An introduction to switched systems

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Outline

1. Introduction
   - Switched systems
   - Examples of switched systems
   - Switching signals
   - Stability of switched systems

2. Stability analysis under arbitrary switching signals
   - Framework
   - Commuting matrices
   - Common quadratic Lyapunov functions
   - Worst trajectory
   - Converse Lyapunov theorems
   - Classification of stable planar switched systems with two modes

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   - To go further
   - References
Introduction
Switched systems

Classical setting for autonomous differential equations:
\[ \dot{x}(t) = f(x(t)), \]
\[ f : \mathbb{R}^d \rightarrow \mathbb{R}^d \] a locally Lipschitz function.
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Switched system: equation under the form,
\[ \dot{x}(t) = f_{\sigma(t)}(x(t)), \]
where one has \( N \) (smooth) vector fields \( f_1, \ldots, f_N \) and a (piecewise constant) switching signal \( \sigma : \mathbb{R}_+ \to [1, N] \).
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Interaction between the continuous variable \( x \in \mathbb{R}^d \) and the discrete state \( \sigma \in [1, N] \).
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Important models in physics, engineering, applied maths...
Example: simple model for the average temperature of a heated room.

\[ \dot{\theta}(t) = -\rho(\theta(t) - \theta_{\text{ext}}), \quad \text{if the heater is off,} \]
\[ \dot{\theta}(t) = -\rho(\theta(t) - \theta_{\text{ext}}) + \beta, \quad \text{if the heater is on.} \]
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Switched system:
\[ \dot{\theta}(t) = -\rho(\theta(t) - \theta_{ext}) + \beta \sigma(t), \quad \sigma(t) \in \{0, 1\}. \]
Introduction
Examples of switched systems

**Example:** simple model for a petrol engine (see [Eastop, McConkey; 1993]).

- Four cycles: isentropic compression / expansion, isovolumetric heating / cooling
- Different dynamics for each cycle.
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- Different dynamics for each cycle.

\[
\dot{x}(t) = f_{\sigma(t)}(x(t)),
\]

\[\sigma(t) \in \{1, 2, 3, 4\}.\]
**Example**: robot controlled by a wireless network (see [Jungers, Heemels; 2015]).

\[ \dot{x}(t) = f(x(t), u(t)) \]

- \( u(t) \): control input sent via a wireless network to the robot.
- \( \dot{x}(t) = f(x(t), 0) \): “natural” dynamics of the robot.
Introduction

Examples of switched systems

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- \(u(t)\): control input sent via a wireless network to the robot.
- \(\dot{x}(t) = f(x(t), 0)\): “natural” dynamics of the robot.
- Failures in the network:
  \[
  \dot{x}(t) = f(x(t), \sigma(t)u(t)), \quad \sigma(t) \in \{0, 1\}.
  \]
Example: optimal control of a point unit mass.

- Dynamics: $\ddot{x}(t) = u(t)$, $u(t) \in [-1, 1]$.
- Goal: move the mass to the origin at rest in minimal time.
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- Goal: move the mass to the origin at rest in minimal time.
- Optimal choice of $u$ (for $x_0 > 0$ and $v_0$ “not too negative”): $u(t) = -1$ up to a certain time $t_\ast$, then $u(t) = 1$ until it arrives at the origin at rest.
- System with optimal control: switched system between $\ddot{x}(t) = -1$ and $\ddot{x}(t) = 1$. 

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Example: **optimal control** of a point unit mass.

- **Dynamics:** $\ddot{x}(t) = u(t)$, $u(t) \in [-1, 1]$.
- **Goal:** move the mass to the origin *at rest* in minimal time.
- **Optimal choice of $u$ (for $x_0 > 0$ and $v_0$ “not too negative”):** $u(t) = -1$ up to a certain time $t_*$, then $u(t) = 1$ until it arrives at the origin at rest.
- **System with optimal control:** switched system between $\ddot{x}(t) = -1$ and $\ddot{x}(t) = 1$.
- **Very often, optimal control problems lead to switched systems!**
- **More on optimal control:** M2 course (2\textsuperscript{nd} semester) “Geometric Control”.

