On the variational formulation of some Mean field game systems

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Mean field games (MFG) theory analyzes :

- optimal control problems
- with (infinitely) many identical controlers

In other words it is a mathematical modeling approach to continuous-time systems which involve a large number of agents

MFG were introduced by Lasry and Lions and by Huang, Caines and Malhamé in 2006 to describe Nash equilibria in differential games with infinitely many players

Typical features of the model :

- players act according to the same principles (they are indistinguishable and have the same optimization criteria)
- players have individually a minor (infinitesimal) influence, but their strategy takes into account the mass of co-players

Roughly : players are particles but have strategies (more sophisticated than particle physics or similar economics models); however, they only consider the statistical state of the mass of co-players (less sophisticated than agents of N-players games)

Goal : introduce a macroscopic description through a mean field approach as the number of players $N \to +\infty$

Motivations :

- Problems arising in economy :
  - financial markets (Price formation and dynamic equilibria, Formation of volatility) (Lasry, Lions, 2006)
  - general economic equilibrium with rational expectations (Guéant, Lasry, and Lions, 2007)
- Dynamics of population models :
  - crowd motion : mexican wave "la ola", ... (Guéant, Lasry, Lions - Lachapelle, ...)
  - academic behavior (Besancenot, Courtault, El Dika...)
- Engineering literature : Large Population Stochastic Wireless Power Control Problem (Huang, Caines, Malhamé, 2003, Mériaux, Lasaulce...)

Different approaches

- limit of N-player (stochastic) differential games as $N \to +\infty$
  $\to$ analogy with the Mean Field theories in statistical physics (kinetic theory of gases, Boltzmann and Vlasov equations) and quantum mechanics and quantum Chemistry (Hartree-Fock models...)

- direct definition of (stochastic) differential games with infinitely many identical players (applications to N-player games),
  $\to$ approach from game theory

- potential games : games arising as necessary conditions for optimal control problems of PDE equations
  $\to$ related to optimal transportation problems

The 3 approaches yield the same MFG system with unknown (ϕ, m) written here in a second order form with degenerate diffusion and a local coupling

$$\begin{cases} -\partial_t \phi - A_{ij}\partial_{ij}\phi + H(x, D\phi) = f(x, m(x,t)) \\ \partial_t m - \partial_{ij}(A_{ij} m) - \mathrm{div}(m D_p H(x, D\phi)) = 0 \\ m(0) = m_0, \ \phi(x, T) = \phi_T(x) \end{cases}$$

where

- $A : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is symmetric and nonnegative
- the Hamiltonian $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is convex in the second variable
- the coupling $f : \mathbb{R}^d \times [0, +\infty) \to [0, +\infty)$ is increasing with respect to the second variable
- m_0 is a probability density
- $\phi_T : \mathbb{R}^d \to \mathbb{R}$ is a given function

The optimal control problem of a typical small player

$\phi(t,x)$ is the value function

Dynamics :

$$dX_s = v_s ds + \Sigma(X_s)dB_s,$$

where (v_s) is the control, (B_s) is a Brownian motion and $\Sigma\Sigma^T = A$

Cost :

$$\mathbb{E}\left[\int_0^T H^*(X_s, -v_s) + f(X_s, m(s, X_s))\, ds + \phi_T(X_T)\right]$$

$\forall t \in [0, T]$ $m(t,x)$ denotes the density of population of small players at position x

the optimal control is formally given by the feedback $(t,x) \to -D_pH(x, D\phi(t,x))$

the second equation in (MFG) is the Kolmogorov equation of the process (X_s) when the small player plays in an optimal way

In the MFG systems with uniformly parabolic diffusions (typically $A_{ij}\partial_{ij}\phi = \Delta\phi$) the solutions are smooth, (at least if the coupling is nonlocal and regularizing or if it has a “small growth")

