An introduction to switched systems

Guilherme Mazanti

Rentrée des masters de la FMJH
IHÉS – 9 septembre 2016

LMO, Université Paris-Sud

Outline

Introduction
Switched systems

- Classical setting for autonomous differential equations:
$$\dot{x}(t) = f(x(t)),$$
$f : \mathbb{R}^d \to \mathbb{R}^d$ a locally Lipschitz function.

Introduction

Switched systems

- Classical setting for autonomous differential equations:
$$\dot{x}(t) = f(x(t)),$$
$f : \mathbb{R}^d \to \mathbb{R}^d$ a locally Lipschitz function.
- Switched system: equation under the form,
$$\dot{x}(t) = f_{\sigma(t)}(x(t)),$$
where one has N (smooth) vector fields $f_1, \dots, f_N$ and a (piecewise constant) switching signal $\sigma : \mathbb{R}_+ \to [\![1, N]\!]$.

Introduction
Switched systems

- Classical setting for autonomous differential equations:
$$\dot{x}(t) = f(x(t)),$$
$f : \mathbb{R}^d \to \mathbb{R}^d$ a locally Lipschitz function.
- Switched system: equation under the form,
$$\dot{x}(t) = f_{\sigma(t)}(x(t)),$$
where one has N (smooth) vector fields $f_1, \dots, f_N$ and a (piecewise constant) switching signal $\sigma : \mathbb{R}_+ \to [\![1, N]\!]$.
- Interaction between the continuous variable $x \in \mathbb{R}^d$ and the discrete state $\sigma \in [\![1, N]\!]$.

Introduction

Switched systems

- Classical setting for autonomous differential equations:
$$\dot{x}(t) = f(x(t)),$$
$f : \mathbb{R}^d \to \mathbb{R}^d$ a locally Lipschitz function.
- Switched system: equation under the form,
$$\dot{x}(t) = f_{\sigma(t)}(x(t)),$$
where one has N (smooth) vector fields $f_1, \dots, f_N$ and a (piecewise constant) switching signal $\sigma : \mathbb{R}_+ \to [\![1, N]\!]$.
- Interaction between the continuous variable $x \in \mathbb{R}^d$ and the discrete state $\sigma \in [\![1, N]\!]$.
- Important models in physics, engineering, applied maths...

Introduction

Examples of switched systems

Example: simple model for the average temperature of a heated room.

$$\dot{\theta}(t) = -\rho(\theta(t) - \theta_{\text{ext}}), \qquad \text{if the heater is off,}$$
$$\dot{\theta}(t) = -\rho(\theta(t) - \theta_{\text{ext}}) + \beta, \qquad \text{if the heater is on.}$$

Introduction

Examples of switched systems

Example: simple model for the average temperature of a heated room.

$$\begin{aligned} \dot{\theta}(t) &= -\rho(\theta(t) - \theta_{\text{ext}}), && \text{if the heater is off,} \\ \dot{\theta}(t) &= -\rho(\theta(t) - \theta_{\text{ext}}) + \beta, && \text{if the heater is on.} \end{aligned}$$

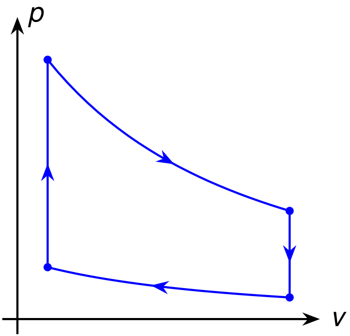
Switched system:

$$\dot{\theta}(t) = -\rho(\theta(t) - \theta_{\text{ext}}) + \beta\sigma(t), \qquad \sigma(t) \in \{0, 1\}.$$

Introduction

Examples of switched systems

Example: simple model for a petrol engine (see [Eastop, McConkey; 1993]).

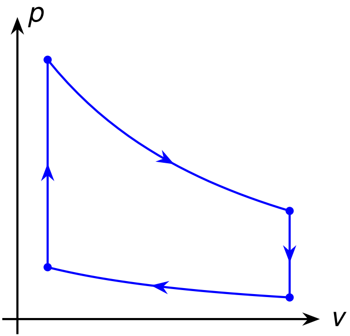


- Four cycles: isentropic compression / expansion, isovolumetric heating / cooling
- Different dynamics for each cycle.

Introduction

Examples of switched systems

Example: simple model for a petrol engine (see [Eastop, McConkey; 1993]).



- Four cycles: isentropic compression / expansion, isovolumetric heating / cooling
- Different dynamics for each cycle.

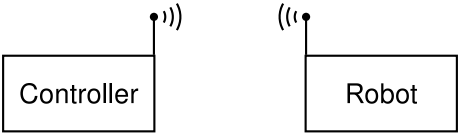
$$\dot{x}(t) = f_{\sigma(t)}(x(t)),$$

$$\sigma(t) \in \{1, 2, 3, 4\}.$$

Introduction

Examples of switched systems

Example: robot controlled by a wireless network (see [Jungers, Heemels; 2015]).



