Optimal control methods for the stability of switched systems

Paolo MASON (CNRS, L2S)
joint work with Y. Chitour and M. Gaye

PGMO Days 2016, EDF Lab, Paris - Saclay

November 9th 2016

Linear switched systems

A linear switched systems has the form

$$\dot{x}(t) = A(t)x(t) \qquad x \in \mathbb{R}^n,\ A(t) \in \mathcal{A} \subset \mathbb{R}^{n\times n},$$

where

- $A(\cdot)$ is the switching law (e.g. piecewise constant)
- $\mathcal{A}$ compact and possibly convex.

Example : $\mathcal{A} = \{A_1, A_2\}$ or
$\mathcal{A} = conv\{A_1, A_2\} = \{\lambda A_1 + (1-\lambda)A_2 \ : \ \lambda \in [0,1]\}$

Stability and Lyapunov exponent

Maximal Lyapunov exponent of $\mathcal{A}$ defined as

$$\rho(\mathcal{A}) = \sup_{A(\cdot),x(0)} \left(\limsup_{t\to\infty} \frac{1}{t} \log \|x(t)\| \right).$$

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$\rho(\mathcal{A}) < 0$

(S) **unif. glob. asymp. stable** $=$ **unif. expon. stable**,
i.e. $\exists M, \lambda > 0$ s.t. $\forall x(0) \in \mathbb{R}^n$, $t \geq 0$, $A(\cdot)$ $\|x(t)\| \leq Me^{-\lambda t}\|x(0)\|$.

$\rho(\mathcal{A}) = 0$

- (S) **stable:** all trajectories are bounded and there exists a trajectory not converging to 0,
- (S) **marginally unstable:** $\exists$ unbounded traj. with non-exponential growth.

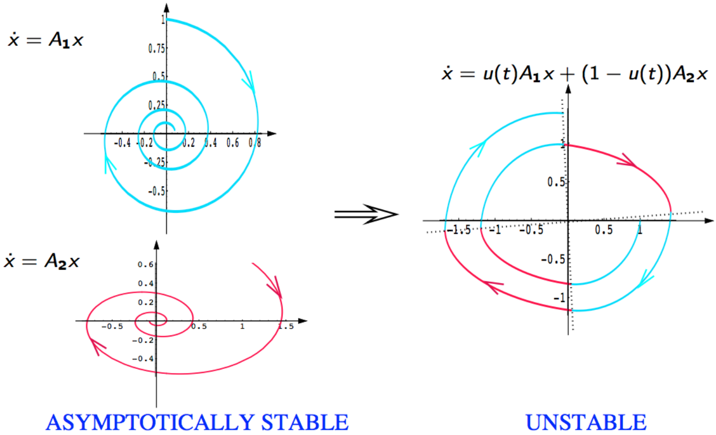
$\rho(\mathcal{A}) > 0$

(S) **unstable:** $\exists$ traj. going to ∞ exponentially.

Nontrivial stability problem, even in dimension 2!!!

$$\dot{x} = \sigma(t)A_1x + (1-\sigma(t))A_2x \qquad x \in \mathbb{R}^2,\ \sigma(t) \in \{0,1\}$$

A_1, A_2 Hurwitz 2×2 matrices.

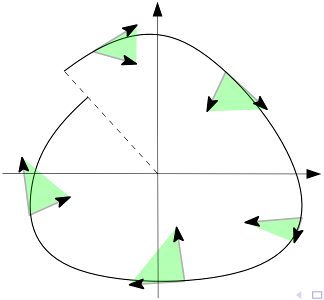


The case of planar systems

$$\dot{x} = \sigma(t)A_1 x + (1 - \sigma(t))A_2 x \qquad x \in \mathbb{R}^2,\ \sigma(t) \in \{0, 1\}$$

has been completely solved (e.g. U. Boscain 2002, Balde-Boscain-Mason 2009): necessary and sufficient conditions.

$\rightarrow$ method based on the analysis of the "most unstable trajectories" (worst trajectory)



The case $\rho(\mathcal{A}) = 0$

The translation,

$$\mathcal{A} \rightsquigarrow \mathcal{A} - \rho(\mathcal{A}) Id_n \qquad x(t) \rightsquigarrow x(t) e^{-\rho(\mathcal{A})t}$$

brings back to the case $\rho(\mathcal{A}) = 0$.