Analysis by PDE methods : see Cardaliaguet, Lasry, Lions and Porretta (2012), Lasry and Lions (2006, 2007), Gomes, Pimentel and Sánchez-Morgado (2013), Lions (2011), Porretta (2013)

Analysis by stochastic techniques : see Carmona and Delarue (2013), Huang, Malhamé and Caines (2006)

The case of couplings with an arbitrary growth has been discussed in Cardaliaguet, Lasry, Lions and Porretta (2012) (for quadratic hamiltonians and smooth solutions) and in Porretta (2013) (for more general hamiltonians but weak solutions)

Since we consider degenerate parabolic equations we face a lack of regularity $\Rightarrow$ break down of the uniformly parabolic techniques

IDEA : use convex optimization methods from optimal transport problems (see Benamou and Brenier (2000), Cardaliaguet, Carlier and Nazaret (2012))

These techniques were already used to study first order MFG systems (i.e., $A \equiv 0$): see Cardaliaguet (2013), Cardaliaguet and Graber (2013), Graber (2013)

We show the existence and uniqueness of a weak solution for the degenerate MFG system

Assumptions

(H1) $f : \mathbb{T}^d \times [0, +\infty) \to \mathbb{R}$ is continuous in both variables, increasing wrt the second variable m, and $\exists q > 1, C_1$ s.t.

$$\frac{1}{C_1}|m|^{q-1} - C_1 \leq f(x,m) \leq C_1|m|^{q-1} + C_1 \qquad \forall m \geq 0$$

Moreover $f(x,0) = 0 \qquad \forall x \in \mathbb{T}^d$

(H2) $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous in both variables, convex and differentiable in the second variable, with D_pH continuous in both variable, and $\exists r > 1, C_2 > 0$ s.t.

$$\frac{1}{rC_2}|\xi|^r - C_2 \leq H(x,\xi) \leq \frac{C_2}{r}|\xi|^r + C_2 \qquad \forall (x,\xi) \in \mathbb{T}^d \times \mathbb{R}^d$$

Assumptions

(H3) $A : \mathbb{T}^d \to \mathbb{R}^{d\times d}$ has symmetric nonnegative values and there exists $\Sigma : \mathbb{T}^d \to \mathbb{R}^{d\times d}$ Lipschitz continuous such that $\Sigma\Sigma^T = A$: $\exists C_3 > 0$ s.t.

$$|\Sigma(x) - \Sigma(y)| \le C_3|x-y| \qquad \forall x, y \in \mathbb{T}^d,\ \xi \in \mathbb{R}^d$$

Moreover, either $r \ge p$ or $A \equiv 0$, where $\frac{1}{p} + \frac{1}{q} = 1$

(H4) $\phi_T : \mathbb{T}^d \to \mathbb{R}$ is of class $\mathcal{C}^2$, while $m_0 : \mathbb{T}^d \to \mathbb{R}$ is a C^1 positive density

The Fenchel conjugate H^* of H wrt the second variable

$$H^*(x, w) = \sup_{p \in \mathbb{R}^d} \langle w, p \rangle - H(x, p)$$

is continuous and

$$\frac{1}{r' C_2} |\xi|^{r'} - C_2 \leq H^*(x, \xi) \leq \frac{C_2}{r'} |\xi|^{r'} + C_2 \qquad \forall (x, \xi) \in \mathbb{T}^d \times \mathbb{R}^d \ ,$$

where r' is s.t. : $\dfrac{1}{r} + \dfrac{1}{r'} = 1$

Let $$F(x,m) = \begin{cases} \int_0^m f(x,\tau)d\tau & \text{if } \ m \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

Then F is continuous on $\mathbb{T}^d \times (0,+\infty)$, derivable and strictly convex in m and

$$\frac{1}{qC_1}|m|^q - C_1 \leq F(x,m) \leq \frac{C_1}{q}|m|^q + C_1 \qquad \forall m \geq 0$$