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Introduction
Switching signals

\[ \dot{x}(t) = f_{\sigma(t)}(x(t)), \quad \sigma(t) \in [1, N]. \]

The switching signal \( \sigma \) can be:

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The **switching signal** \( \sigma \) can be:

- **controlled** or **uncontrolled**;
- **constrained** or **arbitrary**;
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- **random** or **deterministic**.

In this talk: uncontrolled, time-dependent, arbitrary, deterministic switching signals.
In this talk, we focus on the stability of the origin of linear switched systems:

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Example: \( N = 2, \)
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A_1 = \begin{pmatrix} -1 & 9 \\ -1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & -1 \\ 9 & -1 \end{pmatrix}.
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In this talk, we focus on the stability of the origin of linear switched systems:

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Eigenvalues: $\lambda_{1,2} = -1 \pm 3i$. 

![Graph of system dynamics](image-url)
Introduction

Stability of switched systems
Switching between stable subsystems may lead to instability!

Similarly, unstable subsystems may sometimes be stabilized by a suitable switching law.

It is important to provide criteria in order to characterize the stability of switched systems.
Stability analysis under arbitrary switching signals

Framework

\[ \dot{x}(t) = A_{\sigma(t)} x(t), \quad \sigma(t) \in [1, N]. \]

- \(\sigma : \mathbb{R} \to [1, N]\) can be any piecewise constant function (with finitely many discontinuities on any bounded interval),

\[\sigma(t) \in J_1, N^K.\]
Stability analysis under arbitrary switching signals

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- or, more generally, any function in \( L^\infty(\mathbb{R}, [1, N]) \).
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- or, more generally, any function in \( L^\infty(\mathbb{R}, \left[1, N\right]) \).
- Goal: provide conditions on \( A_1, \ldots, A_N \) such that all solutions of the system converge to 0 for every switching signal \( \sigma \).
- Obvious necessary condition: all systems \( \dot{x}(t) = A_i x(t) \) must be exponentially stable for \( i \in \left[1, N\right] \) (i.e., \( A_i \) Hurwitz).
Introduction

Stability analysis

Conclusion

Stability analysis under arbitrary switching signals

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- **Remark:** Thanks to Fenichel’s Uniformity Lemma (see [Colonius, Kliemann; 2000 – Lemma 5.2.7]), asymptotic and exponential stability are equivalent.
Stability analysis under arbitrary switching signals

Commuting matrices

\[ \dot{x}(t) = A_{\sigma(t)} x(t), \quad \sigma(t) \in [1, N]. \]

**Theorem (Narendra, Balakrishnan; 1994)**

Assume that \(A_1, \ldots, A_N\) are Hurwitz and that \(A_i A_j = A_j A_i\) for every \(i, j \in [1, N]\). Then the system is exponentially stable for all switching signals and a Lyapunov function is given by

\[ V(x) = x^T P x, \quad \text{where} \]

\[ P = \int_{0}^{+\infty} e^{A_N^T t_N} \cdots \int_{0}^{+\infty} e^{A_2^T t_2} \int_{0}^{+\infty} e^{A_1^T t_1} e^{A_1 t_1} dt_1 e^{A_2 t_2} dt_2 \cdots e^{A_N t_N} dt_N \]
Stability analysis under arbitrary switching signals

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Under commutativity, the necessary condition of having all \( A_i \)'s Hurwitz is also sufficient!
Stability analysis under arbitrary switching signals

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Stability analysis under arbitrary switching signals

Common quadratic Lyapunov functions

\[ \dot{x}(t) = A_{\sigma(t)}x(t), \quad \sigma(t) \in [1, N]. \]

- Usual technique for stability analysis: look for a common quadratic Lyapunov function (CQLF) \( V(x) = x^T P x, \)

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- i.e., look for symmetric positive definite \( P \) such that \( V \) is a Lyapunov function for every isolated subsystem \( \dot{x}(t) = A_i x(t), \)
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**Stability analysis under arbitrary switching signals**

