Minimal time mean field games

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based on joint works with Samer Dweik and Filippo Santambrogio

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Outline

Introduction

Macroscopic models for pedestrian flow

- Goal: propose and study a nice mean-field game (MFG) model for pedestrian flow in a certain domain $\Omega \subset \mathbb{R}^d$.
- Macroscopic models for pedestrian flow: based on the conservation law

$$\partial_t m + \operatorname{div}(mv) = 0.$$

  - $m(t) \in \mathcal{P}(\Omega)$: density of people in time t, a Borel probability measure.
  - $v(t, x, m)$: velocity field.

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  - $m(t) \in \mathcal{P}(\Omega)$: density of people in time t, a Borel probability measure.
  - $v(t, x, m)$: velocity field.
- How do people choose v?
- Our MFG approach:
  - people solve an optimal control problem: minimize the time to reach a certain target;
  - a person's dynamics and the optimal control problem depend on the average behavior of other people: maximal speed should depend in a decreasing way on congestion.

Introduction

The model

We consider in this talk the minimal time mean field game:

- People move in $\overline{\Omega} \subset \mathbb{R}^d$, where Ω is non-empty, open, and bounded.
- Goal: leave $\overline{\Omega}$ through $\Gamma \subset \partial\Omega$, non-empty and closed.
- Initially: $m_0 \in \mathcal{P}(\overline{\Omega})$.
- Dynamics: people choose their speed up to a maximal value

$$\dot{\gamma}(t) = K(m_t, \gamma(t))u(t),$$
$$\gamma(t) \in \overline{\Omega}, \; u(t) \in \overline{B}(0, 1) = \text{closed unit ball}.$$

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Typically:

$$K(\mu, x) = g\left[\int_{\overline{\Omega}} \chi(x - y)\eta(y)\, d\mu(y)\right],$$

- χ: convolution kernel.
- η: cut-off function to discount people who already left.
- g: positive decreasing function.

Introduction

The model

Most MFGs in the literature consider optimization criteria in fixed time T (same for all agents) or in infinite time horizon.

Our model:

- Optimization criterion: agents want to leave $\overline{\Omega}$ through Γ in minimal time.

$$\begin{aligned}\inf\{T \geq 0 \mid\ & \dot\gamma(t) = K(m_t, \gamma(t))u(t),\ u : \mathbb{R}_+ \to \overline{B}(0, 1),\\ & \gamma(0) = x,\ \gamma(T) \in \Gamma,\ \gamma(t) \in \overline{\Omega} \text{ for } t \in [0, T],\\ & \dot\gamma(t) = 0 \text{ for } t > T\}.\end{aligned}$$

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- For simplicity *(and to get stronger results)*, $\Gamma = \partial\Omega$ in this talk (room with no walls).

Main question: characterize the evolution of the density m.

Existence of an equilibrium

The Lagrangian approach

- Eulerian approach: $m : \mathbb{R}_+ \to \mathcal{P}(\overline{\Omega})$ is a curve on the set of measures. Motion is described by the density and the velocity field of the population.

Existence of an equilibrium

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- Eulerian approach: $m : \mathbb{R}_+ \to \mathcal{P}(\overline{\Omega})$ is a curve on the set of measures. Motion is described by the density and the velocity field of the population.
- Lagrangian approach: $Q \in \mathcal{P}(\mathcal{C})$, where $\mathcal{C} = \mathcal{C}(\mathbb{R}_+, \overline{\Omega})$, is a measure on the set of curves. Motion is described by the trajectory of each agent.

Lagrangian framework for mean field games already used in the literature, cf. e.g. the survey in [Benamou, Carlier, Santambrogio; 2017].

Link between Eulerian and Lagrangian: $m_t = e_{t\#}Q$, where $e_t : \mathcal{C} \to \overline{\Omega}$ is the evaluation at time t of a curve, $e_t(\gamma) = \gamma(t)$.

Existence of an equilibrium

The Lagrangian approach

Definition

A measure $Q \in \mathcal{P}(\mathcal{C})$ is a (Lagrangian) equilibrium of the mean field game with initial condition $m_0 \in \mathcal{P}(\overline{\Omega})$ if $e_{0\#}Q = m_0$ and Q-almost every $\gamma \in \mathcal{C}$ is optimal for

$$\inf\{T \geq 0 \mid \dot{\gamma}(t) = K(e_{t\#}Q, \gamma(t))u(t),\ u : \mathbb{R}_+ \to \overline{B}(0, 1),$$
$$\gamma(0) = x,\ \gamma(T) \in \partial\Omega, \dot{\gamma}(t) = 0 \text{ for } t > T\}.$$

In the sequel, we consider

- the existence of a Lagrangian equilibrium;
- the characterization of equilibria by the MFG system;
- the L^p regularity of m.

