Mixed-integer Bilevel programs for designing robust networks. An application to wind power collection

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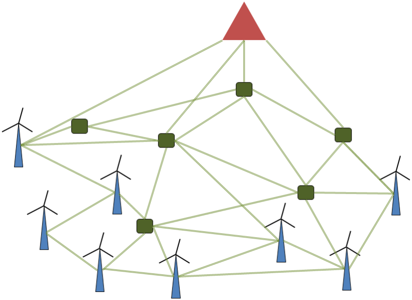
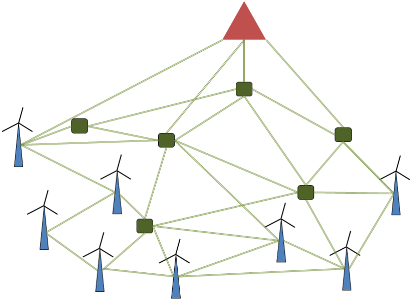
Input Network



- Substation (Root)

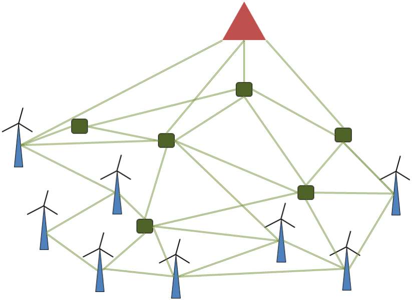
Input Network



- Substation (Root)
- Wind Turbines (Terminals)

Uniform energy production

Input Network



- Substation (Root)
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- Interconnection nodes (Steiner nodes)

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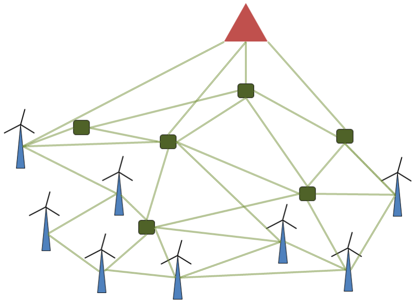
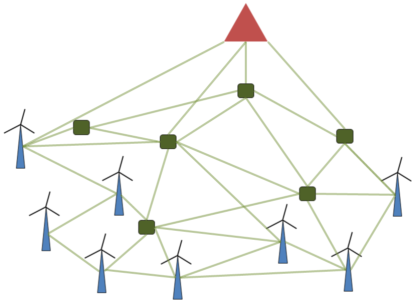


- Substation
  (Root)
- Wind Turbines
  (Terminals)
- Interconnection nodes
  (Steiner nodes)
- Cables
  (Arcs)

Capacity and cost functions

Input Network



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We want to route a unit of flow from the root to each terminal

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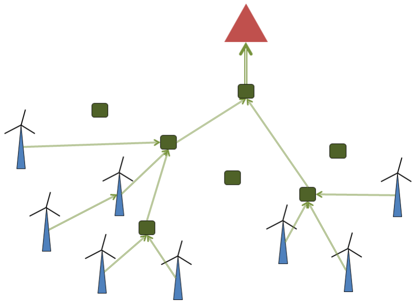

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Constraints and Notations

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- Protection of k' arcs allowed

Those arcs cannot be deleted

Constraints and Notations

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Notations :

- T : set of terminals
- r : root

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- T : set of terminals
- r : root
- c_{ij} : cost the arc (i, j)
- u_{ij} : capacity on the arc (i, j)

Robustness

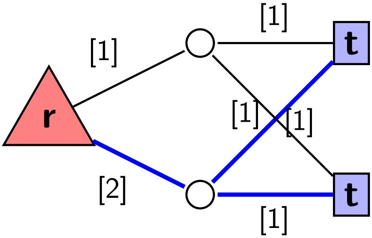
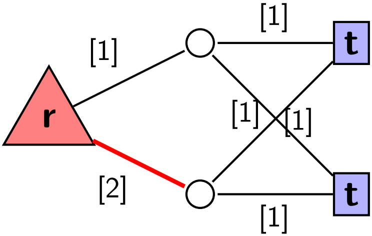


Figure : Example of a network which is not 1-robust

Robustness



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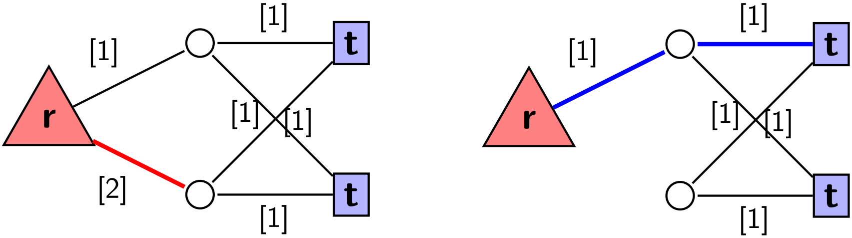

