Polynomial Feedback Laws for Infinite-Dimensional Bilinear Optimal Control Problems

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PGMO Days
Palaiseau
November 14, 2017
We consider the following **bilinear optimal control problem**:

\[
\inf_{u \in L^2(0,\infty)} \mathcal{J}(u, y_0) := \int_0^\infty \frac{1}{2} \|y(t)\|^2_Y + \frac{\beta}{2} |u(t)|^2 dt,
\]

where:

\[
\begin{cases}
\dot{y}(t) = Ay(t) + Ny(t)u(t) + Bu(t), \\
y(0) = y_0 \in Y,
\end{cases}
\]

with associated **value function**:

\[\mathcal{V}(y_0) := \inf_{u \in L^2(0,\infty)} \mathcal{J}(u, y_0)\]

**Main assumption:** \(\exists F \in \mathcal{L}(Y, \mathbb{R})\) such that \(\dot{y} = Ay + BFy\) is exponentially stable.

**Goal:** analysing theoretically and numerically polynomial **feedback laws** derived from **Taylor expansions** of \(\mathcal{V}\) around 0.
1 Theory

2 Numerics
1 Theory

2 Numerics
We construct a polynomial $\mathcal{V}_p$ of the form:

$$
\mathcal{V}_p(y_0) = \frac{1}{2} \mathcal{T}_2(y_0, y_0) + \frac{1}{3!} \mathcal{T}_3(y_0, y_0, y_0) + \cdots + \frac{1}{p!} \mathcal{T}_p(y_0, \ldots, y_0),
$$

where $\mathcal{T}_k : Y^k \to \mathbb{R}$ is a bounded multilinear form (of order $k$). We show a posteriori that $\mathcal{V}_p$ is a Taylor expansion of $\mathcal{V}$ around 0.

More precisely,

- $\mathcal{T}_2$ is constructed as the solution to an algebraic Riccati equation (ARE)
- $\mathcal{T}_3, \mathcal{T}_4, \ldots$ are constructed as solutions to (linear) generalized Lyapunov equations (GLE).
First step: **formal** derivation of the **Hamilton-Jacobi-Bellman** equation, by dynamic programming.

**Proposition**

Assume that there exists a neighborhood $Y_0$ of 0 such that

1. Problem $P(y_0)$ has a continuous solution $u$, $\forall y_0 \in D(A) \cap Y_0$
2. The value function is continuously differentiable on $Y_0$.

Then, for all $y_0 \in D(A) \cap Y_0$,

$$
DV(y_0)Ay_0 + \frac{1}{2}\|y_0\|_Y^2 - \frac{1}{2\beta}(DV(y_0)(Ny_0 + B))^2 = 0. \quad \text{(HJB)}
$$

Moreover, for all continuous solutions $u$ to problem $P(y_0)$,

$$
u(0) = -\frac{1}{\beta}DV(y_0)(Ny + B).
$$
Taylor expansion

The equations characterizing \((T_k)_{k=2,3,...}\) are then obtained by **formal differentiation** of the HJB equation and identification with \((D^k \mathcal{V}(0))_{k=2,3,...}\).

**Remark:** \(\mathcal{V}(0) = 0, \ D\mathcal{V}(0) = 0\).

Differentiating the HJB equation a first time w.r.t. \(y\) in the direction \(z_1 \in \mathcal{D}(A)\) yields

\[ D^2 \mathcal{V}(y)(Ay, z_1) + D\mathcal{V}(y)Az_1 + \langle y, z_1 \rangle Y \]
\[ - \frac{1}{\beta} (D^2 \mathcal{V}(y)(Ny + B, z_1) + D\mathcal{V}(y)Nz_1)(D\mathcal{V}(y)(Ny + B)) = 0. \]
Differentiating a second time, we obtain

\[
D^3 \mathcal{V}(y)(Ay, z_1, z_2) + D^2 \mathcal{V}(y)(Az_2, z_1) + D^2 \mathcal{V}(y)(Az_1, z_2) + \langle z_1, z_2 \rangle \gamma \\
- \frac{1}{\beta} \left( D^2 \mathcal{V}(y)(Ny + B, z_1) + D\mathcal{V}(y)Nz_1 \right) \left( D^2 \mathcal{V}(y)(Ny + B, z_2) + D\mathcal{V}(y)Nz_2 \right) \\
- \frac{1}{\beta} \left( D^3 \mathcal{V}(y)(Ny + B, z_1, z_2) \right) \left( D\mathcal{V}(y)(Ny + B) \right) \\
- \frac{1}{\beta} \left( D^2 \mathcal{V}(y)(Nz_2, z_1) + D^2 \mathcal{V}(y)(Nz_1, z_2) \right) \left( D\mathcal{V}(y)(Ny + B) \right) = 0.
\]