Let F^* be the Fenchel conjugate of F wrt the second variable then $F^*(x,a) = 0$ for $a \leq 0$, moreover,

$$\frac{1}{pC_1}|a|^p - C_1 \leq F^*(x,a) \leq \frac{C_1}{p}|a|^p + C_1 \qquad \forall a \geq 0 \, ,$$

where p is s.t. : $1/p + 1/q = 1$

Optimal control problems in duality

An optimal control problem for a backward HJ equation :

the state variable ϕ is controlled by a distributed control $\alpha : (0, T) \times \mathbb{T}^d \to \mathbb{R}$ in order to minimize the criterium

$$\int_0^T \int_{\mathbb{T}^d} F^* \left(x, \alpha(t,x)\right)\ dxdt - \int_{\mathbb{T}^d} \phi(0,x) dm_0(x)$$

when the state ϕ is driven by

$$\begin{cases} -\partial_t \phi - A_{ij}(x)\partial_{ij}\phi + H(x, D\phi) = \alpha(t,x) \\ \phi(x, T) = \phi_T(x) \end{cases}$$

Precisely : $$\inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi)$$

where $\mathcal{K}_0 := \{\phi \in \mathcal{C}^2([0,T] \times \mathbb{T}^d) |\ \phi(T,x) = \phi_T(x)\}$ and

$$\mathcal{A}(\phi) = \int_0^T \int_{\mathbb{T}^d} F^* \left(x, -\partial_t\phi(t,x) - A_{ij}\partial_{ij}\phi + H(x, D\phi(t,x))\right)\ dxdt - \int_{\mathbb{T}^d} \phi(0,x) dm_0(x)$$

The second optimal control problem is an optimal control problem for a Fokker-Plank equation :

we control the state variable m through a vector field $v : (0, T) \times \mathbb{T}^d \to \mathbb{R}^d$ in order to minimize

$$\int_0^T \int_{\mathbb{T}^d} m(t,x)H^*(x, -v(t,x)) + F(x, m(t,x))\, dxdt + \int_{\mathbb{T}^d} \phi_T(x)m(T,x)dx$$

when m solves the Fokker-Plank equation

$$\partial_t m - \partial_{ij}(A_{ij}(x)m) + \mathrm{div}(mv) = 0$$

with $m(0) = m_0$

Precisely : $$\inf_{(m,w)\in\mathcal{K}_1} \mathcal{B}(m,w)$$

where

$$\mathcal{K}_1 := \Big\{ (m,w) \in L^1((0,T)\times\mathbb{T}^d) \times L^1((0,T)\times\mathbb{T}^d,\mathbb{R}^d) \Big|$$
$$m(t,x) \geq 0 \text{ a.e.}, \int_{\mathbb{T}^d} m(t,x)dx = 1 \text{ for a.e. } t \in (0,T), \text{ and}$$
$$\partial_t m - \partial_{ij}(A_{ij}(x)m) + \mathrm{div}(w) = 0 \text{ in } (0,T)\times\mathbb{T}^d, \qquad m(0) = m_0$$
$$\text{in the sense of distribution} \Big\}$$

i.e. for any test function $\zeta \in C_c^\infty([0,T)\times\mathbb{T}^d)$,

$$-\int_{\mathbb{T}^d} \zeta(0)m_0 - \int_0^T \int_{\mathbb{T}^d} m\partial_t\zeta + A_{ij}m\partial_{ij}\zeta + \langle w, \nabla\zeta\rangle = 0$$