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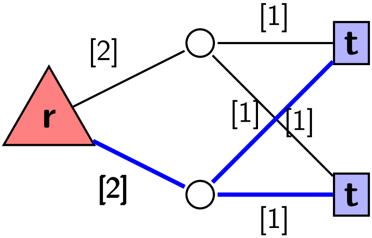
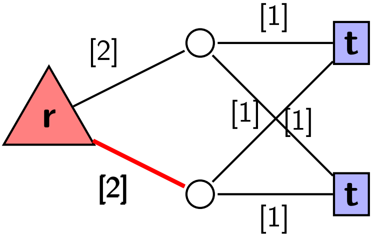


Figure : Example of a network which is 1-robust

Robustness



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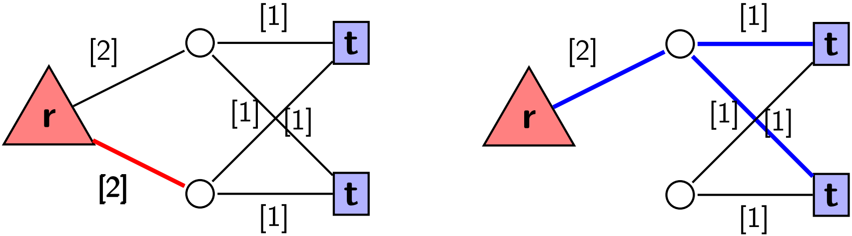

Figure : Example of a network which is 1-robust

Graph transformation

Graph transformation

- We add a sink s to the graph
- We add the edges (t, s) with $u_{ts} = 1$ and $c_{ts} = 0 \quad \forall t \in T$

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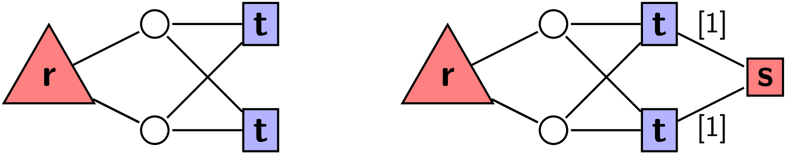

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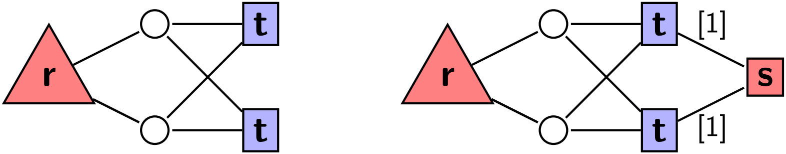

Figure : Example of a graph transformation

We can route a unit of flow to each terminal on the original graph

$\iff$

The maximum flow on the transformed graph is $|T|$

Bilevel formulation

Variables :

- y_{ij} : 0-1 variable equal to 1 if the arc (i, j) is selected in the final network

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Notation :

- $B = \{b \in \{0,1\}^E \mid \sum_{(i,j) \in E} b_{ij} \leq k \; ; \; b_{ts} = 0 \; \forall t \in T\}$

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Remark

We work on the transformed graph.

Bilevel formulation with a min-max second level

Bilevel formulation with an attacker and a defender:

$$\begin{array}{llll} \min\limits_{y,p \in \{0,1\}^E} & \sum\limits_{(i,j)\in E} c_{ij} y_{ij} & & \\ \text{s.t.} & f(y,p) & \geq & |T| \\ & \sum\limits_{(i,j)\in E} p_{ij} & \leq & k' \end{array}$$

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- The defender designs a network (and protects k' arcs)

Bilevel formulation with a min-max second level

Bilevel formulation with an attacker and a defender:

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$$
\begin{array}{lllll}
\text{where } f(y,p) = \min_{b\in B} \max_{x} & \sum_{j\in\Gamma^+(r)} x_{rj} & & & \\
\text{s.t} & \sum_{i\in\Gamma^-(j)} x_{ij} - \sum_{k\in\Gamma^+(j)} x_{jk} & = & 0 & \forall j \in V\setminus\{r,s\} \\
 & x_{ij} & \leq & u_{ij}y_{ij} & \forall (i,j)\in E \\
 & x_{ij} & \leq & u_{ij}(1-b_{ij}+p_{ij}) & \forall (i,j)\in E \\
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- The defender designs a network (and protects k' arcs)
- The attacker chooses the arcs to delete

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- The defender designs a network (and protects k' arcs)
- The attacker chooses the arcs to delete
- The defender computes a max flow on the residual network