For \( y = 0 \), using the representation \( D^2 \mathcal{V}(0)(z_1, z_2) = \langle z_1, \Pi z_2 \rangle \), where \( \Pi : Y \to Y \), we obtain an algebraic Riccati equation:

\[
A^* \Pi + \Pi A + \text{Id} - \frac{1}{\beta} \Pi BB^* \Pi = 0. \quad \text{(ARE)}
\]

It has a unique self-adjoint and non-negative solution.
Taylor expansion

Differentiating a third time, we obtain for \( y = 0 \):

\[
D^3 \mathcal{V}(0)(Az_3, z_1, z_2) + D^3 \mathcal{V}(0)(Az_2, z_1, z_3) + D^3 \mathcal{V}(0)(Az_1, z_2, z_3)
\]

\[ - \frac{1}{\beta} \left( D^3 \mathcal{V}(0)(B, z_1, z_3) + D^2 \mathcal{V}(0)(Nz_3, z_1) + D^2 \mathcal{V}(0)(Nz_1, z_3) \right) D^2 \mathcal{V}(0)(B, z_2) \]

\[ - \frac{1}{\beta} \left( D^3 \mathcal{V}(0)(B, z_2, z_3) + D^2 \mathcal{V}(0)(Nz_3, z_2) + D^2 \mathcal{V}(0)(Nz_2, z_3) \right) D^2 \mathcal{V}(0)(B, z_1) \]

\[ - \frac{1}{\beta} \left( D^3 \mathcal{V}(0)(B, z_1, z_2) + D^2 \mathcal{V}(0)(Nz_2, z_1) + D^2 \mathcal{V}(0)(Nz_1, z_2) \right) D^2 \mathcal{V}(0)(B, z_3) = 0. \]

We set: \( A_\Pi = A - \frac{1}{\beta} BB^* \Pi \), we obtain:

\[
D^3 \mathcal{V}(0)(A_\Pi z_1, z_2, z_3) + D^3 \mathcal{V}(0)(z_1, A_\Pi z_2, z_3) + D^3 \mathcal{V}(0)(z_1, z_2, A_\Pi z_3)
\]

\[ = \frac{1}{2\beta} \mathcal{R}_3(z_1, z_2, z_3), \quad \forall (z_1, z_2, z_3) \in \mathcal{D}(A)^3, \]

where the trilinear form \( \mathcal{R}_3 : \mathcal{Y}^3 \to \mathbb{R} \) is determined by \( \Pi, N, \) and \( B \).
Further differentiation provides a linear equation, called **generalized Lyapunov equation**, with the following structure:

$$D^k \mathcal{V}(0)(A \Pi z_1, z_2, \ldots, z_k) + \ldots + D^k \mathcal{V}(0)(z_1, \ldots, z_{k-1}, A \Pi z_k) = \frac{1}{2\beta} \mathcal{R}_k(z_1, \ldots, z_k), \quad \forall (z_1, \ldots, z_k) \in \mathcal{D}(A)^k \quad (\text{GLE}(k))$$

where the multilinear form $\mathcal{R}_k: Y^k \to \mathbb{R}$ is determined by $\Pi$, $D^3 \mathcal{V}(0), \ldots, D^{k-1} \mathcal{V}(0)$, $N$, and $B$.

**Remark:** end of the formal computations.

**Theorem**

*There exists a unique sequence $(T_k)_{k=3,4,\ldots}$ of symmetric bounded multilinear forms such that $T_k: Y^k \to \mathbb{R}$ is a solution to GLE(k).*
Feedback law