Existence of an equilibrium

Main result

Theorem

Assume that $K : \mathcal{P}(\overline{\Omega}) \times \overline{\Omega} \to \mathbb{R}_+$ *is Lipschitz continuous and*

$$K_{\max} = \sup_{\substack{\mu \in \mathcal{P}(\overline{\Omega}) \\ x \in \overline{\Omega}}} K(\mu, x) < +\infty, \qquad K_{\min} = \inf_{\substack{\mu \in \mathcal{P}(\overline{\Omega}) \\ x \in \overline{\Omega}}} K(\mu, x) > 0.$$

Then there exists a Lagrangian equilibrium $Q \in \mathcal{P}(\mathcal{C})$ *for this game.*

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Distance in $\mathcal{P}(\overline{\Omega})$: Wasserstein distance

$$W_1(\mu, \nu) = \min_{\substack{\gamma \in \mathcal{P}(\overline{\Omega} \times \overline{\Omega})\\ \pi_{1\#}\gamma = \mu,\ \pi_{2\#}\gamma = \nu}} \int_{\overline{\Omega} \times \overline{\Omega}} |x - y| \, d\gamma(x, y).$$

Remark: If $K(\mu, x) = g\left[\int_{\overline{\Omega}} \chi(x - y)\eta(y)\, d\mu(y)\right]$ with g, χ and η Lipschitz and $g > 0$, then K satisfies the above hypothesis.

Existence of an equilibrium

Ideas of the proof

- Translate the definition of equilibrium into a fixed point.

  $\mathbf{Opt}(Q) \subset \mathcal{C}$: set of all optimal trajectories for Q. For $m_0 \in \mathcal{P}(\overline{\Omega})$, define

  $$F_{m_0}(Q) = \{\widetilde{Q} \mid {e_0}_{\#}\widetilde{Q} = m_0 \text{ and } \widetilde{Q}(\mathbf{Opt}(Q)) = 1\}.$$

  Equilibrium with initial condition $m_0 \iff$ fixed point of the set-valued map F_{m_0}, i.e., $Q \in F_{m_0}(Q)$.

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  Equilibrium with initial condition $m_0 \iff$ fixed point of the set-valued map F_{m_0}, i.e., $Q \in F_{m_0}(Q)$.

- Prove required properties of F_{m_0} to apply Kakutani fixed point theorem: upper semi-continuous, $F_{m_0}(Q)$ non-empty, compact and convex for all Q in a compact convex subset of $\mathcal{P}(\mathcal{C})$.

  We use properties of the value function

  $$\begin{aligned}\varphi_Q(t_0, x_0) = \inf\{T \geq 0 \mid & \dot\gamma(t) = K(e_{t\#}Q, \gamma(t))u(t), \\ & u : \mathbb{R}_+ \to \overline{B}(0,1),\ \gamma(t_0) = x_0,\ \gamma(t_0 + T) \in \partial\Omega, \\ & \dot\gamma(t) = 0 \text{ for } t > t_0 + T\}.\end{aligned}$$

  inf is a min, φ bounded and Lipschitz in (t_0, x_0, Q).

Existence of an equilibrium

Remarks

- The assumption $\Gamma = \partial\Omega$ is not needed in this proof.
- We can replace $\overline{\Omega}$ by a compact metric space X, and Γ by a non-empty closed subset of X. Useful for considering MFGs on networks.

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- The assumption $\Gamma = \partial\Omega$ is not needed in this proof.
- We can replace $\overline{\Omega}$ by a compact metric space X, and Γ by a non-empty closed subset of X. Useful for considering MFGs on networks.
- The Lagrangian approach is easier to apply than the Eulerian. Allows one to prove existence with very few properties of optimal trajectories.
- Drawback: no information on $m_t = e_{t\#}Q$. Does it satisfy a continuity equation? With which velocity field? Is it absolutely continuous, L^p?

Goal of the sequel: characterize φ_Q and m as solutions of a system of PDEs, and study the L^p regularity of m.

The MFG system

Hypotheses

In the sequel, Q is an equilibrium and we write $\varphi = \varphi_Q$ to simplify.