Dualisation of the third-level problem

Third-level problem (Max Flow)

$$
\begin{array}{llll}
\max_{x} & \sum_{j \in \Gamma^+(r)} x_{rj} & & \\
\text{s.t} & \sum_{i \in \Gamma^-(j)} x_{ij} - \sum_{k \in \Gamma^+(j)} x_{jk} = 0 & \forall j \in V \setminus \{r, s\} & (\mu) \\
& x_{ij} \leq u_{ij} \hat{y}_{ij} & \forall (i,j) \in E & (\lambda) \\
& x_{ij} \leq u_{ij}(1 - \hat{b}_{ij} + \hat{p}_{ij}) & \forall (i,j) \in E & (\gamma) \\
& x_{ij} \geq 0 & \forall (i,j) \in E &
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$$

$\hat{y}$, $\hat{p}$ and $\hat{b}$ are fixed at this point

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\end{array}
$$

$$
x_{ij} \leq u_{ij}(1 - \hat{b}_{ij} + \hat{p}_{ij}) \Rightarrow \begin{cases} x_{ij} \leq 0 \text{ if } b_{ij} = 1 \text{ and } p_{ij} = 0 \\ \\ \text{Redundant constraint with } x_{ij} \leq u_{ij}\hat{y}_{ij} \\ \text{otherwise} \end{cases}
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\end{array}
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Dual (Min Cut)

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\begin{array}{llcll}
\min\limits_{\lambda,\mu,\gamma} & \sum\limits_{(i,j)\in E} u_{ij}\hat{y}_{ij}\lambda_{ij} + u_{ij}(1 - \hat{b}_{ij} + p_{ij})\gamma_{ij} & & & \\
\text{s.t} & \lambda_{ij} + \gamma_{ij} - \mu_i + \mu_j & \geq & 0 & \forall (i,j) \in E \\
& \mu_r & = & 1 & \\
& \mu_s & = & 0 & \\
& \lambda_{ij}, \gamma_{ij} & \geq & 0 & \forall (i,j) \in E
\end{array}
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\min\limits_{\lambda,\mu,\gamma} & \sum\limits_{(i,j)\in E} u_{ij}\hat{y}_{ij}\lambda_{ij} + u_{ij}(1-\hat{b}_{ij}+p_{ij})\gamma_{ij} & & & \\
\text{s.t} & \lambda_{ij}+\gamma_{ij}-\mu_i+\mu_j & \geq & 0 & \forall (i,j)\in E \\
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\text{s.t} & \lambda_{ij}+\gamma_{ij}-\mu_i+\mu_j \quad \geq \quad 0 & & \forall (i,j)\in E \\
 & \mu_r \quad = \quad 1 & & \\
 & \mu_s \quad = \quad 0 & & \\
 & 0\leq \lambda_{ij},\gamma_{ij}\leq 1 & 0\leq \mu_v \leq 1 & \forall (i,j)\in E \quad \forall v\in V
\end{array}
$$

Find a min cut after the deletion of some edges $\{(i,j)\in E|\hat{b}_{ij}=1\}$

Dualisation of the third-level problem

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\min\limits_{\lambda,\mu,\gamma} & \sum\limits_{(i,j)\in E} u_{ij}\hat{y}_{ij}\lambda_{ij} + u_{ij}(1-\hat{b}_{ij}+p_{ij})\gamma_{ij} & & & \\
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Find a min cut after the deletion of some edges $\{(i,j) \in E | \hat{b}_{ij} = 1\}$

- μ : Variable which defines the cut partition

Dualisation of the third-level problem

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Find a min cut after the deletion of some edges $\{(i,j)\in E|\hat{b}_{ij}=1\}$

- μ : Variable which defines the cut partition
- γ : Edges of the cutset which are deleted

Dualisation of the third-level problem

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\end{array}
$$

Find a min cut after the deletion of some edges $\{(i,j)\in E|\hat{b}_{ij}=1\}$

- μ : Variable which defines the cut partition
- γ : Edges of the cutset which are deleted
- λ : Other edges of the cutset

Second-level problem

Resulting Second-Level

$$
\begin{array}{llll}
\min\limits_{b\in B} & \min\limits_{\lambda,\mu,\gamma} & \sum\limits_{(i,j)\in E} u_{ij}\hat{y}_{ij}\lambda_{ij} + u_{ij}(1+\hat{p}_{ij})\gamma_{ij} - u_{ij}b_{ij}\gamma_{ij} & \\
& \text{s.t} & \lambda_{ij} + \gamma_{ij} - \mu_i + \mu_j \geq 0 & \forall (i,j)\in E \\
& & \mu_r = 1 & \\
& & \mu_s = 0 & \\
& & 0 \leq \lambda,\gamma,\mu \leq 1 &
\end{array}
$$

Second-level problem

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\text{s.t} & \sum\limits_{(i,j)\in E} b_{ij} & \leq & k & \\
 & b_{ts} & = & 0 & \forall t \in T \\
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 & \mu_r & = & 1 & \\
 & \mu_s & = & 0 & \\
 & 0 \leq \lambda, \gamma, \mu \leq 1, \quad b \in \{0,1\} & & &
\end{array}
$$

