Exact algorithms: from Semidefinite to Hyperbolic programming

Simone Naldi and Daniel Plaumann

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Polyhedra

Finite \# linear inequalities:

\[ \ell_i(x) \geq 0, \quad i = 1, \ldots, m \]

One can associate a diagonal matrix:

\[ A(x) = \text{diag}(\ell_i(x)) \]

\[ f = \ell_1(x)\ell_2(x) \cdots \ell_m(x) = \text{det} A(x) \]

Feasibility Problem: Is \( \mathcal{P} \neq \emptyset \) ? (\( \exists x \in \mathbb{R}^n \) s.t. \( \ell_i(x) \geq 0, \forall i \))

Linear Programming (LP): Compute \( \inf \ell(x) \) s.t. \( x \in \mathcal{P} \)

Solutions are "rational"

Combinatorics of boundary – active constraints "\( \ell_i = 0 \)"
Spectrahedra

Start with a symmetric linear matrix:

\[ A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n \]

Infinite \# linear ineq. \[ y^T A(x) y \geq 0 \]

Finite \# Non-linear inequalities:
\[ \{ \text{Principal Minors of } A(x) \} \geq 0 \]

\[ f = \det A(x) \]

Feasibility Problem (LMI): \[ \exists x \in \mathbb{R}^n \text{ s.t. } A(x) \succeq 0 \]

Semidefinite Progr. (SDP): Compute \[ \inf \ell(x) \text{ s.t. } x \in S \]

Irrational solutions - Algebraic degree = \( \delta(\text{size, var, rank}) \)

Combinatorics of boundary – co-rank of \( A(x) \)
This talk: Hyperbolicity cones

Input data: a polynomial and a vector
\[ f \in \mathbb{R}[x_1, \ldots, x_n], e \in \mathbb{R}^n \]

Finite \# Non-linear inequalities:
Coeff. of \( t \mapsto f(te - x) \)

\( f \) hyperbolic w.r. to \( e \)

\( C(f, e) \) hyperbolicity cone

Feasibility Problem? By definition \( e \in \text{Int}(C(f, e)) \) !!

Hyperbolic Progr. (HP): Compute \( \inf \ell(x) \) s.t. \( x \in C(f, e) \)

Algebraic degree = ?(\text{deg}, n, \text{mult}) ("irrational" solutions)

Combinatorics of boundary – \textbf{multiplicity} of \( x \)
Hyperbolic polynomials

**Definition of hyperbolic polynomial**

$f \in \mathbb{R}[x]_d$ is hyperbolic w.r.t. $e = (e_1, \ldots, e_n) \in \mathbb{R}^n$ if

- $f(e) \neq 0$
- $\forall a \in \mathbb{R}^n \ t \mapsto ch_a(t) := f(te - a)$ has only real roots

If such $e$ exists, $f$ is called a hyperbolic polynomial.

Fundamental examples:

1. $f = x_1 \cdots x_d$
   
   $ch_a(t) = \prod_i (te_i - a_i)$

2. $f = \det A(x)$, $A(x)$ sym.
   
   $ch_a(t) = \det(tI_d - A(a))$

General case (here $d = 4$):

Brändén (2010): hyperb. polynomials without determinantal representations
Cones and Multiplicity

**Hyperbolicity cone**

The hyperbolicity cone of \( f \in \mathbb{R}[x]_d \) (w.r.t. \( e \)) is

\[
C(f, e) = \{ a \in \mathbb{R}^n : ch_a(t) = 0 \Rightarrow t \geq 0 \}
\]

**Multiplicity:** For \( a \in \mathbb{R}^n \), we define

\[
\text{mult}(a) := \text{multiplicity of 0 as root of } ch_a(t) = f(te - a)
\]

**Multiplicity set:** For \( m \leq d \), \( \Gamma_m = \{ a \in \mathbb{R}^n : \text{mult}(a) \geq m \} \)

**Remark:** The set \( \Gamma_m \) is real algebraic.

Indeed, if \( ch_a(t) = t^d + g_1(a)t^{d-1} + \cdots + g_{d-1}(a)t + g_d(a) \) then

\[
\Gamma_m = \{ a : g_i(a) = 0, i \geq d - m + 1 \}
\]
Recap

<table>
<thead>
<tr>
<th>Cone</th>
<th>Polynomial</th>
<th>Optimization</th>
<th>Boundary</th>
</tr>
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<tbody>
<tr>
<td><img src="image1" alt="Hyperbolic" /></td>
<td>Hyperbolic</td>
<td>HP</td>
<td>Multiplicity</td>
</tr>
<tr>
<td><img src="image2" alt="Polynomial" /></td>
<td>$f = \det A(x)$</td>
<td>SDP</td>
<td>Co-rank of $A(x)$</td>
</tr>
<tr>
<td><img src="image3" alt="Optimization" /></td>
<td>$f = \prod \ell_i(x)$</td>
<td>LP</td>
<td>Active constraints</td>
</tr>
</tbody>
</table>

Two different approaches:
- “Interior” approach (e.g. classical interior-point methods)
- “Boundary” approach (more suitable for algebraic methods)

General goal: certification of information on the solution
Generalized Lax conjecture

Every hyperbolicity cone is a spectrahedron, that is \( \exists A_1, \ldots, A_n \) such that

\[
C(f, e) = \{ x \in \mathbb{R}^n : A_1 x_1 + \ldots + A_n x_n \preceq 0 \}
\]

Helton-Vinnikov, Lewis-Parrilo-Ramana – True for \( n = 3 \), with the stronger result that \( f \) hyperbolic \( \Rightarrow f = \det(x_1 A_1 + x_2 A_2 + x_3 A_3) \)

By Brändén's counterexamples, the Lax conjecture cannot be proved by proving that every hyperbolic polynomial admits a determinantal representation.