In addition to the previous hypotheses, we also assume:

Hypotheses

- $K : \mathcal{P}(\overline{\Omega}) \times \mathbb{R}^d \to \mathbb{R}_+^*$ is given by
$$K(\mu, x) = g\left[\int_{\overline{\Omega}} \chi(x-y)\eta(y)d\mu(y)\right],$$
$g \in \mathcal{C}^{1,1}(\mathbb{R}_+, \mathbb{R}_+^*)$ is bounded, $\chi \in \mathcal{C}^{1,1}(\mathbb{R}^d, \mathbb{R}_+)$, and $\eta \in \mathcal{C}^{1,1}(\mathbb{R}^d, \mathbb{R}_+)$ with $\eta(x) = 0$ and $\nabla\eta(x) = 0$ for $x \in \partial\Omega$.
- Ω satisfies the uniform exterior sphere condition: $\mathbb{R}^d \setminus \Omega$ is a union of closed balls with the same radius.

The MFG system

Main result

Theorem

Under the previous assumptions, φ and m solve the MFG system

$$\begin{cases} \partial_t m(t,x) - \operatorname{div}_x \left[K(m_t,x)\widehat{\nabla\varphi}(t,x)m(t,x)\right] = 0, & \mathbb{R}_+^* \times \Omega, \\ -\partial_t \varphi(t,x) + \left|\nabla_x \varphi(t,x)\right| K(m_t,x) - 1 = 0, & \mathbb{R}_+ \times \Omega, \\ m(0,x) = m_0(x), & \overline{\Omega}, \\ \varphi(t,x) = 0, & \mathbb{R}_+ \times \partial\Omega. \end{cases}$$

$\widehat{\nabla\varphi}(t,x) = \frac{\nabla_x\varphi(t,x)}{|\nabla_x\varphi(t,x)|}$ (more on this latter).

Continuity equation satisfied in the sense of distributions,
Hamilton–Jacobi equation satisfied in the viscosity sense.

Velocity field for the continuity equation:

$$v(t,x,m_t) = -K(m_t,x)\widehat{\nabla\varphi}(t,x).$$

The MFG system

Ideas of the proof

Hamilton–Jacobi equation: standard techniques on optimal control using a dynamic programming principle.

Continuity equation: more subtle. The velocity field is $v = -K\widehat{\nabla\varphi}$, we need a meaning for $\widehat{\nabla\varphi} = \frac{\nabla_x\varphi}{|\nabla_x\varphi|}$.

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Proposition

If $\gamma \in \mathbf{Opt}(Q)$ *and* φ *is differentiable at* $(t, \gamma(t))$*, then* $\nabla_x\varphi(t, \gamma(t)) \neq 0$ *and*

$$\dot{\gamma}(t) = -K(m_t, \gamma(t)) \frac{\nabla_x\varphi(t, \gamma(t))}{|\nabla_x\varphi(t, \gamma(t))|}.$$

φ is Lipschitz, hence differentiable a.e., but it may be nowhere differentiable along a particular trajectory.

The MFG system

Ideas of the proof

We still need more properties of φ and the optimal trajectories, obtained by applying Pontryagin Maximum Principle, which yields:

Proposition

If $\gamma \in \mathbf{Opt}(Q)$, *then* $\gamma \in \mathcal{C}^{1,1}([0, \varphi(0, \gamma(0))), \Omega)$, *the optimal control* $u \in \mathcal{C}^{1,1}([0, \varphi(0, \gamma(0))), \mathbb{S}^{d-1})$, *and*

$$\begin{cases} \dot\gamma(t) = K(m_t, \gamma(t)) u(t), \\ \dot u(t) = -\operatorname{Proj}_{T_{u(t)}\mathbb{S}^{d-1}} \nabla_x K(m_t, \gamma(t)). \end{cases}$$

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Moreover, if Q is a Lagrangian equilibrium, then

- $(t, x) \mapsto K(e_{t\#}Q, x) = g\left[\int_{\mathbf{Opt}(Q)} \chi(x - \gamma(t)) \eta(\gamma(t)) \, \mathrm{d}Q(\gamma)\right]$ *is also $\mathcal{C}^{1,1}$;*
- φ *is semiconcave* (using [Cannarsa, Sinestrari; 2004]).

The MFG system
Ideas of the proof

Definition

We say that φ admits a normalized gradient at $(t, x) \in \mathbb{R}_+ \times \Omega$ if

$$\left\{ \frac{p_1}{|p_1|} \in \mathbb{S}^{d-1} \;\middle|\; p_1 \neq 0 \text{ and } \exists p_0 \in \mathbb{R} \text{ s.t. } (p_0, p_1) \in D^+\varphi(t, x) \right\}$$

contains exactly one element ($D^+\varphi$: super-differential of φ). In this case, the unique element of this set is denoted by $\widehat{\nabla\varphi}(t, x)$.

