cdot, z)(x; h) \leq L\|z\|^{-m} \exp(\|z\|^2/2)\|h\| \quad \forall x \in \mathbb{B}_{1/L}(\bar{x}) \; \forall z : \|z\| \geq L\}$$

Here (Clarke's directional derivative of partial function),

$$d^C g(\cdot, z)(x; h) := \limsup_{y \to x,\, t \downarrow 0} \frac{g(y + th, z) - g(y, z)}{t}$$

Definition

Let $K \subseteq X^*$ be a closed cone. We say that K has w^*-compact sole if there exists $x_0 \in X$ such that

- $\langle x^*, x^0 \rangle > 0 \quad \forall x^* \in K \setminus \{0\}$
- the set $\{x^* \in K \mid \langle x^*, x^0 \rangle = 1\}$ is w^*- compact.

If X is finite-dimensional, then for any closed cone $K \subseteq X$ one has

$$\operatorname{int} K \neq \emptyset \iff K^* \text{ has } w^*\text{- compact sole}$$

Theorem

Assume that $g : X \times \mathbb{R}^m \to \mathbb{R}$ is locally Lipschitz and convex in the second argument. Moreover, let $\xi \sim \mathcal{N}(\mu, \Sigma)$ and fix a point $\bar{x}$ satisfying $g(\bar{x}, \mu) < 0$. Finally, suppose that for some $L > 0$ the dual C_L^* of the L-cone of nice directions at $\bar{x}$ has a w^*-compact sole. Then,

$$\partial\varphi(\bar{x}) \subseteq \bigcap_{\substack{F \subseteq X \\ \dim F < \infty}} \left\{ \int_{v \in \mathbb{S}^{m-1}} \partial_x e(\bar{x}, v) d\mu_\zeta(v) - C_L^* + F^\perp \right\}$$

Here, ∂ refer to the Mordukhovich subdifferential, μ_ζ is the uniform distribution on $\mathbb{S}^{m-1}$ and

$$e(x, v) := \mu_\chi\{r \ge 0 \mid g(x, \mu + rLv) \le 0\}, \quad (x, v) \in X \times \mathbb{S}^{m-1}; \quad (LL^T = \Sigma),$$

where μ_χ is the χ-distribution with m degrees of freedom.

Example

In the non-differentiable example before, we have (for $L > 0$ large enough) that

$$\partial\varphi(\bar{x}) = \{0\}, \; C_L = (-\infty, 0], \; \partial_x e(\bar{x}, v) = \{0\} \text{ for } \mu_\zeta - a.e. \; v,$$

whence the inclusion in the Theorem reads here as: $\{0\} \subseteq (-\infty, 0]$.

Theorem

Assume that $g : X \times \mathbb{R}^m \to \mathbb{R}$ is locally Lipschitz and convex in the second argument. Moreover, let $\xi \sim \mathcal{N}(\mu, \Sigma)$ and fix a point $\bar{x}$ satisfying $g(\bar{x}, \mu) < 0$. Finally, suppose that $C_L = \mathbb{R}^n$ for some $L > 0$ or that the set $\{z \mid g(\bar{x}, z) \leq 0\}$ is bounded. Then, φ is locally Lipschitzian around $\bar{x}$ and

$$\partial^C \varphi(\bar{x}) \subseteq \int_{v \in \mathbb{S}^{m-1}} \partial_x^C e(\bar{x}, v) d\mu_\zeta(v); \quad (\partial^C = \textit{Clarke subdifferential}).$$

For locally Lipschitzian functions f one always has that $\emptyset \neq \partial f(\bar{x})$ and

$$\#\partial f(\bar{x}) = 1 \iff f \text{ strictly differentiable at } \bar{x}$$

Corollary

In addition to the assumptions above, assume that $\#\partial_x^C e(\bar{x}, v) = 1$ for μ_ζ-a.e. v. Then, φ is strictly differentiable at $\bar{x}$ and

$$\nabla \varphi(\bar{x}) = \int_{v \in \mathbb{S}^{m-1}} \nabla_x e(\bar{x}, v) d\mu_\zeta(v)$$