Precisely : $$\inf_{(m,w)\in\mathcal{K}_1} \mathcal{B}(m,w)$$

where

$$\mathcal{K}_1 := \Big\{ (m,w) \in L^1((0,T)\times\mathbb{T}^d) \times L^1((0,T)\times\mathbb{T}^d,\mathbb{R}^d) \Big| \\ m(t,x) \geq 0 \text{ a.e.}, \int_{\mathbb{T}^d} m(t,x)dx = 1 \text{ for a.e. } t \in (0,T), \text{ and} \\ \partial_t m - \partial_{ij}(A_{ij}(x)m) + \mathrm{div}(w) = 0 \text{ in } (0,T)\times\mathbb{T}^d, \qquad m(0) = m_0 \\ \text{in the sense of distribution} \Big\}$$

$$\mathcal{B}(m,w) = \int_0^T \int_{\mathbb{T}^d} m(t,x) H^*\left(x, -\frac{w(t,x)}{m(t,x)}\right) + F(x, m(t,x))\, dxdt \\ + \int_{\mathbb{T}^d} \phi_T(x) m(T,x) dx$$

for $m(t,x) = 0$, $m(t,x)H^*\left(x, -\frac{w(t,x)}{m(t,x)}\right) = \begin{cases} +\infty & \text{if } w(t,x) \neq 0 \\ 0 & \text{if } w(t,x) = 0 \end{cases}$

Lemma (CGPT 15)

We have

$$\inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi) = - \min_{(m,w) \in \mathcal{K}_1} \mathcal{B}(m, w)$$

Moreover, the minimum in the right-hand side is achieved by a unique pair $(\bar{m}, \bar{w}) \in \mathcal{K}_1$ *satisfying*
$(\bar{m}, \bar{w}) \in L^q((0, T) \times \mathbb{T}^d) \times L^{\frac{r'q}{r'+q-1}}((0, T) \times \mathbb{T}^d)$

IDEA : Use the Fenchel-Rockafellar duality Theorem

Relaxation

In general, we do not expect problem

$$\inf_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi)$$

to have a solution

IDEA : we consider an equivalent relaxed problem and see that it has a solution

Let $\gamma = \frac{rp(1+d)}{d-r(p-1)}$ if $p < 1 + \frac{d}{r}$ and $\gamma = +\infty$ if $p > 1 + \frac{d}{r}$

$$\mathcal{K} := \Big\{ (\phi, \alpha) \in L^\gamma((0,T)\times\mathbb{T}^d) \times L^p((0,T)\times\mathbb{T}^d) \Big| \; D\phi \in L^r((0,T)\times\mathbb{T}^d)$$

and which satisfies in the sense of distribution

$$-\partial_t \phi - A_{ij}(x)\partial_{ij}\phi + H(x, D\phi) \le \alpha, \qquad \phi(T,\cdot) \le \phi_T \Big\}$$

Here "in the sense of distribution" means that for any nonnegative test function $\zeta \in C_c^\infty((0,T]\times\mathbb{T}^d)$,

$$-\int_{\mathbb{T}^d}\zeta(T)\phi_T+\int_0^T\int_{\mathbb{T}^d}\phi\partial_t\zeta+\langle D\zeta, AD\phi\rangle+\zeta(\partial_i A_{ij}\partial_j\phi+H(x,D\phi))\le \int_0^T\int_{\mathbb{T}^d}\alpha\zeta$$

Due to the presence of second order derivatives we do not expect the function ϕ to be BV (as in the first order case)

What about trace properties ? ϕ has a trace in a weak sense

We extend the functional $\mathcal{A}$ to $\mathcal{K}$

$$\mathcal{A}(\phi,\alpha) = \int_0^T \int_{\mathbb{T}^d} F^*(x,\alpha(x,t))\,dxdt - \int_{\mathbb{T}^d} \phi(x,0)m_0(x)\,dx \quad \forall(\phi,\alpha)\in\mathcal{K}$$

Proposition (CGPT 15)

We have

$$\inf_{\phi\in\mathcal{K}_0} \mathcal{A}(\phi) = \inf_{(\phi,\alpha)\in\mathcal{K}} \mathcal{A}(\phi,\alpha)$$

Inequality $\inf_{\phi\in\mathcal{K}_0} \mathcal{A}(\phi) \geq \inf_{(\phi,\alpha)\in\mathcal{K}} \mathcal{A}(\phi,\alpha)$ holds obviously