The MFG system

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Proposition

Let Q be an equilibrium and $\gamma \in \mathbf{Opt}(Q)$. Then, for every $t \in (0, \varphi(0, \gamma(0)))$, φ admits a normalized gradient at $(t, \gamma(t))$ and

$$\dot{\gamma}(t) = -K(m_t, \gamma(t))\widehat{\nabla\varphi}(t, \gamma(t)).$$

Moreover, $\widehat{\nabla\varphi}$ is continuous on a set of full $m_t|_\Omega$ measure.

As a consequence, *m* satisfies the continuity equation.

L^p regularity of the distribution of agents

The function η

$$K(\mu, x) = g\left[\int_{\overline{\Omega}} \chi(x - y)\eta(y)\,\mathrm{d}\mu(y)\right]$$

- χ has an interpretation in a pedestrian flow model: people look around to evaluate the density.

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The function η

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- χ has an interpretation in a pedestrian flow model: people look around to evaluate the density.
- η represents the fact that we do not want to take into account those who already left (and remain in $\partial\Omega$). Ideally, $\eta = \mathbb{1}_\Omega$.
- But K is not continuous w.r.t. μ with this choice of η (e.g., if μ is a Dirac mass on the boundary). The previous arguments do not apply.
- Goal: obtain estimates on φ and m that do not depend on η and pass to the limit along a sequence $\eta_n \to \mathbb{1}_\Omega$.

L^p regularity of the distribution of agents
L^p estimates on m

In order to provide L^p estimates on m that are independent of η, we make use of some extra assumptions.

Hypotheses

g is decreasing and η is given by $\eta(x) = \eta_0(d(x, \partial\Omega))$, where $\eta_0 \in \mathcal{C}^{1,1}(\mathbb{R}_+, [0, 1])$ is non-decreasing and $\eta_0(0) = \dot{\eta}_0(0) = 0$.

m itself cannot be L^p on $\overline{\Omega}$, since agents concentrate in the boundary. We consider in the sequel only the restriction of m to Ω.

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m itself cannot be L^p on $\overline{\Omega}$, since agents concentrate in the boundary. We consider in the sequel only the restriction of m to Ω.

Theorem

Let $p \in (1, +\infty]$. Under the previous assumptions, $\exists C > 0$ independent of η s.t. $m_0 \in L^p(\Omega, \mathbb{R}_+) \implies m_t \in L^p(\Omega, \mathbb{R}_+)$ and

$$\|m_t\|_{L^p} \leq Ce^{Ct} \|m_0\|_{L^p}.$$

L^p regularity of the distribution of agents

L^p estimates on m

Formally, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega m_t^p = p \int_\Omega m_t^{p-1} \operatorname{div} \left[K \widehat{\nabla \varphi} m_t \right]$$

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$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega m_t^p &= p\int_\Omega m_t^{p-1}\,\mathrm{div}\left[K\widehat{\nabla\varphi}m_t\right]\\ &= -p\int_\Omega \nabla\left(m_t^{p-1}\right)\cdot\widehat{\nabla\varphi}Km_t\\ &\quad + p\int_{\partial\Omega} Km_t^p\widehat{\nabla\varphi}\cdot n\end{aligned}$$

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It suffices to obtain an upper bound indep. of η on $\operatorname{div} \left[\widehat{\nabla\varphi} K\right]$.

L^p regularity of the distribution of agents

L^p estimates on m

Formally,

$$\operatorname{div}\left[\widehat{\nabla\varphi} K\right] = \widehat{\nabla\varphi} \cdot \nabla K + K \frac{\Delta\varphi - \widehat{\nabla\varphi} D^2\varphi \widehat{\nabla\varphi}}{\left|\nabla\varphi\right|}$$

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- $K(\mu, x) = g\left[\int_{\overline{\Omega}} \chi(x - y)\eta(y)\, d\mu(y)\right] \implies K$ and ∇K are bounded, indep. of η satisfying the previous assumptions.

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- $K(\mu, x) = g\left[\int_\Omega \chi(x-y)\eta(y)\, d\mu(y)\right] \implies K$ and ∇K are bounded, indep. of η satisfying the previous assumptions.
- Ok if we provide an upper bound on $\Delta\varphi - \widehat{\nabla\varphi} D^2\varphi \widehat{\nabla\varphi}$ and a lower bound on $\left|\nabla\varphi\right|$, both indep. of η.
- Lower bound on $\left|\nabla\varphi\right|$: ok. We already needed one before to show that $\nabla_x\varphi \neq 0$ on differentiability points of φ along optimal trajectories, and it turns out not to depend on η.