Resulting formulation

Resulting Bilevel Formulation

$$\min_{y,p} \quad \sum_{(i,j)\in E} c_{ij} y_{ij}$$

$$\text{s.t.} \quad f(y,p) \geq |T|$$

$$\sum_{(i,j)\in E} p_{ij} \leq \quad k'$$

$$\text{where } f(y,p) = \min_{b,\lambda,\mu,\gamma} \sum_{(i,j)\in E} u_{ij} y_{ij} \lambda_{ij} + u_{ij}\gamma_{ij} + u_{ij} p_{ij} \gamma_{ij} - u_{ij} b_{ij} \gamma_{ij}$$

$$\begin{array}{llll} s.t. & \sum_{(i,j)\in E} b_{ij} & \leq & k \\ & b_{ts} & = & 0 \quad \forall t \in T \\ & \lambda_{ij} + \gamma_{ij} - \mu_i + \mu_j & \geq & 0 \quad \forall (i,j) \in E \\ & \mu_r & = & 1 \\ & \mu_s & = & 0 \\ & 0 \leq \lambda, \gamma, \mu \leq 1, & b \in \{0,1\} & \end{array}$$

Resulting formulation

Notations :

- $\mathcal{F}_2$: Convex hull of the 2[nd]-level constraints

$$\mathcal{F}_2 = conv((b,\lambda,\mu,\gamma) \in \left\{ \begin{array}{llll} \sum_{(i,j)\in E} b_{ij} & \leq & k & \\ b_{ts} & = & 0 & \forall t \in T \\ \lambda_{ij} + \gamma_{ij} - \mu_i + \mu_j \geq 0 & & & \forall (i,j) \in E \\ \mu_r & = & 1 & \\ \mu_s & = & 0 & \\ 0 \leq \lambda,\gamma,\mu \leq & & 1, & b \in \{0,1\} \end{array} \right\})$$

Resulting formulation

Notations :

- $\mathcal{F}_2$: Convex hull of the 2nd-level constraints
- $g(y, p, \lambda, \gamma, b) = \sum_{(i,j)\in E}(u_{ij}y_{ij}\lambda_{ij} + u_{ij}\gamma_{ij} + u_{ij}p_{ij}\gamma_{ij} - u_{ij}b_{ij}\gamma_{ij})$

Resulting formulation

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- $\mathcal{A}$: the set of extreme points of $\mathcal{F}_2$

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- $\mathcal{A}$: the set of extreme points of $\mathcal{F}_2$
- $(\hat{b}^a, \hat{\lambda}^a, \hat{\mu}^a, \hat{\gamma}^a)$: respectively the values of (b,λ,μ,γ) at the extreme point $a \in \mathcal{A}$

Resulting formulation

Resulting Bilevel Formulation

$$\begin{aligned}
\min_{y,p} \quad & \sum_{(i,j)\in E} c_{ij} y_{ij} \\
\text{s.t.} \quad & f(y,p) \geq |T| \\
& \sum_{(i,j)\in E} p_{ij} \leq k' \\
\text{where } f(y,p) = \quad & \min_{b,\lambda,\mu,\gamma} \quad g(y,p,\lambda,\gamma,b) \\
& s.t. \quad (b,\lambda,\mu,\gamma) \in \mathcal{F}_2
\end{aligned}$$

Resulting formulation

$\mathcal{F}_2$ does not depend on y or p.

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Then ...

$$g(y,p,\lambda,\gamma,b) \quad \leq \quad g(y,p,\hat{\lambda}^a,\hat{\gamma}^a,\hat{b}^a) \qquad \forall a \in \mathcal{A}$$
$$(b,\lambda,\mu,\gamma) \in \mathcal{F}_2$$

Resulting formulation

$\mathcal{F}_2$ does not depend on y or p.

Then ...

$$g(y,p,\lambda,\gamma,b) \quad \leq \quad g(y,p,\hat{\lambda}^a,\hat{\gamma}^a,\hat{b}^a) \qquad \forall a \in \mathcal{A}$$
$$(b,\lambda,\mu,\gamma) \in \mathcal{F}_2$$

is equivalent to

$$g(y,p,\lambda,\gamma,b) \quad \leq \quad \min_{b,\lambda,\mu,\gamma} g(y,p,\lambda,\gamma,b)$$
$$(b,\lambda,\mu,\gamma) \in \mathcal{F}_2$$

regardless of the value of y or p

Resulting formulation

Thus ...