"Most unstable trajectories" for $\rho(\mathcal{A}) = 0$

$\rightsquigarrow$ "Most unstable trajectories" for any $\rho(\mathcal{A})$

$\Rightarrow$ Qualitative properties of the "most unstable trajectories" for $\rho(\mathcal{A}) = 0$ may be used to compute the "most unstable trajectories" in the case $\rho(\mathcal{A}) \in \mathbb{R}$

$\Rightarrow$ may lead to general numerical tests for stability.

Barabanov norms

Thus, assume $\rho(\mathcal{A}) = 0$ and $\mathcal{A}$ compact and convex.

Definition

$\mathcal{A}$ is irreducible if $\nexists$ nontrivial subspace $\{0\} \subsetneq V \subsetneq \mathbb{R}^n$, invariant $\forall A \in \mathcal{A}$. Otherwise we say that $\mathcal{A}$ is reducible.

N.B. $\mathcal{A}$ reducible $\iff \exists$ basis s.t. $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad \forall A \in \mathcal{A}$

Theorem (N. Barabanov, 1988)

If $\mathcal{A}$ is irreducible then $\exists$ a norm $v : \mathbb{R}^n \to [0, +\infty)$:

(1) $v(x(t)) \leq v(x(0))$ for every $A(\cdot)$ and $x(0)$,

(2) $\forall x(0)\ \exists\ x(\cdot)$ traj. of the system s.t. $v(x(t)) \equiv v(x(0)) \quad \forall t \geq 0$.

A traj. satisfying (2) is said to be extremal $\to$ "most unstable trajectory"

Extremal solutions as optimal trajectories

$$\dot{x}(t) = A(t)x(t)$$

Adjoint equation :

$$\dot{p}(t) = -A(t)^T p(t) \qquad (A)$$

Theorem

For any extremal traj. $x(\cdot)$ corresponding to $A(\cdot) \in \mathcal{A}$, for any $T \geq 0$ and $\hat{p} \in \partial v(x(T))$ $\exists$ nontrivial solution $p(\cdot)$ of (A) s.t., for $t \in [0, T]$

$$\begin{array}{lc} (C1) & \max_{A \in \mathcal{A}} p^T(t)Ax(t) = p^T(t)A(t)x(t) = 0, \\ (C2) & p(t) \in \partial v(x(t)), \\ (C3) & p(T) = \hat{p}. \end{array}$$

Moreover, $\exists p(\cdot)$ solution of (A) s.t. (C1) and (C2) are satisfied for any $t \geq 0$.

Sketch of the proof of the theorem.

Let $x(\cdot)$ extr., $\hat{p} \in \partial v(x(T))$ and consider the OCP in Mayer form:

$$\text{(OP)} \qquad \begin{array}{c} \min \hat{p}^T \cdot z(T) \quad \text{among all traj. } z(\cdot) \text{ of} \\ \dot{z}(t) = A(t)z(t), \quad z(0) = x(0), \quad A(t) \in \mathcal{A} \end{array}$$

Since $\hat{p} \in \partial v(x(T))$ it is easy to see that the extremal trajectory $x(\cdot)$ is a solution of (OP) on $[0, T]$. A simple application of the Pontryagin maximum principle leads to (C1) while (C2) comes easily from the def. of $\partial v(x(t))$.

The existence of a solution $p(\cdot)$ of the adjoint equation s.t. (C1)-(C2) hold for all $t > 0$ is obtained by extracting a converging subsequence $p_k(\cdot) \to p(\cdot)$ of adjoint trajectories on $[0, T_k]$, $T_k \to \infty$ as above.

General properties

Proposition

If the Barabanov norm $v(\cdot)$ is differentiable at x_0 then it is differentiable at $x(t)$ for any $t > 0$, where $x(\cdot)$ is extremal and $x(0) = x_0$.

Assume in addition:
(C) uniqueness of the solution $(x(\cdot), p(\cdot))$ satisfying the max. cond. (C1) in positive time from every initial point (x_0, p_0) (easily checkable, often)
Then

Proposition

- Assume (C) holds and $\mathcal{A}$ is made of non-singular matrices. If there are two trajectories reaching x_0 then x_0 is a point of diff. of $v(\cdot)$
- If (C) holds true and if x_0 is a point of differentiability of $v(\cdot)$, then there exists a unique extremal trajectory starting at x_0.

Poincaré-Bendixson type results

If $n = 3$ the Barabanov sphere is a nonsmooth manifold of dim. 2.

Flow is not "smooth": discontinuities, non-uniqueness phenomena...