Let $(\phi,\alpha)\in\mathcal{K}$, then

$$\mathcal{A}(\phi,\alpha) \;\geq\; -\inf_{(m,w)\in\mathcal{K}_1} \mathcal{B}(m,w) = \inf_{\phi\in\mathcal{K}_0} \mathcal{A}(\phi)$$

Proposition (CGPT 15)

The relaxed problem has at least one solution $(\bar{\phi}, \bar{\alpha}) \in \mathcal{K}$ *which is bounded below by a constant depending on* $\|\phi_T\|_{C^2}$, *on* $\|A_{ij}\|_{C^0}$ *and on* $\|H(\cdot, D\phi_T)\|_\infty$.

Therefore, thanks to the equivalence, we have a infimum for $\mathcal{A}$ over $\mathcal{K}_0$ and a minimum for $\mathcal{B}$ over $\mathcal{K}_1$

$$\min_{(\phi,\alpha)\in\mathcal{K}} \mathcal{A}(\phi,\alpha) = \inf_{\phi\in\mathcal{K}_0} \mathcal{A}(\phi) = -\min_{(m,w)\in\mathcal{K}_1} \mathcal{B}(m,w)$$

Existence and uniqueness

We say that a pair $(\phi, m) \in L^\gamma((0,T)\times\mathbb{T}^d) \times L^q((0,T)\times\mathbb{T}^d)$ is a weak solution to (MFG) if

(i) $D\phi \in L^r,\ mH^*(\cdot, D_pH(\cdot, D\phi)) \in L^1 \quad \text{and} \quad mD_pH(\cdot, D\phi)) \in L^1$

(ii) The following inequality holds in the sense of distribution

$$-\partial_t\phi - \partial_i(A_{ij}(x)\partial_j\phi) + (\partial_i A_{ij})\partial_j\phi + H(x, D\phi) \le f(x,m) \quad \text{in } (0,T)\times\mathbb{T}^d,$$

with $\phi(T,\cdot) \le \phi_T$

(iii) The following equation holds in the sense of distribution

$$\partial_t m - \partial_{ij}(A_{ij}(x)m) - \mathrm{div}(mD_pH(x, D\phi))) = 0 \text{ in } (0,T)\times\mathbb{T}^d, m(0) = m_0$$

(iv) The following equality holds :

$$\int_0^T \int_{\mathbb{T}^d} m(t,x)\left(f(x, m(t,x)) + H^*(x, D_pH(x, D\phi)(t,x))\right) dxdt + \int_{\mathbb{T}^d} m(T,x)\phi_T(x) - m_0(x)\phi(0,x)dx = 0$$

Theorem (CGPT 15)

There exists a weak solution $(\bar\phi, \bar m)$ to (MFG).

Moreover this solution is unique in the following sense: if $(\bar\phi, \bar m)$ and $(\bar\phi', \bar m')$ are two solutions, then $\bar m = \bar m'$ a.e. and $\bar\phi = \bar\phi'$ in $\{\bar m > 0\}$.

Finally, there exists a solution which is bounded below by a constant depending on $\|\phi_T\|_{C^2}$, on $\|A_{ij}\|_{C^0}$ and on $\|H(\cdot, D\phi_T)\|_\infty$.

Idea of the Proof :

Let $(\bar{m}, \bar{w}) \in \mathcal{K}_1$ be a minimizer of $\min_{(m,w)\in\mathcal{K}_1} \mathcal{B}(m, w)$
and $(\bar{\phi}, \bar{\alpha}) \in \mathcal{K}$ be a minimizer of $\min_{(\phi,\alpha)\in\mathcal{K}} \mathcal{A}(\phi, \alpha)$

Then $\bar{w} = -\bar{m} D_p H(\cdot, D\bar{\phi})$ a.e. and $\bar{\alpha}(t, x) = f(x, \bar{m}(t, x))$ a.e.
and $(\bar{\phi}, \bar{m})$ is a weak solution