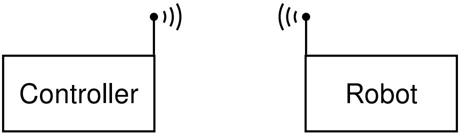
$$\dot{x}(t) = f(x(t), u(t))$$

- $u(t)$: control input sent via a wireless network to the robot.
- $\dot{x}(t) = f(x(t), 0)$: "natural" dynamics of the robot.

Introduction

Examples of switched systems

Example: robot controlled by a wireless network (see [Jungers, Heemels; 2015]).



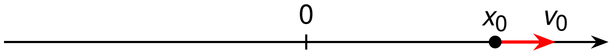
$$\dot{x}(t) = f(x(t), u(t))$$

- $u(t)$: control input sent via a wireless network to the robot.
- $\dot{x}(t) = f(x(t), 0)$: "natural" dynamics of the robot.
- Failures in the network:
$$\dot{x}(t) = f(x(t), \sigma(t)u(t)), \qquad \sigma(t) \in \{0, 1\}.$$

Introduction

Examples of switched systems

Example: optimal control of a point unit mass.

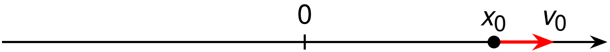


- Dynamics: $\ddot{x}(t) = u(t)$, $u(t) \in [-1, 1]$.
- Goal: move the mass to the origin *at rest* in minimal time.

Introduction

Examples of switched systems

Example: optimal control of a point unit mass.

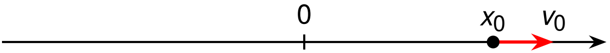


- Dynamics: $\ddot{x}(t) = u(t)$, $u(t) \in [-1, 1]$.
- Goal: move the mass to the origin *at rest* in minimal time.
- Optimal choice of u (for $x_0 > 0$ and v_0 "not too negative"): $u(t) = -1$ up to a certain time t_*, then $u(t) = 1$ until it arrives at the origin at rest.
- System with optimal control: switched system between $\ddot{x}(t) = -1$ and $\ddot{x}(t) = 1$.

Introduction

Examples of switched systems

Example: optimal control of a point unit mass.



- Dynamics: $\ddot{x}(t) = u(t)$, $u(t) \in [-1, 1]$.
- Goal: move the mass to the origin *at rest* in minimal time.
- Optimal choice of u (for $x_0 > 0$ and v_0 "not too negative"): $u(t) = -1$ up to a certain time t_*, then $u(t) = 1$ until it arrives at the origin at rest.
- System with optimal control: switched system between $\ddot{x}(t) = -1$ and $\ddot{x}(t) = 1$.
- Very often, optimal control problems lead to switched systems!
- More on optimal control: M2 course (2^{nd} semester) "Geometric Control".

Introduction

Switching signals

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \qquad \sigma(t) \in [\![1, N]\!].$$

The switching signal σ can be:

- controlled or uncontrolled;

Introduction

Switching signals

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \qquad \sigma(t) \in [\![1, N]\!].$$

The switching signal σ can be:

- controlled or uncontrolled;
- constrained or arbitratry;

Introduction
Switching signals

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \qquad \sigma(t) \in [\![1, N]\!].$$

The switching signal σ can be:

- controlled or uncontrolled;
- constrained or arbitratry;
- state-dependent or time-dependent;

Introduction

Switching signals

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \qquad \sigma(t) \in [\![1, N]\!].$$

The switching signal σ can be:

- controlled or uncontrolled;
- constrained or arbitratry;
- state-dependent or time-dependent;
- random or deterministic.

Introduction

Switching signals

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \qquad \sigma(t) \in [\![1, N]\!].$$

The switching signal σ can be:

- controlled or uncontrolled;
- constrained or arbitratry;
- state-dependent or time-dependent;
- random or deterministic.

In this talk: uncontrolled, time-dependent, arbitrary, deterministic switching signals.

Introduction
Stability of switched systems

In this talk, we focus on the stability of the origin of linear switched systems:

$$\dot{x}(t) = A_{\sigma(t)}x(t).$$

Introduction

Stability of switched systems

In this talk, we focus on the stability of the origin of linear switched systems:

$$\dot{x}(t) = A_{\sigma(t)} x(t).$$

Example: $N = 2$,

$$A_1 = \begin{pmatrix} -1 & 9 \\ -1 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -1 & -1 \\ 9 & -1 \end{pmatrix}.$$

Introduction

Stability of switched systems

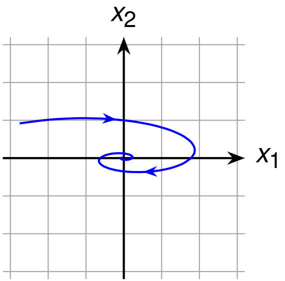
In this talk, we focus on the stability of the origin of linear switched systems:

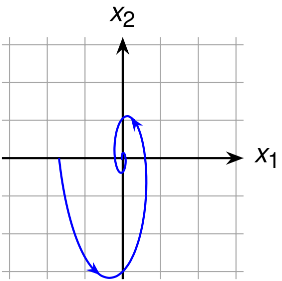
$$\dot{x}(t) = A_{\sigma(t)} x(t).$$

Example: $N = 2$,

$$A_1 = \begin{pmatrix} -1 & 9 \\ -1 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -1 & -1 \\ 9 & -1 \end{pmatrix}.$$

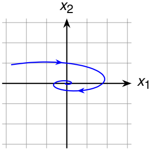
Eigenvalues: $\lambda_{1,2} = -1 \pm 3i$.

