Interdiction Games and Monotonicity

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Bilevel Optimization

General bilevel optimization problem

$$\min_{x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}} F(x, y) \tag{1}$$

$$G(x, y) \leq 0 \tag{2}$$

$$y \in \arg \min_{y' \in \mathbb{R}^{n_2}} \{ f(x, y') : g(x, y') \leq 0 \} \tag{3}$$

- **optimistic** vs pessimistic
- Stackelberg game: two-person sequential game

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Leader

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Leader

Follower

- **optimistic** vs pessimistic
- Stackelberg game: two-person sequential game

Interdiction Games (IGs)

- special case of bilevel optimization problems
- leader and follower have **opposite objective functions**
- leader **interdicts** items of follower
  - type of interdiction: linear or **discrete**, cost increase or **destruction**
  - interdiction **budget**
- two-person, zero-sum sequential game
- studied mostly for network-based problems in the follower

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(a) Linear, cost increase

(b) Discrete, destruction

Figure : Early Applications of Interdiction, following [Livy, 218BC]

Interdiction Games (IGs)

We focus on:

$$\min_{x \in X} \max_{y \in \mathbb{R}^{n_2}} \; d^T y \tag{4}$$

$$Q\, y \le Q_0 \tag{5}$$

$$0 \le y_j \le u_j(1 - x_j), \qquad \forall j \in N \tag{6}$$

$$y_j \text{ integer}, \qquad \forall j \in J_y \tag{7}$$

- $X = \{x \in \mathbb{R}^{n_1} : Ax \le b,\ x_j \text{ integer } \forall j \in J_x,\ x_j \text{ binary } \forall j \in N\}$ (feasible interdiction policies).
- n_1 and n_2 are the number of leader (x) and follower (y) variables, resp.
- d, Q, Q_0, u, A, b are given rational matrices/vectors of appropriate size.
- u: finite upper bounds on the follower variables y_j that can be interdicted.
- $y_N = (y_j)_{j \in N}$ variables that can be interdicted, and $y_R = (y_j)_{j \in R}$ the remaining follower variables. Associated $Q = (Q_N, Q_R)$ and $d^T = (d_N^T, d_R^T)$.

Example: Knapsack Interdiction Problem

item set $N = \{1, \dots, n\}$

$$\min_x \max_y \; p^T y \tag{8}$$

$$v^T x \leq C_l \tag{9}$$

$$w^T y \leq C_f \tag{10}$$

$$x_j + y_j \leq 1, \qquad \forall j \in N \tag{11}$$

$$x \in \{0,1\}^n \tag{12}$$

$$y \in \{0,1\}^n \tag{13}$$

- DeNegre [2011], Caprara et al. [2016]. Σ_p^2-complete (Caprara et al. [2014]).
- Corporate Strategy Problem (DeNegre [2011])
  - Companies A and B. A dominates the market. B wants to enter the market
  - Whenever A and B target the same region, campaign of B is not effective
  - Items are, e.g., demographic/geographic regions
  - Cost and benefit for each target region
  - A wants to prevent B in maximizing its gain (subject to available budget)

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INTERDICTION GAMES WITH MONOTONICITY PROPERTY

Interdiction Problems with Monotonicity Property

Downward Monotonicity

We assume $Q_N \geq 0$, hence the feasible follower solutions satisfy:

"if $\hat{y} = (\hat{y}_N, \hat{y}_R)$ is a feasible follower for a given x and $y' = (y'_N, \hat{y}_R)$ satisfies integrality constraints and $0 \leq y'_N \leq \hat{y}_N$, then y' is **also feasible** for x".

In particular, if all y are binary and $R = \emptyset$:

Independent Systems

$\mathcal{S} := \{S \subseteq N : Q\,\chi_S \leq Q_0\} \subseteq 2^N$ forms an **independent system**, where χ_S denotes the 0-1 incidence vector of S.

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Even with Monotonicity the Problems Remain Hard...

Complexity

- Even when the follower is a pure LP, the problem remains NP-hard (Zenklusen [2010], Dinitz and Gupta [2013]).
- In general, already knapsack interdiction is Σ_2-hard (Caprara et al. [2013]).

Examples

Interdicting:

- set packing problem
- (multidimensional) knapsack problem
- prize-collecting Steiner tree
- orienteering problem

PROBLEM REFORMULATION

Problem Reformulation

For a given $x \in X$ we define the value function:

$$\Phi(x) = \max_{y \in \mathbb{R}^{n_2}} d^T y \tag{14}$$
$$Q\, y \leq Q_0 \tag{15}$$
$$0 \leq y_j \leq u_j(1 - x_j), \qquad \forall j \in N \tag{16}$$
$$y_j \text{ integer}, \qquad \forall j \in J_y \tag{17}$$

so that problem can be restated in the $\mathbb{R}^{n_1+1}$ space as

$$\min_{x \in \mathbb{R}^{n_1}, w \in \mathbb{R}} w \tag{18}$$
$$w \geq \Phi(x) \tag{19}$$
$$Ax \leq b \tag{20}$$
$$x_j \text{ integer}, \qquad \forall j \in J_x \tag{21}$$
$$x_j \in \{0, 1\}, \qquad \forall j \in N. \tag{22}$$

Try to replace the constraints (19) by linear constraints.