**Polynomial** $\mathcal{V}_p$ of degree $p$:

$$
\mathcal{V}_p(y) = \sum_{k=2}^{p} \frac{1}{k!} T_k(y, \ldots, y).
$$

**Feedback law** $u_p$ of order $p$:

$$
u_p: y \in Y \mapsto u_p(y) = -\frac{1}{\beta} D\mathcal{V}_p(y)(Ny + B).
$$

**Closed-loop system** of order $p$:

$$
\dot{y}_p(t) = Ay_p(t) + (Ny_p(t) + B)u_p(y_p(t)), \quad y(0) = y_0.
$$

**Open-loop control** $U_p(y_0)$ generated by the feedback $u_p$ and $y_0$:

$$
U_p(y_0; t) = u_p(y_p(t)).
$$
Theoretical results

**Theorem**

For all $p \geq 2$, there exist two constants $\delta > 0$ and $C > 0$ such that for all $y_0 \in Y$ with $\|y_0\|_Y \leq \delta$:

1. The closed-loop system is well-posed and generates an open-loop control in $L^2(0, \infty)$ such that:

   $$ J(U_p(y_0), y_0) \leq V(y_0) + C\|y_0\|^{p+1}_Y. $$

2. It holds:

   $$ |V(y_0) - V_p(y_0)| \leq C\|y_0\|^{p+1}. $$

Remark: local result, $\delta$ decreases as $\beta$ decreases and $p$ increases.
1 Theory

2 Numerics
**Numerical approach**

1. **Discretize** the operators $A$, $N$, and $B$ in such a way that the bilinear structure is preserved (e.g. with finite differences).

2. Find a **reduced-order** model with a generalization of the balanced truncation method:

   \[
   \inf_{u \in L^2(0,\infty)} J(u, y_0) := \int_0^\infty \frac{1}{2} \| C_r y_r(t) \|_{\mathbb{R}^n}^2 + \frac{\beta}{2} |u(t)|^2 dt,
   \]

   where:

   \[
   \begin{aligned}
   \dot{y}_r(t) &= A_r y_r(t) + N_r y_r(t)u(t) + B_r u(t), \\
   y_r(0) &= y_{0,r} \in Y.
   \end{aligned}
   \]

3. Solve the reduced GLE with a **tensor-calculus technique**.
Lyapunov equations

The associated reduced GLE of order $k$:

$$T_{k,r}(A\Pi,rz_1,z_2,\ldots,z_k) + \ldots + T_{k,r}(z_1,\ldots,z_{k-1},A\Pi,rz_k) = \frac{1}{2\beta} R_{k,r}(z_1,\ldots,z_k)$$

is equivalent to a linear system with $r^k$ variables. Solution:

$$T_{k,r}(z_1,\ldots,z_k) = -\int_0^\infty R_{k,r}(e^{A\Pi,r}t z_1,\ldots,e^{A\Pi,r}t z_k)dt.$$

An approximation is given by:

$$\sum_{i=-\ell}^{\ell} w_i R_{k,r}(e^{A\Pi,r}t_i z_1,\ldots,e^{A\Pi,r}t_i z_k),$$

for an appropriate choice of points $t_i$ and weights $w_i$. 
Fokker-Planck equation

Controlled Fokker-Planck equation:

\[ \frac{\partial \rho}{\partial t} = \nu \Delta \rho + \nabla \cdot (\rho \nabla G) + u \nabla \cdot (\rho \nabla \alpha_j) \quad \text{in } \Omega \times (0, \infty), \]

\[ 0 = (\nu \nabla \rho + \rho \nabla G) \cdot \vec{n} \quad \text{on } \Gamma \times (0, \infty), \]

\[ \rho(x, 0) = \rho_0(x) \quad \text{in } \Gamma, \]

where \( \Omega \in \mathbb{R}^d \) denotes a bounded domain with smooth boundary \( \Gamma \).

For all \( t \), \( \rho(\cdot, t) \) is the probability density function of \( X_t \), sol. to

\[ dX(t) = -\nabla_x V(X(t), t) dt + \sqrt{2\nu} dW_t, \]

where the potential \( V \) is controlled by \( u \):

\[ V(x, t) = G(x) + u(t) \alpha(x), \quad \forall x \in \Omega, \forall t \geq 0. \]
The uncontrolled Fokker-Planck equation is known to converge to its stationary distribution $\rho_\infty$. 

(a) Ground potential

(b) Stationary distribution
Optimal control problem:
\[
\inf_{u \in L^2(0, \infty)} \int_0^\infty \frac{1}{2} \| \rho(\cdot, t) - \rho_\infty(\cdot) \|_{L^2(\Omega)}^2 + \beta |u(t)|^2 dt,
\]
where \( \rho \) satisfies the Fokker-Planck equation.

Under regularity assumptions on \( G \) and \( \alpha \), the problem can be reformulated, so that it falls in the abstract framework.