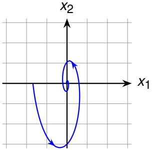


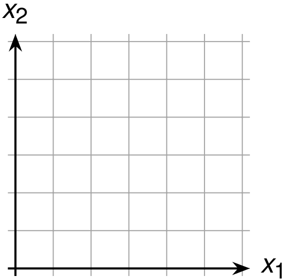


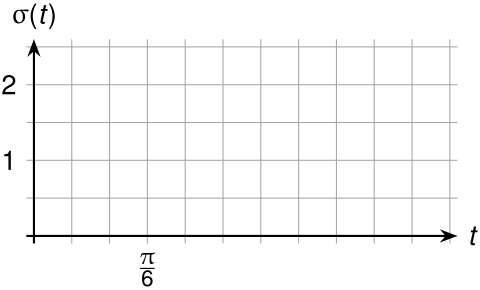
Introduction

Stability of switched systems









Introduction

Stability of switched systems

- Switching between stable subsystems may lead to instability!
- Similarly, unstable subsystems may sometimes be stabilized by a suitable switching law.
- It is important to provide criteria in order to characterize the stability of switched systems.

Stability analysis under arbitrary switching signals

Framework

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

- $\sigma : \mathbb{R} \to [\![1, N]\!]$ can be any piecewise constant function (with finitely many discontinuities on any bounded interval),

Stability analysis under arbitrary switching signals

Framework

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

- $\sigma : \mathbb{R} \to [\![1, N]\!]$ can be any piecewise constant function (with finitely many discontinuities on any bounded interval),
- or, more generally, any function in $L^\infty(\mathbb{R}, [\![1, N]\!])$.

Stability analysis under arbitrary switching signals

Framework

$$\dot{x}(t) = A_{\sigma(t)}x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

- $\sigma : \mathbb{R} \to [\![1, N]\!]$ can be any piecewise constant function (with finitely many discontinuities on any bounded interval),
- or, more generally, any function in $L^\infty(\mathbb{R}, [\![1, N]\!])$.
- Goal: provide conditions on $A_1, \ldots, A_N$ such that all solutions of the system converge to 0 for every switching signal σ.
- Obvious necessary condition: all systems $\dot{x}(t) = A_i x(t)$ must be exponentially stable for $i \in [\![1, N]\!]$ (i.e., A_i Hurwitz).

Stability analysis under arbitrary switching signals
Framework

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

- $\sigma : \mathbb{R} \to [\![1, N]\!]$ can be any piecewise constant function (with finitely many discontinuities on any bounded interval),
- or, more generally, any function in $L^\infty(\mathbb{R}, [\![1, N]\!])$.
- Goal: provide conditions on $A_1, \dots, A_N$ such that all solutions of the system converge to 0 for every switching signal σ.
- Obvious necessary condition: all systems $\dot{x}(t) = A_i x(t)$ must be exponentially stable for $i \in [\![1, N]\!]$ (i.e., A_i Hurwitz).
- Remark: Thanks to Fenichel's Uniformity Lemma (see [Colonius, Kliemann; 2000 – Lemma 5.2.7]), asymptotic and exponential stability are equivalent.

Stability analysis under arbitrary switching signals
Commuting matrices

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

Theorem (Narendra, Balakrishnan; 1994)

Assume that $A_1, \ldots, A_N$ are Hurwitz and that $A_i A_j = A_j A_i$ for every $i, j \in [\![1, N]\!]$. Then the system is exponentially stable for all switching signals and a Lyapunov function is given by $V(x) = x^{\mathrm{T}} P x$, where

$$P = \int_0^{+\infty} e^{A_N^{\mathrm{T}} t_N} \cdots \int_0^{+\infty} e^{A_2^{\mathrm{T}} t_2} \int_0^{+\infty} e^{A_1^{\mathrm{T}} t_1} e^{A_1 t_1} \, dt_1 \, e^{A_2 t_2} \, dt_2 \cdots e^{A_N t_N} \, dt_N$$

Stability analysis under arbitrary switching signals

Commuting matrices

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

Theorem (Narendra, Balakrishnan; 1994)

Assume that $A_1, \dots, A_N$ are Hurwitz and that $A_i A_j = A_j A_i$ for every $i, j \in [\![1, N]\!]$. Then the system is exponentially stable for all switching signals and a Lyapunov function is given by $V(x) = x^T P x$, where

$$P = \int_0^{+\infty} e^{A_N^T t_N} \cdots \int_0^{+\infty} e^{A_2^T t_2} \int_0^{+\infty} e^{A_1^T t_1} e^{A_1 t_1} dt_1 \, e^{A_2 t_2} dt_2 \cdots e^{A_N t_N} dt_N$$

Under commutativity, the necessary condition of having all A_i's Hurwitz is also sufficient!