Benders-Like Reformulation

Find (sufficiently large) M_j's and reformulate the follower [Wood, 2010]

$$\Phi(x) = \max\{d^T y - \sum_{j \in N} M_j x_j y_j : y \in Y\}, \tag{23}$$

where

$$Y = \{y \in \mathbb{R}^{n_2} : Q\, y \le Q_0, \quad 0 \le y_j \le u_j \ \forall j \in N, \quad y_j \text{ integer } \forall j \in J_y\}.$$

Let $\hat{Y}$ be extreme points of $\operatorname{conv} Y$.

Benders-Like Reformulation

$$\min_{x \in \mathbb{R}^{n_1}, w \in \mathbb{R}} w \tag{24}$$

$$w \ge d^T \hat{y} - \sum_{j \in N} M_j x_j \hat{y}_j \qquad \forall \hat{y} \in \hat{Y} \tag{25}$$

$$Ax \le b \tag{26}$$

$$x_j \text{ integer}, \qquad \forall j \in J_x \tag{27}$$

$$x_j \text{ binary}, \qquad \forall j \in N. \tag{28}$$

The Choice of M_j's is Crucial

Theorem

For Interdiction Games with Monotonicity $M_j = d_j$*, i.e., we have:*

$$
\begin{aligned}
\min_{x \in \mathbb{R}^{n_1}, w \in \mathbb{R}} \; & w \\
& w \geq \sum_{j \in R} d_j \hat{y}_j + \sum_{j \in N} d_j \hat{y}_j (1 - x_j) && \forall \hat{y} \in \hat{Y} \\
& Ax \leq b \\
& x_j \text{ integer,} && \forall j \in J_x \\
& x_j \text{ binary,} && \forall j \in N.
\end{aligned}
$$

The formulation can be solved by cutting planes (DeNegre [2011] for knapsack interdiction) or branch-and-cut (our approach).

NEW INTERDICTION CUTS

New Interdiction Cuts

Assumption 2

All follower variables y_N are binary, i.e., $N \subseteq J_y$ and $u = 1$.

Theorem

Take any $\hat{y} \in \hat{Y}$. Let $a, b \in N$ with $\hat{y}_a = 1$, $\hat{y}_b = 0$, $d_a < d_b$ and $Q_a \geq Q_b$. Then the following **lifted interdiction cut** is valid:

$$w \geq \sum_{j \in R} d_j \hat{y}_j + \sum_{j \in N} d_j \hat{y}_j (1 - x_j) + (d_b - d_a)(1 - x_b).$$

It can be generalized to two distinct collections of items $S_a = \{a_1, \ldots, a_K\} \subset N$ and $S_b = \{b_1, \ldots, b_K\} \subset N$.

$$w \geq \sum_{j \in R} d_j \hat{y}_j + \sum_{j \in N} d_j \hat{y}_j (1 - x_j) + \sum_{k=1}^{K} (d_{b_k} - d_{a_k})(1 - x_{b_k}).$$

New Interdiction Cuts, cont.

Theorem

Take any $\hat{y} \in \hat{Y}$. Let $a, b \in N$ with $\hat{y}_a = 1$, $\hat{y}_b = 0$ and $Q_a \geq Q_b$. Then the following **modified interdiction cut** is valid:

$$w \geq \sum_{j \in R} d_j \hat{y}_j + \sum_{j \in N} d_j \hat{y}_j (1 - x_j) + d_b(x_a - x_b). \tag{29}$$

The proof is based on disjunction arguments that for any $x \in X$, either $x_a - x_b \leq 0$ or ≥ 1.

It can be generalized to two distinct collections of items $S_a = \{a_1, \ldots, a_K\} \subset N$ and $S_b = \{b_1, \ldots, b_K\} \subset N$.

$$w \geq \sum_{j \in R} d_j \hat{y}_j + \sum_{j \in N} d_j \hat{y}_j (1 - x_j) + \sum_{k=1}^{K} d_{b_k}(x_{a_k} - x_{b_k}).$$

New Interdiction Cuts, cont.

Theorem

Let $i, s \in N$ be two distinct items such that $A_i \leq A_s$, $Q_i \leq Q_s$, and $d_i \geq d_s$. Then the following **dominance inequality**

$$x_s \leq x_i$$

is satisfied by at least one optimal solution.