Partial (Clarke-) subdifferential of $e(x,v)$

Theorem (v. Ackooij / H. 2015)

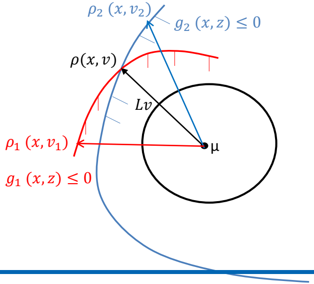
For $g(x,z) := \max_{i=1,\dots,p} g_i(x,z)$ *and* $\xi \sim \mathcal{N}(\mu,\Sigma)$ *suppose that*

- $g_i \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^m)$ *and convex in the second argument*
- $C = \mathbb{R}^n$ *(all directions nice);* $g_i(\bar{x},\mu) < 0$ *for* $i = 1,\dots,n$ *(Slater point)*

Then, $\partial_x^C e(\bar{x},v) = \text{Co}\left\{ -\frac{\chi(\rho(\bar{x},v))}{\langle \nabla_z g_i(\bar{x},\rho(\bar{x},v)Lv), Lv\rangle} \nabla_x g_i(\bar{x},\rho(\bar{x},v)Lv) : i \in I(v) \right\}$

for $e(x,v) := \mu_\chi\{r \geq 0 \mid g(x,\mu + rLv) \leq 0\}, \quad (LL^T = \Sigma) \quad ((x,v) \in \mathbb{R}^n \times \mathbb{S}^{m-1})$.

Here, $I(v) := \{i \mid \rho(\bar{x},v) = \rho_i(\bar{x},v)\}$ *and* χ *is the density of the Chi-distribution with* m *d.f.*



If $\mu_\zeta(\{v \in \mathbb{S}^{n-1} \mid \#I(v) \geq 2\}) = 0$ then

φ is strictly differentiable at $\bar{x}$.

Corollary

In addition to the assumptions of the previous theorem assume the following constraint qualification:

$$\operatorname{rank}\{\nabla_z g_i(\bar{x},z),\nabla_z g_j(\bar{x},z)\}=2\quad\forall i\neq j\in\mathcal{I}(z)\ \forall z:g(\bar{x},z)\leq 0,$$

where, $\mathcal{I}(z):=\{i\mid g_i(\bar{x},z)=0\}$.

Then, φ *is strictly differentiable at* $\bar{x}$*. If this condition holds locally around* $\bar{x}$*, then* φ *is continuously differentiable. Moreover the gradient formula*

$$\nabla\varphi\left(\bar{x}\right)=-\int\limits_{v\in\mathbb{S}^{m-1}}\frac{\chi\left(\rho\left(\bar{x},v\right)\right)}{\left\langle\nabla_z g_{i^*(v)}\left(\bar{x},\rho\left(\bar{x},v\right)Lv\right),Lv\right\rangle}\nabla_x g_{i^*(v)}\left(\bar{x},\rho\left(\bar{x},v\right)Lv\right)d\mu_\zeta(v)$$

holds true. Here, $i^*(v):=\{i|\rho(\bar{x},v)=\rho_i(\bar{x},v)\}$.

For a single smooth inequality ($p=1$ above) the constraint qualification is trivially fulfilled. So $C=\mathbb{R}^m$ (all directions nice) implies continuous differentiability of φ. This condition takes the explicit form:

$$\forall h\in\mathbb{R}^n\exists L>0:$$
$$\langle\nabla_x g(x,z),h\rangle\leq L\|z\|^{-m}\exp(\|z\|^2/2)\|h\|\quad\forall x\in\mathbb{B}_{1/L}(\bar{x})\ \forall z:\|z\|\geq L$$