$$
\begin{aligned}
g(y, p, \lambda, \gamma, b) &\leq g(y, p, \hat{\lambda}^a, \hat{\gamma}^a, \hat{b}^a) \qquad \forall a \in \mathcal{A} \\
g(y, p, \lambda, \gamma, b) &\geq |T| \\
(b, \lambda, \mu, \gamma) &\in \mathcal{F}_2
\end{aligned}
$$

Resulting formulation

Thus ...

$$\begin{aligned} g(y,p,\lambda,\gamma,b) &\leq g(y,p,\hat{\lambda}^a,\hat{\gamma}^a,\hat{b}^a) \qquad \forall a \in \mathcal{A} \\ g(y,p,\lambda,\gamma,b) &\geq |T| \\ (b,\lambda,\mu,\gamma) &\in \mathcal{F}_2 \end{aligned}$$

is equivalent to

$$\begin{aligned} \min_{b,\lambda,\mu,\gamma} g(y,p,\lambda,\gamma,b) &\geq |T| \\ (b,\lambda,\mu,\gamma) &\in \mathcal{F}_2 \end{aligned}$$

Resulting formulation

$$
(P)\left\{
\begin{array}{llll}
\min\limits_{y,b,\lambda,\mu,\gamma} & \sum\limits_{(i,j)\in E} c_{ij}y_{ij} & & \\
\text{s.t} & \sum\limits_{(i,j)\in E} p_{ij} \leq k' & & \\
& g(y,p,\lambda,\gamma,b) \geq |T| & & \\
& g(y,p,\lambda,\gamma,l) \leq g(y,p,\hat{\lambda}^a,\hat{\gamma}^a,\hat{b}^a) & & \forall a \in \mathcal{A} \\
& (b,\lambda,\mu,\gamma) \in \mathcal{F}_2 & & \\
& y_{ij}, p_{ij} \in \{0,1\} & & \forall (i,j) \in E
\end{array}
\right.
$$

Solve the bilevel formulation

How to solve this problem?

Constraints generation based on the extreme points

Solve the bilevel formulation

How to solve this problem?

Constraints generation based on the extreme points

Master problem (P_R)

(P) initialized
without the constraints
on the extreme points

Slave problem (S)

$$\min_{b,\lambda,\mu,\gamma} \quad g(\hat{y}, \hat{p}, \lambda, \gamma, b)$$
$$(b, \lambda, \mu, \gamma) \in \mathcal{F}_2$$

Solve the bilevel formulation

Constraints generation

Solve the bilevel formulation

Constraints generation

1. Solve (P_R) and let $\hat{y}$ and $\hat{p}$ be the values of the optimal solution.

Solve the bilevel formulation

Constraints generation

1. Solve (P_R) and let $\hat{y}$ and $\hat{p}$ be the values of the optimal solution.
2. Solve (S) with $\hat{y}$ and $\hat{p}$ and let $(\hat{b}, \hat{\lambda}, \hat{\gamma})$ be the optimal values of (b, λ, γ).

Solve the bilevel formulation

Constraints generation

1. Solve (P_R) and let $\hat{y}$ and $\hat{p}$ be the values of the optimal solution.
2. Solve (S) with $\hat{y}$ and $\hat{p}$ and let $(\hat{b}, \hat{\lambda}, \hat{\gamma})$ be the optimal values of (b, λ, γ).
3. If $g(\hat{y}, \hat{p}, \hat{\lambda}, \hat{\gamma}, \hat{b}) \geq |T|$, go to 5.

Solve the bilevel formulation

Constraints generation

1. Solve (P_R) and let $\hat{y}$ and $\hat{p}$ be the values of the optimal solution.
2. Solve (S) with $\hat{y}$ and $\hat{p}$ and let $(\hat{b}, \hat{\lambda}, \hat{\gamma})$ be the optimal values of (b, λ, γ).
3. If $g(\hat{y}, \hat{p}, \hat{\lambda}, \hat{\gamma}, \hat{b}) \geq |T|$, go to 5.
4. Add the constraint $g(y, p, \lambda, \gamma, l) \leq g(y, p, \hat{\lambda}, \hat{\gamma}, \hat{b})$ to (P_R) and go to 1.

Solve the bilevel formulation

Constraints generation

1. Solve (P_R) and let $\hat{y}$ and $\hat{p}$ be the values of the optimal solution.
2. Solve (S) with $\hat{y}$ and $\hat{p}$ and let $(\hat{b}, \hat{\lambda}, \hat{\gamma})$ be the optimal values of (b, λ, γ).
3. If $g(\hat{y}, \hat{p}, \hat{\lambda}, \hat{\gamma}, \hat{b}) \geq |T|$, go to 5.
4. Add the constraint $g(y, p, \lambda, \gamma, l) \leq g(y, p, \hat{\lambda}, \hat{\gamma}, \hat{b})$ to (P_R) and go to 1.
5. $\hat{y}$ is the optimal value of y in (P).