In general, if the matrices of $\mathcal{A}$ are non-singular, then it is possible to show the existence of periodic trajectories.

Assume, in addition:
(C) uniqueness of the solution $(x(\cdot), p(\cdot))$ satisfying the maximization condition (C1) in positive time from every initial point (x_0, p_0)

Theorem

If the matrices in $\mathcal{A}$ are non-singular and (C) holds, then all extremal trajectories converge to a limit cycle.

"Barabanov" switched systems for $n = 3$

For $n = 3$ consider the linear switched system with

$$\mathcal{A} := conv\{A, A + bc^T\} = \{A + ubc^T : u \in [0,1]\},$$

where

- A and $A + bc^T$ are 3×3 Hurwitz matrices with $b, c \in \mathbb{R}^3$.
- (A, b) and (A^T, c) are controllable
- $\rho(\mathcal{M}) = 0$.

One can see that

- $\mathcal{A}$ is irreducible and made of non-singular (Hurwitz) matrices
- (C) holds true in this case

In particular there exists a Barabanov norm $v(\cdot)$, Poincaré-Bendixson holds...

What we can prove:

Theorem

Consider a Barabanov switched system and let S be the unit sphere of $v(\cdot)$. The following alternative holds true:

- either there exists on S a 1-parameter family of periodic trajectories $[0, s^*] \ni s \mapsto \gamma_s(\cdot)$ and each curve $\gamma_s(\cdot)$ has, on its period, four bang arcs for $s \in (0, s_*)$ and two bang arcs for $s \in \{0, s_*\}$;
- or there exists a finite number of periodic trajectories on S, each of them having four bang arcs.

We conjecture that the first alternative never occurs.

Sketch of the proof

The proof of the previous theorem is made of several steps and preliminary results. In particular:

- It can be shown that both $c^T x(t)$ and $b^T l(t)$ must change sign an infinite number of times on any extremal trajectory (otherwise they tend to zero and this gives a contradiction with extremality),
- It can be shown that if $x(\cdot)$ is periodic then $l(\cdot)$ is also periodic with the same period,
- By studying the spectrum of the fundamental matrix one shows that there are two sign changes in each period for $c^T x(\cdot)$ and $b^T l(\cdot)$,
- Since switchings occur when $(c^T x(t)) \cdot (b^T l(t))$ changes sign there are at most four bang arcs on each period.
- Periodic trajectories turn out to be in correspondence with zeros of a certain real analytic function. The two alternatives (finite number of periodic trajectories or existence of a continuum of periodic traj.) come from Lojasiewicz's structure theorem

Literature on the extremal flow?

For continuous time switched systems few noteworthy results are known about Barabanov spheres and the extremal flow, namely those by Barabanov (and consequent results by Margaliot)

- Barabanov '93 considers the case $\mathcal{A} := conv\{A, A + bc^T\}$. He claims that there exists a unique periodic trajectory with exactly 4 bang arcs which is also central symmetric. In particular the period of the corresponding switching law coincides with half the period of such a trajectory (only two bang arcs)
- Barabanov '08 claims in a much more general context the existence of a unique central symmetric periodic trajectory
- The proofs in the latter two cases are incomplete!! There are some gaps related to the unanswered question

$$?? \; v(x) = \limsup_{t\to\infty} \|\bar{x}(t)\| \quad \text{for some extremal} \quad \bar{x}(\cdot) \; ??$$

Numerical construction of a Barabanov sphere

Consider the matrices (derived from an example by Margaliot '06)

$$A_1 = \begin{pmatrix} -0.6989 & 1.16994 & -1.21696 \\ -2.09113 & 0.0408355 & -0.0675592 \\ 0.0221938 & 0.0145805 & -0.841936 \end{pmatrix}, \quad A_2 = A_1 + bc^T$$

with $b = \begin{pmatrix} 0.870658 \\ 0.0475324 \\ -0.0203314 \end{pmatrix}$, $c = \begin{pmatrix} 0 \\ 0.577858 \\ 1.35096 \end{pmatrix}$.

If the results by Barabanov are correct then it is possible to compute numerically $\rho(\mathcal{A})$ for $\mathcal{A} = \text{conv}\{A_1, A_2\}$ and check that $\rho(\mathcal{A}) = 0$.