- Control shape function \( \alpha(x) \approx x/12 \).
- Discretization of \( \Omega = (-6, 6) \): \( n = 100 \).
- Reduction: \( r = 21 \) (selection of singular values above \( 10^{-6} \)).
- Results for two initial values (a close one/a further one), different values of \( \beta \).
Numerical results (test case 1)

(a) Initial/stationary distributions
(b) Controls for $\beta = 10^{-3}$
Numerical results (test case 1)

(a) Controls for $\beta = 10^{-4}$

(b) Controls for $\beta = 10^{-5}$
### Numerical results (test case 1)

#### (a) Cost of the controls $u_p$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$J(u_2)$</th>
<th>$J(u_3)$</th>
<th>$J(u_4)$</th>
<th>$J(u_5)$</th>
<th>$J(u_6)$</th>
<th>$J(u_{\text{opt}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \times 10^{-3}$</td>
<td>0.156</td>
<td>0.155</td>
<td>0.155</td>
<td>0.155</td>
<td>0.155</td>
<td>0.154</td>
</tr>
<tr>
<td>$1 \times 10^{-4}$</td>
<td>0.138</td>
<td>0.122</td>
<td>0.120</td>
<td>0.120</td>
<td>0.120</td>
<td>0.119</td>
</tr>
<tr>
<td>$1 \times 10^{-5}$</td>
<td>0.205</td>
<td>0.194</td>
<td>0.104</td>
<td>0.111</td>
<td>0.113</td>
<td>0.095</td>
</tr>
</tbody>
</table>

#### (b) $L^2$-distance between the controls $u_p$ and the optimal control $u_{\text{opt}}$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$p = 2$</th>
<th>$p = 3$</th>
<th>$p = 4$</th>
<th>$p = 5$</th>
<th>$p = 6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \times 10^{-3}$</td>
<td>1.149</td>
<td>0.169</td>
<td>0.119</td>
<td>0.034</td>
<td>0.031</td>
</tr>
<tr>
<td>$1 \times 10^{-4}$</td>
<td>18.50</td>
<td>7.02</td>
<td>3.16</td>
<td>4.01</td>
<td>1.52</td>
</tr>
<tr>
<td>$1 \times 10^{-5}$</td>
<td>90.5</td>
<td>78.0</td>
<td>39.0</td>
<td>42.6</td>
<td>34.3</td>
</tr>
</tbody>
</table>
Numerical results (test case 2)

(a) Initial/stationary distributions

(b) Controls for $\beta = 10^{-2}$
Numerical results (test case 2)

(a) Controls for $\beta = 10^{-3}$

(b) Controls for $\beta = 10^{-4}$
### Numerical results

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$J(u_2)$</th>
<th>$J(u_3)$</th>
<th>$J(u_4)$</th>
<th>$J(u_5)$</th>
<th>$J(u_6)$</th>
<th>$J(u_{\text{opt}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1e^{-2}$</td>
<td>0.788</td>
<td>0.788</td>
<td>0.788</td>
<td>0.788</td>
<td>0.788</td>
<td>0.787</td>
</tr>
<tr>
<td>$1e^{-3}$</td>
<td>0.525</td>
<td>0.511</td>
<td>0.511</td>
<td>0.512</td>
<td>0.510</td>
<td>0.507</td>
</tr>
<tr>
<td>$1e^{-4}$</td>
<td>0.381</td>
<td>0.368</td>
<td>2.689</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>0.246</td>
</tr>
</tbody>
</table>

(a) Cost of the controls $u_p$

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$|u_p - u_{\text{opt}}|_{L^2(0,T)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$p = 2$</td>
</tr>
<tr>
<td>$1e^{-2}$</td>
<td>0.19</td>
</tr>
<tr>
<td>$1e^{-3}$</td>
<td>4.88</td>
</tr>
<tr>
<td>$1e^{-4}$</td>
<td>46.34</td>
</tr>
</tbody>
</table>

(b) $L^2$-distance between the controls $u_p$ and the optimal control $u_{\text{opt}}$
Conclusion

Summary:

- General method for deriving polynomial feedback laws
- Implementation for an infinite-dimensional problem thanks to model reduction
- Good results, but only locally.

P. Benner, T. Damm. Lyapunov equations, energy functionals, and model order reduction of bilinear and stochastic systems, SICON, 2011. → **Model reduction.**