Solve the bilevel formulation

If ...

y^1 and y^2 are non feasible solutions
and $y^2 \subset y^1$ ($y^2_{ij} \leq y^1_{ij} \quad \forall (i,j) \in E$)

Solve the bilevel formulation

If ...

y^1 and y^2 are non feasible solutions
and $y^2 \subset y^1$ ($y^2_{ij} \leq y^1_{ij} \quad \forall (i,j) \in E$)

And ...

The constraint $g(y, p, \lambda, \gamma, b) \leq g(y, p, \hat{\lambda}^a, \hat{\gamma}^a, \hat{b}^a)$
cuts y_1 from the feasible set

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And ...

The constraint $g(y, p, \lambda, \gamma, b) \leq g(y, p, \hat{\lambda}^a, \hat{\gamma}^a, \hat{b}^a)$
cuts y_1 from the feasible set

Then...

The constraint $g(y, p, \lambda, \gamma, b) \leq g(y, p, \hat{\lambda}^a, \hat{\gamma}^a, \hat{b}^a)$
cuts y_2 from the feasible set

Solve the bilevel formulation

If ...

y^1 and y^2 are non feasible solutions
and $y^2 \subset y^1$ ($y^2_{ij} \leq y^1_{ij} \quad \forall (i,j) \in E$)

And ...

The constraint $g(y,p,\lambda,\gamma,b) \leq g(y,p,\hat{\lambda}^a,\hat{\gamma}^a,\hat{b}^a)$
cuts y_1 from the feasible set

Then...

The constraint $g(y,p,\lambda,\gamma,b) \leq g(y,p,\hat{\lambda}^a,\hat{\gamma}^a,\hat{b}^a)$
cuts y_2 from the feasible set

$\rightarrow$ Enhancement of the generated constraints

Flow Formulation

Notations :

- F : set of arc failures scenarios (set of k-combinations of elements in E)

Variables :

- y_{ij} : 0-1 variable equal to 1 if the arc (i, j) is selected in the final network
- p_{ij} : 0-1 variable equal to 1 if the arc (i, j) is protected in the final network
- x_{ij}^{f} : amount of flow routed through the arc (i, j) when the the scenario of breakdowns $f \in F$ occurs

Flow Formulation

Flow Formulation :

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Flow Formulation :

$$\min_{x,y,p} \sum_{(i,j)\in E} c_{ij} y_{ij}$$

Flow Formulation

Flow Formulation :

$$\min_{x,y,p} \sum_{(i,j)\in E} c_{ij} y_{ij}$$

$$\text{s.t.} \sum_{i\in\Gamma^-(j)} x_{ij}^f - \sum_{k\in\Gamma^+(j)} x_{jk}^f = 0 \quad \forall j \in V \setminus \{r,s\}, \quad \forall f \in F$$

Flow conservation constraints

Flow Formulation

Flow Formulation :

$$\begin{aligned}
\min_{x,y,p} \quad & \sum_{(i,j)\in E} c_{ij} y_{ij} \\
\text{s.t.} \quad & \sum_{i\in\Gamma^-(j)} x_{ij}^f - \sum_{k\in\Gamma^+(j)} x_{jk}^f = 0 && \forall j \in V \setminus \{r,s\}, \quad \forall f \in F \\
& \sum_{j\in\Gamma^+(r)} x_{rj}^f = |T| && \forall f \in F
\end{aligned}$$

Flow Formulation

Flow Formulation :

$$
\begin{aligned}
\min_{x,y,p} \quad & \sum_{(i,j)\in E} c_{ij} y_{ij} \\
\text{s.t.} \quad & \sum_{i\in\Gamma^-(j)} x^f_{ij} - \sum_{k\in\Gamma^+(j)} x^f_{jk} = 0 && \forall j \in V \setminus \{r,s\}, \quad \forall f \in F \\
& \sum_{j\in\Gamma^+(r)} x^f_{rj} = |T| && \forall f \in F \\
& \sum_{(i,j)\in E} p_{ij} \leq k'
\end{aligned}
$$