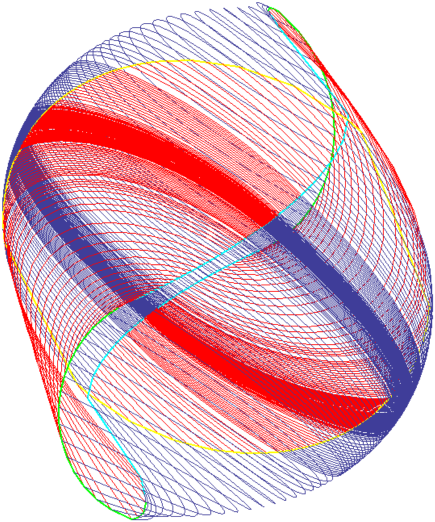
(just compute the max. modulus of eigenvalues of $e^{t_1 A_1} e^{t_2 A_2}$ w.r.t. t_1, t_2)

Numerical construction of a Barabanov sphere

By applying the results by Barabanov in the previous example it is easy to construct the limit cycle.

We also assume some intuitive (partially proved) regularity condition on the Barabanov sphere.

We end up with the picture $\longrightarrow$

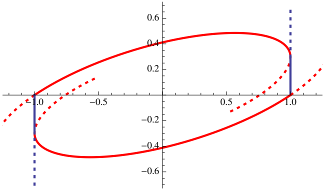


Side question: strict convexity

Example. Let us consider the matrices

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \qquad A_2 \sim \begin{pmatrix} 0.8896 & 3 \\ -0.6 & 0.7 \end{pmatrix}.$$

One can check that $\rho(\mathcal{A}) = 0$. For this example the Barabanov norm is not strictly convex :



Strict convexity

Proposition

If $\mathcal{A}$ does not contain any singular matrix, then the intersection between the corresponding Barabanov sphere with an hyperplane P has empty interior in P.

Corollary

If $n = 2$ and $\mathcal{A}$ does not contain singular matrices, then the Barabanov ball is strictly convex.

A global answer is missing already for $n = 3$!!

A class of switched systems with strictly convex Barabanov ball:

Proposition

If $\mathcal{A}$ is a $\mathcal{C}^1$ (closed) domain of $\mathbb{R}^{n\times n}$ then the corresponding Barabanov ball is strictly convex.

Side question: uniqueness of Barabanov norm

Let $\mathcal{M} := conv\{A_1, A_2, A_3\}$ with

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},\ A_3 = \begin{pmatrix} -\alpha & 1 \\ -1 & -\alpha \end{pmatrix}.$$

The system is irreducible, thus it admits a Barabanov norm.

It is easy to see that, taking $\alpha \geq 1$, any norm $v_\beta(x) := \max\{|x_1|, \beta|x_2|\}$ with $\beta \in [\frac{1}{\alpha}, \alpha]$ is a Barabanov norm of the system.

Uniqueness of Barabanov norm

Barabanov explicitely defined his norm as

$$v(x) = \sup_{x(\cdot), x(0)=x} \limsup_{t \to \infty} \|x(t)\|$$

Is it the unique Barabanov norm?

Problem already studied for discrete time sw. systems (Morris, 2010-2012). For continuous time systems the following result is based on the study of the union of all ω-limits of extremal trajectories :

$$\Omega := \bigcup_{\{x(\cdot): x(t) \in S\}} \omega(x(\cdot)).$$

Theorem

If Ω is connected then the Barabanov norm is unique (up to a multiplicative factor).

Uniqueness of Barabanov norm

Example. Let us consider the set $\mathcal{A} := conv\{A_1, A_2\}$ with

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

$\rightarrow$ the system is irreducible

$\rightarrow$ the system is stable since norm is non-increasing

$\rightarrow$ Not asymptotically stable: there are two intersecting periodic trajectories.

$\rightarrow$ It can be easily seen that extremal traj. must converge to the union of these two periodic traj.

$\rightarrow$ Ω connected, thus uniqueness of the Barabanov norm.

Open questions

A number of question are still totally open, even in dimension 3.

For instance:

- under which conditions
  - the Barabanov ball is strictly convex?
  - is there a periodic trajectory (even for $n > 3$)?
  - $v(x) = \limsup_{t\to\infty} \|\bar{x}(t)\|$ for an extremal trajectory $\bar{x}(t)$?
- If $n = 3$ and $\mathcal{A} = conv\{A_1, A_2\}$, is there always a periodic trajectory with 4 bang arcs?
- If $n \geq 4$ is it possible to find examples with no periodic trajectories? For instance chaotic behaviors?