L^p regularity of the distribution of agents

L^p estimates on m

If $\{e_1, \ldots, e_d\}$ orthonormal basis of $\mathbb{R}^d$ with $e_1 = \widehat{\nabla \varphi}$,

$$\Delta \varphi - \widehat{\nabla \varphi} D^2 \varphi \widehat{\nabla \varphi} = \sum_{j=2}^d \partial_{jj}^2 \varphi.$$

It suffices to obtain an upper bound $D^2 \varphi \leq C \operatorname{Id}_d$. We have it from semiconcavity, but C might depend on η. Solution: refine the semiconcavity proof from [Cannarsa, Sinestrari; 2004] to show that C is independent of η.

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Semiconcavity relies on Lipschitz behavior of K with respect to t and $\mathcal{C}^{1,1}$ behavior with respect to x.

$$K(m_t, x) = g\left[\int_{\mathbf{Opt}(Q)} \chi(x - \gamma(t))\eta(\gamma(t))\, \mathrm{d}Q(\gamma)\right].$$

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$$K(m_t, x) = g\left[\int_{\mathbf{Opt}(Q)} \chi(x - \gamma(t))\eta(\gamma(t))\, \mathrm{d}Q(\gamma)\right].$$

$$\partial_t K(m_t, x) = g'[\cdot] \int_{\mathbf{Opt}(Q)} \big(-\nabla\chi(x - \gamma(t)) \cdot \dot\gamma(t) + \nabla\eta(\gamma(t)) \cdot \dot\gamma(t)\big)\, \mathrm{d}Q(\gamma).$$

L^p regularity of the distribution of agents

L^p estimates on m

If $\{e_1, \ldots, e_d\}$ orthonormal basis of $\mathbb{R}^d$ with $e_1 = \widehat{\nabla\varphi}$,

$$\Delta\varphi - \widehat{\nabla\varphi} D^2\varphi \widehat{\nabla\varphi} = \sum_{j=2}^d \partial_{jj}^2 \varphi.$$

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Semiconcavity relies on Lipschitz behavior of K with respect to t and $\mathcal{C}^{1,1}$ behavior with respect to x.

$$K(m_t, x) = g\left[\int_{\mathbf{Opt}(Q)} \chi(x - \gamma(t))\eta(\gamma(t))\,\mathrm{d}Q(\gamma)\right].$$

$$\partial_t K(m_t, x) = \underbrace{g'[\cdot]}_{\leq 0} \int_{\mathbf{Opt}(Q)} \Big(\underbrace{-\nabla\chi(x - \gamma(t)) \cdot \dot\gamma(t)}_{|\cdot| \leq C \text{ indep. of } \eta} + \underbrace{\nabla\eta(\gamma(t)) \cdot \dot\gamma(t)}_{\leq 0,\text{ no lower bound}}\Big)\,\mathrm{d}Q(\gamma).$$

$\partial_t K \geq -C$, no upper bound. Sufficient to obtain semiconcavity.

L^p regularity of the distribution of agents

The case $\eta = \mathbb{1}_\Omega$

Finally, in the case $\eta = \mathbb{1}_\Omega$, i.e.,

$$K(\mu, x) = g\left[\int_{\Omega} \chi(x-y)\,\mathrm{d}\mu(y)\right],$$

Theorem

Assume $m_0 \in L^\infty$. Then there exists an equilibrium Q and $m_t = e_{t\#}Q$ and φ satisfy the MFG system

$$\begin{cases} \partial_t m(t,x) - \operatorname{div}_x\left[K(m_t,x)\widehat{\nabla\varphi}(t,x)m(t,x)\right] = 0, & \mathbb{R}_+^* \times \Omega, \\ -\partial_t\varphi(t,x) + \left|\nabla_x\varphi(t,x)\right| K(m_t,x) - 1 = 0, & \mathbb{R}_+ \times \Omega, \\ m(0,x) = m_0(x), & \overline{\Omega}, \\ \varphi(t,x) = 0, & \mathbb{R}_+ \times \partial\Omega. \end{cases}$$

This follows by a limit argument with $\eta_n \to \mathbb{1}_\Omega$, using the bounds on $\|m_t\|_{L^p}$ and $|\nabla\varphi|$ and the upper bound on $D^2\varphi$, all indep. of η_n.