Remark

The three families of cuts are pairwise incomparable.

A HEURISTIC FOR INTERDICTION GAMES

Two Step Heuristic

Step 1

Assume, all **profitable items** $N^+ = \{j : d_j > 0\}$ that are not interdicted are taken by the follower, i.e.

$$y_j^*(x) = \begin{cases} u_j(1 - x_j), & \text{if } d_j > 0, \\ 0, & \text{otherwise.} \end{cases}$$

MILP Reformulation

$$(HEU\_REF) \qquad \min \sum_{j \in N^+} d_j u_j (1 - x_j)$$

$$Ax \leq b$$

$$\sum_{j \in N^+} Q_j u_j (1 - x_j) \leq Q_0$$

$$x_j \text{ integer}, \qquad \forall j \in J_x$$

$$x_j \in \{0, 1\}, \qquad \forall j \in N$$

Two Step Heuristic

Step 2

- (HEU_REF) can be infeasible.
- If a solution for (HEU_REF) exists, it provides a **valid upper bound UB** (feasible solution).
- Impose hence a **cutoff constraint** to the Benders-reformulation

$$w \leq UB - 1$$

- If the resulting Benders-reformulation is infeasible $\Rightarrow$ proof that **UB** is optimal!

Enhancement

In the Benders-reformulation one can impose a disjunction: at least one of the constraints in $\sum_{j \in N^+} Q_j u_j (1 - x_j) \leq Q_0$ must be violated.
Single linear constraint (possibly weak):

$$\sum_{j \in N^+} \max_i \{q_{ij}\} \, u_j (1 - x_j) \geq \min_i \{q_{i0} + 1\}$$

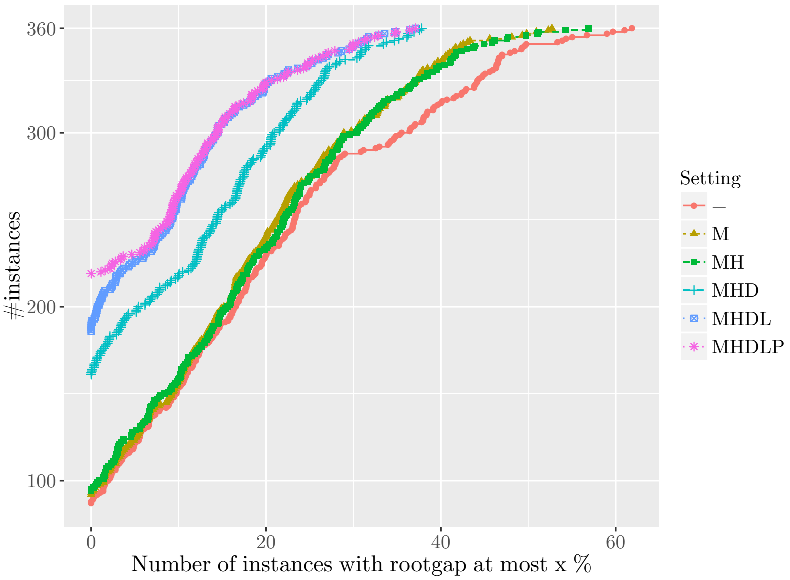
COMPUTATIONAL RESULTS

Computational Results

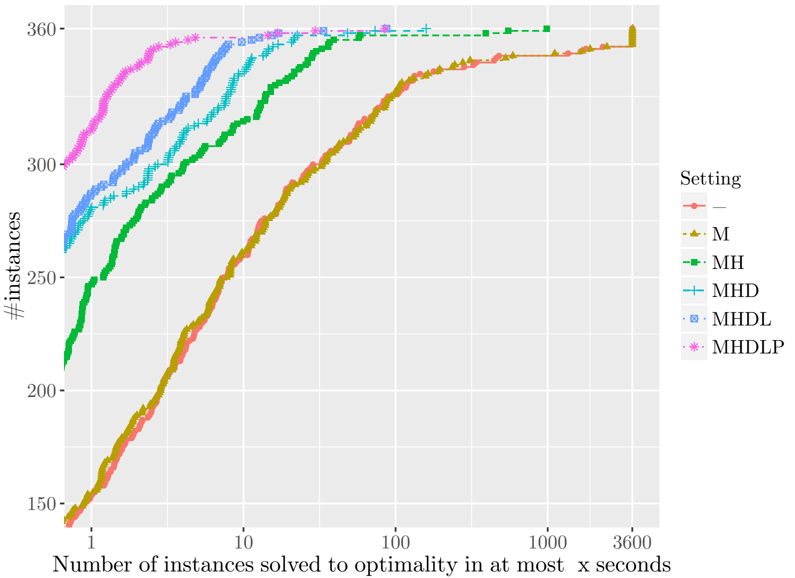
- Python, CPLEX 12.6.3, Intel Xeon E3-1220V2 3.1 GHz, single thread
- Two classes of instances:
  - Knapsack Interdiction Problem (KIP) from Caprara et al. [2016], DeNegre [2011], Tang et al. [2015]
  - Multiple Knapsack Interdiction Problem (MKIP) created from the MKP library SAC-94.
- Settings tested in the following