Flow Formulation

Flow Formulation :

$$\min_{x,y,p} \sum_{(i,j)\in E} c_{ij} y_{ij}$$

$$\text{s.t.} \quad \sum_{i\in\Gamma^-(j)} x^f_{ij} - \sum_{k\in\Gamma^+(j)} x^f_{jk} = 0 \quad \forall j \in V \setminus \{r,s\}, \quad \forall f \in F$$

$$\sum_{j\in\Gamma^+(r)} x^f_{rj} = |T| \quad \forall f \in F$$

$$\sum_{(i,j)\in E} p_{ij} \leq k'$$

$$x^f_{ij} \leq u_{ij} y_{ij} \quad \forall (i,j) \in E, \quad \forall f \in F$$

Capacity constraints

Flow Formulation

Flow Formulation :

$$
\begin{array}{llll}
\min\limits_{x,y,p} & \sum\limits_{(i,j)\in E} c_{ij} y_{ij} & & \\
\text{s.t.} & \sum\limits_{i\in\Gamma^-(j)} x_{ij}^f - \sum\limits_{k\in\Gamma^+(j)} x_{jk}^f & = 0 & \forall j \in V \setminus \{r,s\}, \quad \forall f \in F \\
& \sum\limits_{j\in\Gamma^+(r)} x_{rj}^f = |T| & & \forall f \in F \\
& \sum\limits_{(i,j)\in E} p_{ij} \leq k' & & \\
& x_{ij}^f \leq u_{ij} y_{ij} & & \forall (i,j) \in E, \quad \forall f \in F \\
& x_{ij}^f \leq u_{ij} p_{ij} & & \forall (i,j) \in f, \quad \forall f \in F
\end{array}
$$

No flow on deleted edges

Flow Formulation

Flow Formulation :

$$
\begin{array}{llll}
\min\limits_{x,y,p} & \sum\limits_{(i,j)\in E} c_{ij} y_{ij} & & \\
\text{s.t.} & \sum\limits_{i\in\Gamma^-(j)} x_{ij}^f - \sum\limits_{k\in\Gamma^+(j)} x_{jk}^f = 0 & & \forall j \in V \setminus \{r,s\}, \quad \forall f \in F \\
& \sum\limits_{j\in\Gamma^+(r)} x_{rj}^f = |T| & & \forall f \in F \\
& \sum\limits_{(i,j)\in E} p_{ij} \leq k' & & \\
& x_{ij}^f \leq u_{ij} y_{ij} & & \forall (i,j) \in E, \quad \forall f \in F \\
& x_{ij}^f \leq u_{ij} p_{ij} & & \forall (i,j) \in f, \quad \forall f \in F \\
& x_{ij}^f \in \mathbb{N}, \quad y_{ij} \in \{0,1\} & & \forall (i,j) \in E, \quad \forall f \in F
\end{array}
$$

Flow Formulation

Flow Formulation :

$$
\begin{array}{lllll}
\min\limits_{x,y,p} & \sum\limits_{(i,j)\in E} c_{ij} y_{ij} & & & \\
\text{s.t.} & \sum\limits_{i\in\Gamma^-(j)} x^f_{ij} - \sum\limits_{k\in\Gamma^+(j)} x^f_{jk} & = 0 & & \forall j \in V \setminus \{r,s\}, \quad \forall f \in F \\
& \sum\limits_{j\in\Gamma^+(r)} x^f_{rj} & = & |T| & \forall f \in F \\
& \sum\limits_{(i,j)\in E} p_{ij} & \leq & k' & \\
& x^f_{ij} & \leq & u_{ij} y_{ij} & \forall (i,j) \in E, \quad \forall f \in F \\
& x^f_{ij} & \leq & u_{ij} p_{ij} & \forall (i,j) \in f, \quad \forall f \in F \\
& x^f_{ij} \in \mathbb{N}, \quad y_{ij} \in \{0,1\} & & & \forall (i,j) \in E, \quad \forall f \in F
\end{array}
$$

Too many variables and constraints

Flow Formulation

Columns-and-Constraints generation

Flow Formulation

Columns-and-Constraints generation

1. Declare the variables x_{ij}^{f} for only a small subset of F

Flow Formulation

Columns-and-Constraints generation

1. Declare the variables x_{ij}^{f} for only a small subset of F
2. Solve the flow formulation

Flow Formulation

Columns-and-Constraints generation

1. Declare the variables x_{ij}^f for only a small subset of F
2. Solve the flow formulation
3. Find the k most vital links on the current network

k Most Vital Links problem

Find the k arcs of E whose removal from the graph causes the greatest decrease in the maximum flow from the source to the sink.

Flow Formulation

Columns-and-Constraints generation

1. Declare the variables x_{ij}^f for only a small subset of F
2. Solve the flow formulation
3. Find the k most vital links on the current network
4. If there are k most vital links F, add the variables associated to the scenario of F and go to 2, otherwise stop.

k Most Vital Links problem

Find the k arcs of E whose removal from the graph causes the greatest decrease in the maximum flow from the source to the sink.

