Inverse optimal control: the sub-Riemannian case

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Outline

1. Inverse optimal control
2. Inverse sub-Riemannian problem
3. Levi-Civita pairs
4. Sub-Riemannian case

Outline

Inverse optimal control

(Direct) Optimal control problem

Given a dynamic $\dot{x} = f(x, u)$ and a cost $C(x_u)$: for any pair of points x_0, x_1, find a trajectory x_{u^*} solution of

$$\inf\{C(x_u) \ : \ x_u \text{ traj. s.t. } x_u(0) = x_0, x_u(T) = x_1\}.$$

Inverse optimal control problem

Given $\dot{x} = f(x, u)$ and a set Γ of trajectories: find a cost $C(x_u)$ such that every $\gamma \in \Gamma$ is solution of

$$\inf\{C(x_u) \ : \ x_u \text{ traj. s.t. } x_u(0) = \gamma(0), x_u(T) = \gamma(T)\}.$$

Applications to analysis/modelling of human motor control (physiology)

$\rightarrow$ looking for optimality principles

- Inverse problem: let $\Phi : C \in \mathcal{C} \mapsto \Gamma$ (=set of all minimizing trajectories).

  Inverse optimal control problem = find an inverse Φ^{-1}.

  Well-posed problem?
  - Injectivity Φ?
  - Surjectivity of Φ?
  - Continuity (and stability) of Φ^{-1}?

- Very few general results:
  - Hilbert's 4th problem (*for which metrics the shortest paths are straight lines?*)
  - Inverse problem of calculus of variations (since 70's)
  - Linear-Quadratic case [Kalmann 64, Nori-Frezza 04]

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Special class of problems

- Control-linear systems on M $\quad$ (M = smooth manifold)

$$\dot{x} = \sum_{i=1}^{m} u_i f_i(x), \qquad \text{i.e.} \qquad \dot{x} \in D_x = \mathrm{span}\{f_1(x), \dots, f_m(x)\}$$

(with $\dim \mathrm{Lie}(f_1, \dots, f_m)(x) \equiv \dim M \quad \Rightarrow \quad$ controllable system)

- Quadratic costs (w.r.t. the control)

$$\mathcal{E}_g(x_u) = \int_0^T g_{x_u(t)}(u(t))dt \quad \text{where} \quad g_x(\cdot) \text{ positive quadratic form}$$

Assumption: $\dim D_x = m$ constant $\quad$ ($D \subset TM$ is a distribution)

- if $\dim D_x = \dim M \;\rightarrow\; g$ Riemannian metric on M $\;(D = TM)$
- if $\dim D_x < \dim M \;\rightarrow\; g$ sub-Riemannian metric on D

Inverse sub-Riemannian (or Riemannian) problem

Let M be a manifold and $D = \mathrm{span}\{f_1, \ldots, f_m\}$ a distribution.

Can we recover g in **a unique way** from the given of all minimizers of $\mathcal{E}_g$?

(unique up to a constant since g and cg have same minimizers)

Remarks:

- u minimizes $\displaystyle\int_0^T g_{x_u}(u)dt$

$$\Longleftrightarrow \quad u \text{ minimizes } \int_0^T \sqrt{g_{x_u}(u)}dt \quad \text{and} \quad g_{x_u}(u) = \mathrm{const.}$$

$\rightarrow$ shortest paths $=$ unparameterized minimizers of $\mathcal{E}_g$

- Other possible (more difficult) sR inverse problem: from the shortest paths

Geodesics in sub-Riemannian geometry

Maximum Principle $\Rightarrow$ all minimizers are Pontryagin extremals (= geodesics)

Two kind of sub-Riemannian geodesics:

- Normal geodesics: Hamiltonian equations, locally minimizing
- Abnormal geodesics: only determined by D, independent of g

Definition

Two metrics $g, \widetilde{g}$ on D are:

- **affinely equivalent** if their normal geodesics coincide;
- **projectively equivalent** if their normal geodesics coincide as unparameterized curves.

Remark: If $g = c\widetilde{g}$ ($c =$ const), then $g, \widetilde{g}$ trivially proj. and aff. equivalent

Equivalence vs inverse sR problem

Theorem (J-Maslovskaya-Zelenko, 2016)

For two metrics on D:

- *same shortest paths* $\Rightarrow$ *projectively equivalent metrics*
- *same $\mathcal{E}_g$ minimizers* $\Rightarrow$ *affinely equivalent metrics*

Definition

A metric g on D is (proj. or aff.) **rigid** if it admits no nontrivial (proj. or aff.) equivalent metric

Consequence:

- If all metrics on D are aff. rigid, then the sR inverse problem with $\mathcal{E}_g$ minimizers is injective
- If all metrics on D are proj. rigid, then the sR inverse problem with shortest paths is injective

Outline

Example of nontrivial equivalent metrics

On $M = \mathbb{R}$

- All Riemannian metrics are projectively equivalent.
- Two Riemannian metrics are affinely equivalent $\Leftrightarrow$ constantly proportional

Product space

Let $(M, D, g), (\widetilde{M}, \widetilde{D}, \widetilde{g})$ be sub-Riemannian manifolds.