  - `-` : only basic interdiction cuts separated (exactly);
  - `M` : plus heuristic separation for modified interdiction cuts;
  - `MH` : plus heuristic separation procedure for basic interdiction cuts;
  - `MHD` : plus all dominance inequalities statically added;
  - `MHDL` : plus with the lifted interdiction cut;
  - `MHDLP` : plus preprocessing (heuristic) .

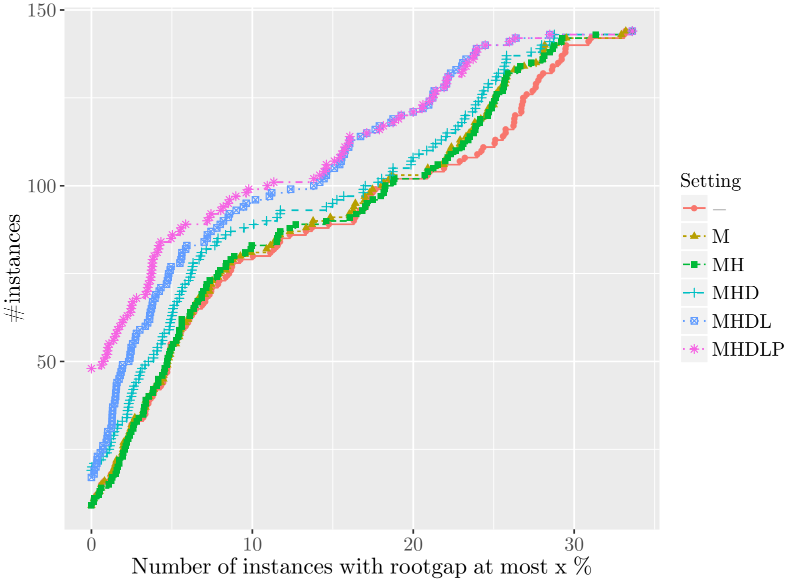
Root-Gap for KIP - Performance Profile



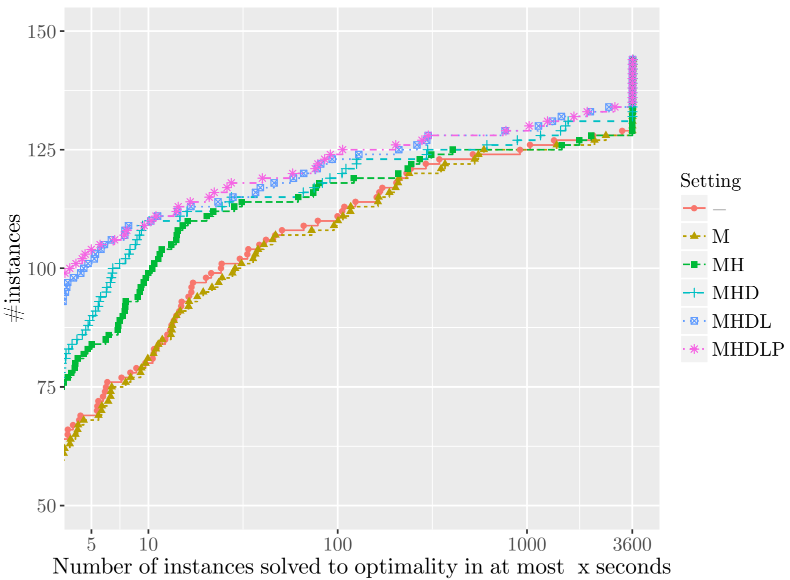
Runtime to Optimality for KIP - Performance Profile



Root-Gap for MKIP - Performance Profile



Runtime to Optimality for MKIP - Performance Profile



Conclusion and Further Work

- What we did ...
  - ... developed a branch-and-cut for interdiction games with monotonicity (Σ_2^p-hard problem)
  - ... previous approaches from the literature outperformed by up to 4 orders of magnitude
  - ... all 360 KIP instances from the literature solved to optimality (incl 27 unsolved ones)
  - ... for only 4 of these 360 more than 10 seconds needed
- Further work ...
  - ... further extensions to semi-interdiction problems
  - ... both, bilevel or min-max
  - ... both, exact and heuristics

Thank you for your Attention!
Questions?

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Literature I

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