Cutset formulation for $k = 1$

Notation :

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- $\mathcal{C}$: set of subsets $C \subset V \setminus \{r\}$ ($r - s$ cut)

Cutset formulation for k = 1

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- $\mathcal{C}$: set of subsets $C \subset V \setminus \{r\}$ ($r - s$ cut)

Variables :

- y_{ij} : 0-1 variable equal to 1 if the arc (i, j) is selected in the final network

Cutset formulation for k=1

$$\min_{y} \sum_{(i,j)\in E} c_{ij} y_{ij}$$

$$\text{s.t.} \quad \sum_{(i,j)\in E} p_{ij} \leq k'$$

Cutset formulation for k=1

$$\min_{y} \quad \sum_{(i,j)\in E} c_{ij} y_{ij}$$

$$\text{s.t.} \quad \sum_{(i,j)\in E} p_{ij} \leq k'$$

$$\sum_{(i,j)\in\delta^-(C)} u_{ij} y_{ij} - \max_{(i,j)\in\delta^-(C)} u_{ij}(y_{ij} - p_{ij}) \geq |T| \quad \forall C \in \mathcal{C}$$

Cutset formulation for k=1

$$\min_{y} \quad \sum_{(i,j)\in E} c_{ij} y_{ij}$$

$$\text{s.t.} \quad \sum_{(i,j)\in E} p_{ij} \quad \leq \quad k'$$

$$\sum_{(i,j)\in \delta^{-}(C)} u_{ij} y_{ij} \quad - \max_{(i,j)\in \delta^{-}(C)} u_{ij}(y_{ij} - p_{ij}) \quad \geq \quad |T| \quad \forall C \in \mathcal{C}$$

$$y, p \in \{0, 1\} \qquad \forall (i,j) \in E$$

Cutset formulation for k=1

$$
\begin{aligned}
\min_{y} \quad & \sum_{(i,j)\in E} c_{ij} y_{ij} \\
\text{s.t.} \quad & \sum_{(i,j)\in E} p_{ij} \leq k' \\
& \sum_{(i,j)\in \delta^-(C)} u_{ij} y_{ij} - \max_{(i,j)\in \delta^-(C)} u_{ij}(y_{ij} - p_{ij}) \geq |T| \quad \forall C \in \mathcal{C} \\
& y, p \in \{0,1\} \qquad \forall (i,j) \in E
\end{aligned}
$$

Remark

For uniform capacities and $k' = 0$,
$\max_{(i,j)\in \delta^-(S)} u_{ij}(y_{ij} - p_{ij})$ becomes a constant.

Cutset formulation for k=1

Final cutset formulation on the transformed graph

$$
\begin{aligned}
\min_{y} \quad & \sum_{(i,j)\in E} c_{ij} y_{ij} \\
\text{s.t.} \quad & \sum_{(i,j)\in E} p_{ij} \leq k' \\
& \sum_{(i,j)\in \delta^-(C)} u_{ij} y_{ij} - B_C \geq |T| && \forall C \in \mathcal{C} \\
& B_C \geq u_{ij}(y_{ij} - p_{ij}) && \forall (i,j) \in \delta^-(C) \\
& y, p \in \{0,1\} && \forall (i,j) \in E
\end{aligned}
$$

Cutset formulation for k=1

Final cutset formulation on the transformed graph

$$
\begin{aligned}
\min_{y} \quad & \sum_{(i,j)\in E} c_{ij} y_{ij} \\
\text{s.t.} \quad & \sum_{(i,j)\in E} p_{ij} \leq k' \\
& \sum_{(i,j)\in \delta^-(C)} u_{ij} y_{ij} - B_C \geq |T| & \forall C \in \mathcal{C} \\
& B_C \geq u_{ij}(y_{ij} - p_{ij}) & \forall (i,j) \in \delta^-(C) \\
& y, p \in \{0,1\} & \forall (i,j) \in E
\end{aligned}
$$

Number exponential of constraints

Cutset Formulation

Constraints generation algorithm

Cutset Formulation

Constraints generation algorithm

1. Initialize the problem with the cutset constraints defined only for the root-cut

Cutset Formulation

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1. Initialize the problem with the cutset constraints defined only for the root-cut
2. Solve the cutset formulation

Cutset Formulation

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1. Initialize the problem with the cutset constraints defined only for the root-cut
2. Solve the cutset formulation
3. Find a set of cuts which violate the cutset constraints

Cutset Formulation

Constraints generation algorithm

1. Initialize the problem with the cutset constraints defined only for the root-cut
2. Solve the cutset formulation
3. Find a set of cuts which violate the cutset constraints
4. If the set of cuts found in 4 is empty, stop. Otherwise, add the constraints associated to each cut in the cutset formulation and go to 2.

Results

- Cutset formulation very efficient on uniform capacities with $k' = 0$

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- The bilevel formulation is way more efficient than the two others on general cases

Results

- Cutset formulation very efficient on uniform capacities with $k' = 0$
- The bilevel formulation is way more efficient than the two others on general cases
- Unlike for the bilevel formulation, the incrementation of k and k' increases importantly the solving time for the flow and the cutset formulations

Thanks

Thank you for your attention !