Determinantal representations: Leykin, Netzer, Plaumann, Sinn, Sturmfels, Thom, Vinzant... with many different techniques

This work: Algebraic approach to HP

We prove that every hyperbolic programming problem is equivalent to linear optimization over some multiplicity locus.

Moreover, computing $\max mult(x)$ over $C(f,e)$ is equivalent to the computation of witness points on connected components of the multiplicity loci (classical problem in R.A.G.)

All the reduced problems have simply exponential complexity with respect to the number $n$ of variables.

Representation of the solution:

$$x_i = \frac{q_i(t)}{q_0(t)}, \quad q(t) = 0 \quad \text{with } q_i \in \mathbb{Q}[t]$$
A test on a hyperbolic quartic

Optimizing random linear forms yields solutions of
  • multiplicity 1 for 64% of the times
  • multiplicity 2 for 36% of the times

One can get rational solutions (multiplicity 2):

\[ x_1 = 0 \quad x_2 = 1/2 \quad x_3 = 0 \]

or irrational smooth boundary solutions (mult 1):

\[
\begin{align*}
x_1 & \in \left[ -\frac{29707767148026666593}{147573952589676412928}, -\frac{29707767148024593931}{147573952589676412928} \right] \approx -0.2013076605 \\
x_2 & \in \left[ -\frac{18765770300641154993}{73786976294838206464}, -\frac{18765770300640685591}{73786976294838206464} \right] \approx -0.2543236116 \\
x_3 & \in \left[ \frac{21153099339285995043}{1180591620717411303424}, \frac{66103435435276711}{36893488147419103232} \right] \approx 0.01791737208
\end{align*}
\]
Based on the remark that

\[ f \text{ hyperbolic w.r. to } e \implies D_e f = \sum_i e_i \frac{\partial f}{\partial x_i} \text{ still hyperbolic} \]

This gives a nested sequence of convex hyperbolicity cones:

\[ \mathcal{C}(f,e) \subset \mathcal{C}(D_e f,e) \subset \cdots \subset \mathcal{C}(D_e^{(d-1)} f,e) \]

(the last one being a half-space), giving a sequence of lower bounds for the linear function to optimize:

\[ \inf_{\mathcal{C}(f,e)} \ell(a) \geq \inf_{\mathcal{C}(D_e f,e)} \ell(a) \geq \cdots \geq \inf_{\mathcal{C}(D_e^{(d-1)} f,e)} \ell(a) \]
The quartic

Quartic hyperbolic polynomial
The quartic

First derivative
Why is Renegar’s method useful from an effective viewpoint?

- At each step of the relaxation, the **degree** of the polynomial decreases by 1.
- The **multiplicity** decreases: the solution becomes “smoother” at each step.
- One of the $\mathcal{C}(D^{(j)}_e f, e)$ could be a section of the PSD cone (solution set of a LMI), in which case a lower bound can be computed by solving a **single SDP**:

Sanyal (2013): spectrahedral repr. for derivative cones of polyhedra
Saunderson (2017): spectrahedral repr. for first derivative of PSD cone
Planar 3–ellipse

Here I minimize a linear function $\ell(x)$ over the 3-ellipse

$$C(f, e) = \{ P \in \mathbb{R}^2 : d(P, P_1) + d(P, P_2) + d(P, P_3) \leq c \}$$

The boundary is given by

$$f = 9x^8 - 72x^7z + 36x^6y^2 - 96x^6yz - \ldots$$

<table>
<thead>
<tr>
<th>Deriv.</th>
<th>$x^*$</th>
<th>Mult.</th>
<th>$\ell(x^*)$</th>
<th>deg($q(t)$)</th>
<th>Alg.Deg. of $x^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(0.750, 0.000, 0.250)</td>
<td>2</td>
<td>5.500000</td>
<td>56</td>
<td>1</td>
</tr>
<tr>
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<td>(0.759, −0.018, 0.258)</td>
<td>1</td>
<td>5.499158</td>
<td>42</td>
<td>30</td>
</tr>
<tr>
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<td>5.456196</td>
<td>30</td>
<td>26</td>
</tr>
<tr>
<td>3</td>
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<td>5.392044</td>
<td>20</td>
<td>20</td>
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<tr>
<td>4</td>
<td>(0.981, −0.254, 0.273)</td>
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<td>12</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>(1.336, −0.762, 0.426)</td>
<td>1</td>
<td>5.090555</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

Still OK for 4–ellipse, becoming hard for $\geq 5$–ellipse
Summary

Hyperbolic polynomials (resp. HP) is a rich class of real polynomials, defining highly-structured optimization problems.

It is important to develop an effective approach

to hyperbolic polynomials, independent on their determinantal representability

to hyperbolicity cones, independent on Lax conjecture

Algebraic alternatives to interior-point methods for polynomial optimization

Complexity of HP: polynomial in the number of variables with fixed degree?

True for generic SDP
Thanks!