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- which is equivalent to finding a symmetric positive definite \( P \) such that \( A_i^T P + PA_i \) is negative definite for every \( i \in [1, N] \).
Stability analysis under arbitrary switching signals
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- Advantages: one can algorithmically determine if \( A_1, \ldots, A_N \) admit a CQLF. Some theoretical criteria also exist.
Stability analysis under arbitrary switching signals

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  - which is equivalent to finding a symmetric positive definite \( P \) such that \( A_i^TP + PA_i \) is negative definite for every \( i \in [1, N]. \)
- Advantages: one can algorithmically determine if \( A_1, \ldots, A_N \) admit a CQLF. Some theoretical criteria also exist.
- Major disadvantage: the existence of a CQLF is only a sufficient condition for exponential stability of the switched system.
Stability analysis under arbitrary switching signals

Common quadratic Lyapunov functions

\[ \dot{x}(t) = A_{\sigma(t)}x(t), \quad \sigma(t) \in [1, N]. \]

**Theorem (Liberzon, Hespanha, Morse; 1999)**

Let \( \mathfrak{g} \) be the Lie algebra generated by \( \{A_1, \ldots, A_N\} \). If \( \mathfrak{g} \) is solvable, then the system admits a CQLF, and it is thus exponentially stable.
Stability analysis under arbitrary switching signals

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**Definition**

Let \( g \) be a Lie algebra. Set \( g^{(0)} = g \), \( g^{(k+1)} = [g^{(k)}, g^{(k)}] \) for \( k \in \mathbb{N} \). We say that \( g \) is **solvable** if \( g^{(k)} = \{0\} \) for some \( k \in \mathbb{N} \).
Stability analysis under arbitrary switching signals
Common quadratic Lyapunov functions

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Other criteria: [Shorten, Narendra; 1999 and 2002], [Shorten, Narendra, Mason; 2003], [Gurvits, Shorten, Mason; 2007], [Laffey, Šmigoc; 2007]...

More on Lie algebraic stability criteria: [Agrachev, Liberzon; 2001], [Agrachev, Baryshnikov, Liberzon; 2012].
Some exponentially stable systems do not admit CQLF.

**Example:** $N = 2$, $A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & -10 \\ 1 \cdot 10 & -1 \end{pmatrix}$.

This switched system is exponentially stable (see [Dayawansa, Martin; 1999] for a proof), but does not admit a CQLF.
Stability analysis under arbitrary switching signals
Common quadratic Lyapunov functions

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- Indeed, if $V(x) = x^TPx$ is a CQLF, then $A_1^TP + PA_1$ and $A_2^TP + PA_2$ are negative definite.

- Write $P = \begin{pmatrix} 1 & q \\ q & r \end{pmatrix}$. Using Sylvester’s criterion for positive-definiteness, one must have

\[
\frac{(r - 3)^2}{8} + q^2 < 1, \quad \frac{(r - 300)^2}{80000} + \frac{q^2}{100} < 1.
\]

- These two ellipses do not intersect: no CQLF.
Stability analysis under arbitrary switching signals

Worst trajectory

\[ \dot{x}(t) = A_{\sigma(t)} x(t), \quad \sigma(t) \in [1, N]. \]

Another technique for stability analysis: study the worst trajectory.
Stability analysis under arbitrary switching signals

Worst trajectory

\[ \dot{x}(t) = A_{\sigma(t)}x(t), \quad \sigma(t) \in [1, N]. \]

- Another technique for stability analysis: study the worst trajectory.
- Main idea: for a given initial condition \( x_0 \), find the switching signal \( \sigma \) such that the solution goes as far away from the origin as possible, using the techniques of optimal control.
- If such trajectory converges, all other trajectories also converge, and one has stability. Otherwise, the system is unstable.
- Very useful in dimension 2, much harder in dimension \( \geq 3 \).
Stability analysis under arbitrary switching signals

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Stability analysis under arbitrary switching signals

Worst trajectory

Example: \( N = 2, A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & -10 \\ \frac{1}{10} & -1 \end{pmatrix}. \)
Stability analysis under arbitrary switching signals

Worst trajectory

Example: \( N = 2, \ A_1 = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \ A_2 = \begin{pmatrix} -1 & -10 \\ \frac{1}{10} & -1 \end{pmatrix} \).