- $\mathbf{D} = D \oplus \widetilde{D}$ distribution on $\mathbf{M} = M \times \widetilde{M}$
- For $\lambda \in (0,1)$, $g^\lambda = \lambda g + (1-\lambda)\widetilde{g}$ metric on $\mathbf{D}$,

$$g^\lambda_{(x,\widetilde{x})}(u, \widetilde{u}) = \lambda g_x(u) + (1-\lambda)\widetilde{g}_{\widetilde{x}}(\widetilde{u})$$

- $\mathbf{x}(\cdot) = (x, \widetilde{x})(\cdot)$ shortest path of g^λ $\Longleftrightarrow$

  $x(\cdot)$ shortest path of g and $\widetilde{x}(\cdot)$ shortest path of $\widetilde{g}$,

$\Rightarrow$ For any λ, μ, g^λ and g^μ are (proj. and aff.) equivalent

(Generalized) Levi-Civita pair

$g, \widetilde{g}$ **Levi-Civita pair** on D near $x_0 \in M$ if, in coordinates on $U =$ neighb. of x_0,

- D admits a direct sum $D = D_1 \oplus \cdots \oplus D_N$ where

  $$U = U_1 \times \cdots \times U_N \quad \text{and} \quad D_s \text{ is a LBG distribution on } U_s$$

- $g, \widetilde{g}$ have the form of product metrics:
  for each s, $\exists\ g^s_{x_s}$ Riemannian metric on D_s, where x_s coordinates on U_s, s.t.

$$g_x = \sum_{s=1}^{N} \gamma_s(x) g^s_{x_s} \quad \text{and} \quad \widetilde{g}_x = \sum_{s=1}^{N} \lambda_s(x)\gamma_s(x) g^s_{x_s}$$

with $\lambda_s(x), \gamma_s(x)$ of special form and independent of x_i if $\dim U_i > 1$.

- $g, \widetilde{g}$ are projectively equivalent on U
- $g, \widetilde{g}$ are affinely equivalent on U iff all λ_s, γ_s constants

Conjecture

Two metrics are projectively equivalent iff they form a Levi-Civita pair, and affinely equivalent iff moreover all λ_s, γ_s constants.

Existing results

Conjecture true near generic points:

- in Riemannian case $(D = TM)$ [Levi-Civita, 1896], [Eisenhart, 1923]
- in contact and quasi-contact sR cases [Zelenko, 2006]

$\rightarrow$ No product structure on contact distribution

Corollary

If D contact distribution, all metrics on D are rigid

$\rightarrow$ The geodesic flows of Levi-Civita pairs admit quadratic first-integrals

Corollary

- *Generic Riemannian metrics are rigid*
- *If D quasi-contact distribution, generic metrics on D are rigid*

Outline

New results [FJ, S. Maslovskaya, I. Zelenko (2016)]

Theorem

D distribution on M — $g, \tilde{g}$ sR metrics on D

If g and $\tilde{g}$ projectively equivalent, then their normal geodesic flows admit quadratic first-integrals.

Generically, existence of such integrals

$\Rightarrow$ g and $\tilde{g}$ proportional (maybe non constantly)

Definition

g proj. conformally rigid if all proj. equivalent metrics are proportional

Corollary

On a fixed D, generic metrics are affinely rigid and proj. conformally rigid

Graph at $x \in M$ of a frame $f_1, \ldots, f_m$ of D:

$$\text{vertices} = \{1, \ldots, m\}, \quad i \text{ and } j \text{ connected if } [f_i, f_j](x) \notin D_x$$

Theorem

Assume that the graphs of all frames of D are connected.
Then all metrics on D are aff. rigid and proj. conformally rigid

Remark: connected graph $\Rightarrow$ no product structure on D

Corollary

Assume $\dim M \geq m(m+1)/2$ *(i.e. D of step > 2).*
Then, on a generic D, all metrics are aff. rigid and proj. conformally rigid

$\rightarrow$ injectivity of the inverse sR problem for generic distributions of step > 2