Conversely, we can show that any weak solution $(\bar{\phi}, \bar{m})$ of (MFG) is such that :

the pair $(\bar{m}, -\bar{m} D_p H(\cdot, D\bar{\phi}))$ is the minimizer of $\min_{(m,w)\in\mathcal{K}_1} \mathcal{B}(m, w)$

while $(\bar{\phi}, f(\cdot, \bar{m}))$ is a minimizer of $\min_{(\phi,\alpha)\in\mathcal{K}} \mathcal{A}(\phi, \alpha)$

Uniqueness : Since the solution of $\min_{(m,w)\in\mathcal{K}_1} \mathcal{B}(m, w)$ is unique $\bar{m}$ is unique

The proof of the uniqueness of $\bar{\phi}$ is more technical

The planning problem

What happens if we prescribe the terminal distribution of the agents $m(T)$?

Suggested by P.L. Lions, this is an optimal planning problem: can we drive in time T the density of the agents from the initial configuration m_0 to a target configuration m_T in a way which is optimal for the agents' cost ?
$\Longrightarrow$ Optimal transport problem

In this case, the MFG system

$$\begin{cases} -\partial_t \phi - A_{ij}\partial_{ij}\phi + H(x, D\phi) = f(x, m(x,t)) \\ \partial_t m - \partial_{ij}(A_{ij} m) - \mathrm{div}(m D_p H(x, D\phi)) = 0 \end{cases}$$

is complemented with initial and terminal conditions for the density m:

$$m(0) = m_0, \quad m(T) = m_T$$

The final pay-off $\phi(T)$ is not prescribed here; it is a degree of freedom to be used to reach the target

The optimal control problem of a typical small player

This can be seen as an optimal transport problem for the probability densities of the process law

Dynamics :

$$dX_s = v_s ds + \Sigma(X_s) dB_s,$$

$$m_0 = \mathcal{L}(X_0), \quad m_T = \mathcal{L}(X_T)$$

minimizing :

$$\mathbb{E}\left[\int_0^T H^*(X_s, -v_s) + f(X_s, m(s, X_s))\ ds\right]$$

among admissible feedback v_s

Alternatively, the pb. can be seen as an exact controllability problem for the Fokker-Planck equation (+ optimization among admissible drifts v):

Minimize

$$\int_0^T \int_{\mathbb{T}^d} m(t,x) H^* (x, -v(t,x)) + F(x, m(t,x)) \; dxdt$$

when m solves the Fokker-Plank equation

$$\partial_t m - \partial_{ij}(A_{ij}(x) m) + \mathrm{div}(mv) = 0$$

with $m(0) = m_0$ and $m(T) = m_T$

- Existing results :
  - Lions : existence and uniqueness of smooth solutions :
    - for the deterministic case ($A_{ij} = 0$), H is a smooth superlinear and strictly convex Hamiltonian, $f(m)$ nondecreasing and bounded, m_0, m_T smooth and positive
    -for the model case ($H(x,p) = \frac{1}{2}|p|^2$), $A_{ij}\partial_{ij}\phi = \Delta\phi$, $f(m)$ nondecreasing and bounded, m_0, m_T smooth and positive
  - Porretta (2013): existence and uniqueness of weak solutions (through energy estimates) for a more general strictly convex Hamiltonian (the Hamiltonian has quadratic growth but no precise asymptotics is required at infinity), $A_{ij}\partial_{ij}\phi = \Delta\phi$, $f(m)$ nondecreasing and bounded, m_0, m_T smooth and positive
  - Achdou, Camilli, Capuzzo Dolcetta (2012) : numerical results
- Idea : Exploit the variational formulation in order to prove existence and uniqueness of weak solutions in the first order case
  - Ok when m_0, m_T are absolutely continuous wrt Lebesgue measure
  - use optimal transport techniques to deal with the case m_0, m_T are general probability Measures

This is an ongoing collaboration with F.Silva