- Both trajectories turn counterclockwise around the origin.
- The “worst trajectory” should be the one chosen such that, at each point, we pick the vector field directed further away from the origin (to justify this: M2 course “Geometric Control”).
Stability analysis under arbitrary switching signals

Worst trajectory

The vector fields are parallel on the lines defined by the equations

\[ 2x_1 + (\sqrt{161} + 11)x_2 = 0, \quad 2x_1 - (\sqrt{161} - 11)x_2 = 0. \]
Stability analysis under arbitrary switching signals

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Stability analysis under arbitrary switching signals

Converse Lyapunov theorems

\[ \dot{x}(t) = A_{\sigma(t)}x(t), \quad \sigma(t) \in [1, N]. \]

Not all exponentially stable switched systems admit CQLFs.

Theorem (Molchanov, Pyatnitskiy; 1989)

The following are equivalent.
1. The switched system is exponentially stable.
2. The system admits a Lyapunov function of the form
   \[ V(x) = x^T P(x) x, \]
   where \( P(\tau x) = P(x) \) for all \( x \in \mathbb{R}^d \setminus \{0\} \), \( \tau \in \mathbb{R} \setminus \{0\} \).
3. The system admits a piecewise quadratic Lyapunov function
   \[ V(x) = \max_{i \in J_1, m} \langle p_i, x \rangle^2 \]
   for some \( p_1, \ldots, p_m \in \mathbb{R}^d \) such that \( m \geq d \) and \( p_1, \ldots, p_m \) span \( \mathbb{R}^d \).
4. The system admits a piecewise linear Lyapunov function
   \[ V(x) = \max_{i \in J_1, m} |\langle p_i, x \rangle| \]
   for some \( p_1, \ldots, p_m \in \mathbb{R}^d \) as before.
Stability analysis under arbitrary switching signals
Converse Lyapunov theorems

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3. **The system admits a piecewise quadratic Lyapunov function** \( V(x) = \max_{i \in [1, m]} \langle p_i, x \rangle^2 \) **for some** \( p_1, \ldots, p_m \in \mathbb{R}^d \) **such that** \( m \geq d \) **and** \( p_1, \ldots, p_m \) **span** \( \mathbb{R}^d \).
Stability analysis under arbitrary switching signals
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2. The system admits a Lyapunov function of the form \( V(x) = x^\top P(x)x \), where \( P(x)^\top = P(x) = P(\tau x) \) for all \( x \in \mathbb{R}^d \setminus \{0\}, \tau \in \mathbb{R} \setminus \{0\} \).
3. The system admits a piecewise quadratic Lyapunov function \( V(x) = \max_{i \in [1,m]} \langle p_i, x \rangle^2 \) for some \( p_1, \ldots, p_m \in \mathbb{R}^d \) such that \( m \geq d \) and \( p_1, \ldots, p_m \) span \( \mathbb{R}^d \).
4. The system admits a piecewise linear Lyapunov function \( V(x) = \max_{i \in [1,m]} |\langle p_i, x \rangle| \) for some \( p_1, \ldots, p_m \in \mathbb{R}^d \) as before.
Stability analysis under arbitrary switching signals
Converse Lyapunov theorems

\[ \dot{x}(t) = A_{\sigma(t)}x(t), \quad \sigma(t) \in [1, N]. \]

**Theorem (Mason, Boscain, Chitour; 2006)**

The switched system is exponentially stable if and only if it admits a polynomial Lyapunov function.
\[ \dot{x}(t) = A_{\sigma(t)}x(t), \quad \sigma(t) \in [1, N]. \]

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1. The switched system is exponentially stable if and only if it admits a polynomial Lyapunov function.