Stability analysis under arbitrary switching signals

Commuting matrices

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

Theorem (Narendra, Balakrishnan; 1994)

Assume that $A_1, \dots, A_N$ are Hurwitz and that $A_i A_j = A_j A_i$ for every $i, j \in [\![1, N]\!]$. Then the system is exponentially stable for all switching signals and a Lyapunov function is given by $V(x) = x^T P x$, where

$$P = \int_0^{+\infty} e^{A_N^T t_N} \dots \int_0^{+\infty} e^{A_2^T t_2} \int_0^{+\infty} e^{A_1^T t_1} e^{A_1 t_1} dt_1 \, e^{A_2 t_2} dt_2 \cdots e^{A_N t_N} dt_N$$

Under commutativity, the necessary condition of having all A_i's Hurwitz is also sufficient!

Stability analysis under arbitrary switching signals

Common quadratic Lyapunov functions

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

- Usual technique for stability analysis: look for a common quadratic Lyapunov function (CQLF) $V(x) = x^{\mathrm{T}} P x$,

Stability analysis under arbitrary switching signals

Common quadratic Lyapunov functions

$$\dot{x}(t) = A_{\sigma(t)}x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

- Usual technique for stability analysis: look for a common quadratic Lyapunov function (CQLF) $V(x) = x^{\mathrm{T}}Px$,
- i.e., look for symmetric positive definite P such that V is a Lyapunov function for every isolated subsystem $\dot{x}(t) = A_i x(t)$, $i \in [\![1, N]\!]$,

Stability analysis under arbitrary switching signals

Common quadratic Lyapunov functions

$$\dot{x}(t) = A_{\sigma(t)}x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

- Usual technique for stability analysis: look for a common quadratic Lyapunov function (CQLF) $V(x) = x^{\mathrm{T}}Px$,
- i.e., look for symmetric positive definite P such that V is a Lyapunov function for every isolated subsystem $\dot{x}(t) = A_i x(t)$, $i \in [\![1, N]\!]$,
- which is equivalent to finding a symmetric positive definite P such that $A_i^{\mathrm{T}}P + PA_i$ is negative definite for every $i \in [\![1, N]\!]$.

Stability analysis under arbitrary switching signals

Common quadratic Lyapunov functions

$$\dot{x}(t) = A_{\sigma(t)}x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

- Usual technique for stability analysis: look for a common quadratic Lyapunov function (CQLF) $V(x) = x^{\mathrm{T}}Px$,
- i.e., look for symmetric positive definite P such that V is a Lyapunov function for every isolated subsystem $\dot{x}(t) = A_i x(t)$, $i \in [\![1, N]\!]$,
- which is equivalent to finding a symmetric positive definite P such that $A_i^{\mathrm{T}}P + PA_i$ is negative definite for every $i \in [\![1, N]\!]$.
- Advantages: one can algorithmically determine if $A_1, \ldots, A_N$ admit a CQLF. Some theoretical criteria also exist.

Stability analysis under arbitrary switching signals

Common quadratic Lyapunov functions

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

- Usual technique for stability analysis: look for a common quadratic Lyapunov function (CQLF) $V(x) = x^{\mathrm{T}} P x$,
- i.e., look for symmetric positive definite P such that V is a Lyapunov function for every isolated subsystem $\dot{x}(t) = A_i x(t)$, $i \in [\![1, N]\!]$,
- which is equivalent to finding a symmetric positive definite P such that $A_i^{\mathrm{T}} P + P A_i$ is negative definite for every $i \in [\![1, N]\!]$.
- Advantages: one can algorithmically determine if $A_1, \ldots, A_N$ admit a CQLF. Some theoretical criteria also exist.
- Major disadvantage: the existence of a CQLF is only a sufficient condition for exponential stability of the switched system.

Stability analysis under arbitrary switching signals

Common quadratic Lyapunov functions

$$\dot{x}(t) = A_{\sigma(t)}x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

Theorem (Liberzon, Hespanha, Morse; 1999)

Let $\mathfrak{g}$ be the Lie algebra generated by $\{A_1, \dots, A_N\}$. If $\mathfrak{g}$ is solvable, then the system admits a CQLF, and it is thus exponentially stable.

Stability analysis under arbitrary switching signals

Common quadratic Lyapunov functions

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

Theorem (Liberzon, Hespanha, Morse; 1999)

Let $\mathfrak{g}$ be the Lie algebra generated by $\{A_1, \dots, A_N\}$. If $\mathfrak{g}$ is solvable, then the system admits a CQLF, and it is thus exponentially stable.

Definition

Let $\mathfrak{g}$ be a Lie algebra. Set $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}]$ for $k \in \mathbb{N}$. We say that $\mathfrak{g}$ is solvable if $\mathfrak{g}^{(k)} = \{0\}$ for some $k \in \mathbb{N}$.

Stability analysis under arbitrary switching signals

Common quadratic Lyapunov functions

$$\dot{x}(t) = A_{\sigma(t)}x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

Theorem (Liberzon, Hespanha, Morse; 1999)

Let $\mathfrak{g}$ be the Lie algebra generated by $\{A_1, \dots, A_N\}$. If $\mathfrak{g}$ is solvable, then the system admits a CQLF, and it is thus exponentially stable.

Definition

Let $\mathfrak{g}$ be a Lie algebra. Set $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(k+1)} = [\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)}]$ for $k \in \mathbb{N}$. We say that $\mathfrak{g}$ is solvable if $\mathfrak{g}^{(k)} = \{0\}$ for some $k \in \mathbb{N}$.

Other criteria: [Shorten, Narendra; 1999 and 2002], [Shorten, Narendra, Mason; 2003], [Gurvits, Shorten, Mason; 2007], [Laffey, Šmigoc; 2007]...
More on Lie algebraic stability criteria: [Agrachev, Liberzon; 2001], [Agrachev, Baryshnikov, Liberzon; 2012].

Stability analysis under arbitrary switching signals
Common quadratic Lyapunov functions

Some exponentially stable systems do not admit CQLF.

Example: $N = 2$, $A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & -10 \\ \frac{1}{10} & -1 \end{pmatrix}$.

This switched system is exponentially stable (see [Dayawansa, Martin; 1999] for a proof), but does not admit a CQLF.

Stability analysis under arbitrary switching signals

Common quadratic Lyapunov functions

Some exponentially stable systems do not admit CQLF.

Example: $N = 2$, $A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & -10 \\ \frac{1}{10} & -1 \end{pmatrix}$.

This switched system is exponentially stable (see [Dayawansa, Martin; 1999] for a proof), but does not admit a CQLF.

- Indeed, if $V(x) = x^T P x$ is a CQLF, then $A_1^T P + P A_1$ and $A_2^T P + P A_2$ are negative definite.

Stability analysis under arbitrary switching signals

Common quadratic Lyapunov functions

Some exponentially stable systems do not admit CQLF.

Example: $N = 2$, $A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & -10 \\ \frac{1}{10} & -1 \end{pmatrix}$.

This switched system is exponentially stable (see [Dayawansa, Martin; 1999] for a proof), but does not admit a CQLF.

- Indeed, if $V(x) = x^{\mathrm{T}} P x$ is a CQLF, then $A_1^{\mathrm{T}} P + P A_1$ and $A_2^{\mathrm{T}} P + P A_2$ are negative definite.
- Write $P = \begin{pmatrix} 1 & q \\ q & r \end{pmatrix}$. Using Sylvester's criterion for positive-definiteness, one must have
$$\frac{(r-3)^2}{8} + q^2 < 1, \qquad \frac{(r-300)^2}{80000} + \frac{q^2}{100} < 1.$$
- These two ellipses do not intersect: no CQLF.

Stability analysis under arbitrary switching signals

Worst trajectory

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

- Another technique for stability analysis: study the worst trajectory.

Stability analysis under arbitrary switching signals

Worst trajectory

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

- Another technique for stability analysis: study the worst trajectory.
- Main idea: for a given initial condition x_0, find the switching signal σ such that the solution goes as far away from the origin as possible, using the techniques of optimal control.
- If such trajectory converges, all other trajectories also converge, and one has stability. Otherwise, the system is unstable.
- Very useful in dimension 2, much harder in dimension ≥ 3.

Stability analysis under arbitrary switching signals

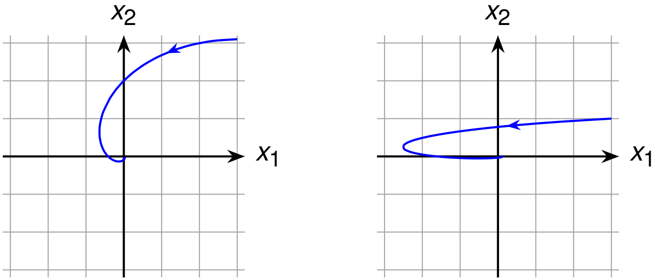
Worst trajectory

Example: $N = 2$, $A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & -10 \\ \frac{1}{10} & -1 \end{pmatrix}$.

Stability analysis under arbitrary switching signals

Worst trajectory

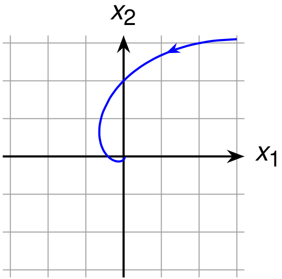
Example: $N = 2$, $A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & -10 \\ \frac{1}{10} & -1 \end{pmatrix}$.

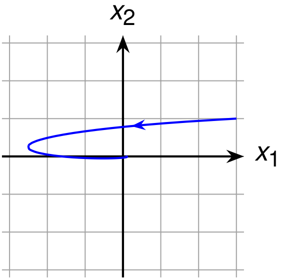


Stability analysis under arbitrary switching signals

Worst trajectory

Example: $N = 2$, $A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & -10 \\ \frac{1}{10} & -1 \end{pmatrix}$.

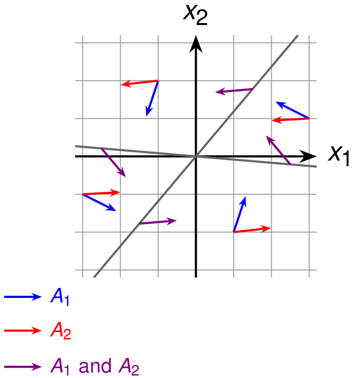




- Both trajectories turn counterclockwise around the origin.
- The "worst trajectory" should be the one chosen such that, at each point, we pick the vector field directed further away from the origin (to justify this: M2 course "Geometric Control").

Stability analysis under arbitrary switching signals

Worst trajectory

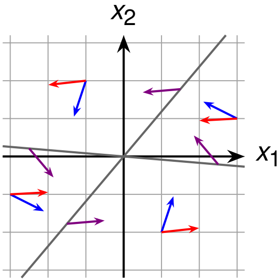


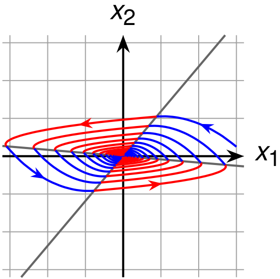
The vector fields are parallel on the lines defined by the equations

$$2x_1 + (\sqrt{161} + 11)x_2 = 0, \qquad 2x_1 - (\sqrt{161} - 11)x_2 = 0.$$

Stability analysis under arbitrary switching signals

Worst trajectory





$\longrightarrow$ A_1

$\longrightarrow$ A_2

$\longrightarrow$ A_1 and A_2

The vector fields are parallel on the lines defined by the equations

$$2x_1 + (\sqrt{161} + 11)x_2 = 0, \qquad 2x_1 - (\sqrt{161} - 11)x_2 = 0.$$

Stability analysis under arbitrary switching signals

Converse Lyapunov theorems

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

Not all exponentially stable switched systems admit CQLFs.

Stability analysis under arbitrary switching signals

Converse Lyapunov theorems

$$\dot{x}(t) = A_{\sigma(t)}x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

Not all exponentially stable switched systems admit CQLFs.

Theorem (Molchanov, Pyatnitskiy; 1989)

The following are equivalent.

1. *The switched system is exponentially stable.*
2. *The system admits a Lyapunov function of the form* $V(x) = x^{\mathrm{T}} P(x) x$*, where* $P(x)^{\mathrm{T}} = P(x) = P(\tau x)$ *for all* $x \in \mathbb{R}^d \setminus \{0\}$*,* $\tau \in \mathbb{R} \setminus \{0\}$*.*

Stability analysis under arbitrary switching signals

Converse Lyapunov theorems

$$\dot{x}(t) = A_{\sigma(t)}x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

Not all exponentially stable switched systems admit CQLFs.

Theorem (Molchanov, Pyatnitskiy; 1989)

The following are equivalent.

1. *The switched system is exponentially stable.*
2. *The system admits a Lyapunov function of the form $V(x) = x^{\mathrm{T}}P(x)x$, where $P(x)^{\mathrm{T}} = P(x) = P(\tau x)$ for all $x \in \mathbb{R}^d \setminus \{0\}$, $\tau \in \mathbb{R} \setminus \{0\}$.*
3. *The system admits a piecewise quadratic Lyapunov function $V(x) = \max_{i \in [\![1,m]\!]} \langle p_i, x\rangle^2$ for some $p_1, \ldots, p_m \in \mathbb{R}^d$ such that $m \geq d$ and $p_1, \ldots, p_m$ span $\mathbb{R}^d$.*

Stability analysis under arbitrary switching signals

Converse Lyapunov theorems

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

Not all exponentially stable switched systems admit CQLFs.

Theorem (Molchanov, Pyatnitskiy; 1989)

The following are equivalent.

1. *The switched system is exponentially stable.*
2. *The system admits a Lyapunov function of the form* $V(x) = x^{\mathsf{T}} P(x) x$, *where* $P(x)^{\mathsf{T}} = P(x) = P(\tau x)$ *for all* $x \in \mathbb{R}^d \setminus \{0\}$, $\tau \in \mathbb{R} \setminus \{0\}$.
3. *The system admits a piecewise quadratic Lyapunov function* $V(x) = \max_{i \in [\![1, m]\!]} \langle p_i, x\rangle^2$ *for some* $p_1, \dots, p_m \in \mathbb{R}^d$ *such that* $m \geq d$ *and* $p_1, \dots, p_m$ *span* $\mathbb{R}^d$.
4. *The system admits a piecewise linear Lyapunov function* $V(x) = \max_{i \in [\![1, m]\!]} \lvert\langle p_i, x\rangle\rvert$ *for some* $p_1, \dots, p_m \in \mathbb{R}^d$ *as before.*

Stability analysis under arbitrary switching signals

Converse Lyapunov theorems

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

Theorem (Mason, Boscain, Chitour; 2006)

1. *The switched system is exponentially stable if and only if it admits a polynomial Lyapunov function.*

Stability analysis under arbitrary switching signals

Converse Lyapunov theorems

$$\dot{x}(t) = A_{\sigma(t)}x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

Theorem (Mason, Boscain, Chitour; 2006)

1. *The switched system is exponentially stable if and only if it admits a polynomial Lyapunov function.*
2. *For every $m \in \mathbb{N}$, there exist exponentially stable switched systems which do not admit polynomial Lyapunov functions of degree $\leq m$.*

Stability analysis under arbitrary switching signals

Converse Lyapunov theorems

$$\dot{x}(t) = A_{\sigma(t)}x(t), \qquad \sigma(t) \in [\![1, N]\!].$$

Theorem (Mason, Boscain, Chitour; 2006)

1. *The switched system is exponentially stable if and only if it admits a polynomial Lyapunov function.*
2. *For every $m \in \mathbb{N}$, there exist exponentially stable switched systems which do not admit polynomial Lyapunov functions of degree $\leq m$.*

Even when one can prove Lyapunov functions exist, they can be hard to compute!

Stability analysis under arbitrary switching signals

Classification of stable planar switched systems with two modes

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad \sigma(t) \in \{1, 2\}, \; x(t) \in \mathbb{R}^2.$$

A complete stability analysis of such system was carried out in [Balde, Boscain, Mason; 2009].

Stability analysis under arbitrary switching signals

Classification of stable planar switched systems with two modes

$$\dot{x}(t) = A_{\sigma(t)} x(t), \qquad \sigma(t) \in \{1, 2\},\ x(t) \in \mathbb{R}^2.$$

A complete stability analysis of such system was carried out in [Balde, Boscain, Mason; 2009].

Define:

$$\delta_X = \mathrm{Tr}(X)^2 - 4\det(X),$$

$$\Gamma(X, Y) = \frac{1}{2}\left(\mathrm{Tr}(X)\,\mathrm{Tr}(Y) - \mathrm{Tr}(XY)\right).$$

Set

$$\tau_i = \begin{cases} \mathrm{Tr}(A_i)/\sqrt{\left|\delta_{A_i}\right|} & \text{if } \delta_{A_1} \neq 0,\ \delta_{A_2} \neq 0, \\ \mathrm{Tr}(A_i)/\sqrt{\left|\delta_{A_j}\right|} & \text{if } \delta_{A_1}\delta_{A_2} = 0 \text{ and } \delta_{A_j} \neq 0, \\ \mathrm{Tr}(A_i)/2 & \text{if } \delta_{A_1} = \delta_{A_2} = 0, \end{cases} \qquad i \in \{1, 2\},$$

Stability analysis under arbitrary switching signals

Classification of stable planar switched systems with two modes

$$k = \frac{2\tau_1\tau_2}{\operatorname{Tr}(A_1)\operatorname{Tr}(A_2)}\left(\operatorname{Tr}(A_1 A_2) - \frac{1}{2}\operatorname{Tr}(A_1)\operatorname{Tr}(A_2)\right),$$

$$\Delta = 4\left(\Gamma(A_1, A_2)^2 - \Gamma(A_1, A_1)\Gamma(A_2, A_2)\right),$$

$$t_i = \begin{cases} \frac{\pi}{2} - \arctan\left(\frac{\operatorname{Tr}(A_1)\operatorname{Tr}(A_2)(k\tau_i + \tau_{3-i})}{2\tau_1\tau_2\sqrt{\Delta}}\right) & \text{if } \delta_{A_i} < 0, \\ \operatorname{arctanh}\left(\frac{2\tau_1\tau_2\sqrt{\Delta}}{\operatorname{Tr}(A_1)\operatorname{Tr}(A_2)(k\tau_i - \tau_{3-i})}\right) & \text{if } \delta_{A_i} > 0, \\ \frac{2\sqrt{\Delta}}{\left(\operatorname{Tr}(A_1 A_2) - \frac{1}{2}\operatorname{Tr}(A_1)\operatorname{Tr}(A_2)\right)\tau_i} & \text{if } \delta_{A_i} = 0, \end{cases}$$

$$R = \frac{2\Gamma(A_1, A_2) + \sqrt{\Delta}}{2\sqrt{\det(A_1)\det(A_2)}} e^{\tau_1 t_1 + \tau_2 t_2}.$$

Stability analysis under arbitrary switching signals

Classification of stable planar switched systems with two modes

Theorem (Balde, Boscain, Mason; 2009)

1. *If* $\Gamma(A_1, A_2) > -\sqrt{\det(A_1)\det(A_2)}$ *and* $\mathrm{Tr}(A_1 A_2) > -2\sqrt{\det(A_1)\det(A_2)}$, *then the switched system admits a CQLF and is thus exponentially stable.*

Stability analysis under arbitrary switching signals

Classification of stable planar switched systems with two modes

Theorem (Balde, Boscain, Mason; 2009)

1. *If* $\Gamma(A_1, A_2) > -\sqrt{\det(A_1)\det(A_2)}$ *and* $\mathrm{Tr}(A_1 A_2) > -2\sqrt{\det(A_1)\det(A_2)}$, *then the switched system admits a CQLF and is thus exponentially stable.*
2. *If* $\Gamma(A_1, A_2) < -\sqrt{\det(A_1)\det(A_2)}$, *then the switched system is unstable.*

Stability analysis under arbitrary switching signals

Classification of stable planar switched systems with two modes

Theorem (Balde, Boscain, Mason; 2009)

1. *If* $\Gamma(A_1, A_2) > -\sqrt{\det(A_1)\det(A_2)}$ *and* $\operatorname{Tr}(A_1 A_2) > -2\sqrt{\det(A_1)\det(A_2)}$, *then the switched system admits a CQLF and is thus exponentially stable.*
2. *If* $\Gamma(A_1, A_2) < -\sqrt{\det(A_1)\det(A_2)}$, *then the switched system is unstable.*
3. *If* $\Gamma(A_1, A_2) = -\sqrt{\det(A_1)\det(A_2)}$, *then the system is stable, but not asymptotically stable.*

Stability analysis under arbitrary switching signals

Classification of stable planar switched systems with two modes

Theorem (Balde, Boscain, Mason; 2009)

1. *If* $\Gamma(A_1, A_2) > -\sqrt{\det(A_1)\det(A_2)}$ *and* $\mathrm{Tr}(A_1 A_2) > -2\sqrt{\det(A_1)\det(A_2)}$, *then the switched system admits a CQLF and is thus exponentially stable.*
2. *If* $\Gamma(A_1, A_2) < -\sqrt{\det(A_1)\det(A_2)}$, *then the switched system is unstable.*
3. *If* $\Gamma(A_1, A_2) = -\sqrt{\det(A_1)\det(A_2)}$, *then the system is stable, but not asymptotically stable.*
4. *If* $\Gamma(A_1, A_2) > \sqrt{\det(A_1)\det(A_2)}$ *and* $\mathrm{Tr}(A_1 A_2) \leq -2\sqrt{\det(A_1)\det(A_2)}$, *then the system is exponentially stable if* $R < 1$, *stable but not asymptotically stable if* $R = 1$, *and unstable if* $R > 1$.

Conclusion

To go further

- This talk has only discussed uncontrolled switching signals. The case of controlled switching is of much importance, in particular stabilization by switching.

Conclusion
To go further

- This talk has only discussed uncontrolled switching signals. The case of controlled switching is of much importance, in particular stabilization by switching.
- Random models for switching signals are also of much importance in practical situations, as well as state-dependent switching.

Conclusion

To go further

- This talk has only discussed uncontrolled switching signals. The case of controlled switching is of much importance, in particular stabilization by switching.
- Random models for switching signals are also of much importance in practical situations, as well as state-dependent switching.
- We have only considered linear systems. The case of non-linear systems (with different equilibrium points!) is also of much importance.

Conclusion
To go further

- This talk has only discussed uncontrolled switching signals. The case of controlled switching is of much importance, in particular stabilization by switching.
- Random models for switching signals are also of much importance in practical situations, as well as state-dependent switching.
- We have only considered linear systems. The case of non-linear systems (with different equilibrium points!) is also of much importance.
- The theory of switched systems is very rich and still developing, with much work to be done.

Conclusion

References

Books:

- Liberzon, Daniel. Switching in systems and control. Systems & Control: Foundations & Applications. Birkhäuser Boston, 2003.
- Z. Sun and S. S. Ge. Switched Linear Systems. Control and Design. Communications and Control Engineering. Springer-Verlag, 2005.

Surveys:

- H. Lin and P. J. Antsaklis. Stability and stabilizability of switched linear systems: a survey of recent results. IEEE Trans. Automat. Control, 54(2):308–322, 2009.
- H. Lin and P. J. Antsaklis. Hybrid dynamical systems: Stability and stabilization. In W. S. Levine, editor, The Control Handbook: Control System Advanced Methods, chap. 30. CRC Press, 2nd ed., 2010.
- R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King. Stability criteria for switched and hybrid systems. SIAM Rev., 49(4):545–592, 2007.