2. For every \( m \in \mathbb{N} \), there exist exponentially stable switched systems which do not admit polynomial Lyapunov functions of degree \( \leq m \).
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Even when one can prove Lyapunov functions exist, they can be hard to compute!
Stability analysis under arbitrary switching signals
Classification of stable planar switched systems with two modes

\[ \dot{x}(t) = A_{\sigma(t)} x(t), \quad \sigma(t) \in \{1, 2\}, \ x(t) \in \mathbb{R}^2. \]

A complete stability analysis of such system was carried out in [Balde, Boscain, Mason; 2009].
Stability analysis under arbitrary switching signals
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Define:

\[ \delta_X = \text{Tr}(X)^2 - 4 \det(X), \]
\[ \Gamma(X, Y) = \frac{1}{2} (\text{Tr}(X) \text{Tr}(Y) - \text{Tr}(XY)). \]

Set

\[ \tau_i = \begin{cases} 
\text{Tr}(A_i)/\sqrt{|\delta_{A_i}|} & \text{if } \delta_{A_1} \neq 0, \delta_{A_2} \neq 0, \\
\text{Tr}(A_i)/\sqrt{|\delta_{A_j}|} & \text{if } \delta_{A_1} \delta_{A_2} = 0 \text{ and } \delta_{A_j} \neq 0, \\
\text{Tr}(A_i)/2 & \text{if } \delta_{A_1} = \delta_{A_2} = 0,
\end{cases} \quad i \in \{1, 2\}, \]
Stability analysis under arbitrary switching signals
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\[ k = \frac{2\tau_1 \tau_2}{\text{Tr}(A_1) \text{Tr}(A_2)} \left( \text{Tr}(A_1 A_2) - \frac{1}{2} \text{Tr}(A_1) \text{Tr}(A_2) \right), \]

\[ \Delta = 4 \left( \Gamma(A_1, A_2)^2 - \Gamma(A_1, A_1) \Gamma(A_2, A_2) \right), \]

\[ t_i = \begin{cases} 
\frac{\pi}{2} - \arctan \left( \frac{\text{Tr}(A_1) \text{Tr}(A_2)(k\tau_i + \tau_{3-i})}{2\tau_1 \tau_2 \sqrt{\Delta}} \right) & \text{if } \delta_{A_i} < 0, \\
\arctanh \left( \frac{2\tau_1 \tau_2 \sqrt{\Delta}}{\text{Tr}(A_1) \text{Tr}(A_2)(k\tau_i - \tau_{3-i})} \right) & \text{if } \delta_{A_i} > 0, \\
\frac{2 \sqrt{\Delta}}{\left( \text{Tr}(A_1 A_2) - \frac{1}{2} \text{Tr}(A_1) \text{Tr}(A_2) \right) \tau_i} & \text{if } \delta_{A_i} = 0,
\end{cases} \]

\[ R = \frac{2\Gamma(A_1, A_2) + \sqrt{\Delta}}{2\sqrt{\det(A_1) \det(A_2)}} e^{\tau_1 t_1 + \tau_2 t_2}. \]
Stability analysis under arbitrary switching signals
Classification of stable planar switched systems with two modes

Theorem (Balde, Boscain, Mason; 2009)

If $\Gamma(A_1, A_2) > -\sqrt{\det(A_1) \det(A_2)}$ and
$\text{Tr}(A_1 A_2) > -2\sqrt{\det(A_1) \det(A_2)}$, then the switched system
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Stability analysis under arbitrary switching signals
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Stability analysis under arbitrary switching signals
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4. If $\Gamma(A_1, A_2) > \sqrt{\det(A_1) \det(A_2)}$ and $\text{Tr}(A_1 A_2) \leq -2\sqrt{\det(A_1) \det(A_2)}$, then the system is exponentially stable if $R < 1$, stable but not asymptotically stable if $R = 1$, and unstable if $R > 1$. 

An introduction to switched systems

Guilherme Mazanti
This talk has only discussed uncontrolled switching signals. The case of controlled switching is of much importance, in particular stabilization by switching.
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We have only considered linear systems. The case of non-linear systems (with different equilibrium points!) is also of much importance.
To go further

- This talk has only discussed uncontrolled switching signals. The case of controlled switching is of much importance, in particular stabilization by switching.

- Random models for switching signals are also of much importance in practical situations, as well as state-dependent switching.

- We have only considered linear systems. The case of non-linear systems (with different equilibrium points!) is also of much importance.

- The theory of switched systems is very rich and still developing, with much work to be done.
Conclusion

References

Books:


Surveys:
