Polynomial Feedback Laws for Infinite-Dimensional Bilinear Optimal Control Problems

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Introduction

We consider the following **bilinear optimal control problem**:

$$\inf_{u\in L^2(0,\infty)} \mathcal{J}(u,y_0) := \int_0^\infty \frac{1}{2}\|y(t)\|_Y^2 + \frac{\beta}{2}|u(t)|^2 \mathrm{d}t, \qquad (P(y_0))$$

$$\text{where: } \begin{cases} \dot{y}(t) = Ay(t) + Ny(t)u(t) + Bu(t), \\ y(0) = y_0 \in Y, \end{cases}$$

with associated **value function**: $\mathcal{V}(y_0) := \inf_{u\in L^2(0,\infty)} \mathcal{J}(u,y_0)$.

Main assumption: $\exists F \in \mathcal{L}(Y,\mathbb{R})$ such that $\dot{y} = Ay + BFy$ is exponentially stable.

Goal: analysing theoretically and numerically polynomial **feedback laws** derived from **Taylor expansions** of $\mathcal{V}$ around 0.

1 Theory

2 Numerics

1 Theory

2 Numerics

Roadmap

We construct a polynomial $\mathcal{V}_p$ of the form:

$$\mathcal{V}_p(y_0) = \frac{1}{2}\mathcal{T}_2(y_0, y_0) + \frac{1}{3!}\mathcal{T}_3(y_0, y_0, y_0) + \ldots + \frac{1}{p!}\mathcal{T}_p(y_0, \ldots, y_0),$$

where $\mathcal{T}_k : Y^k \to \mathbb{R}$ is a **bounded multilinear form** (of order k). We show a posteriori that $\mathcal{V}_p$ is a Taylor expansion of $\mathcal{V}$ around 0.

More precisely,

- $\mathcal{T}_2$ is constructed as the solution to an algebraic Riccati equation (ARE)
- $\mathcal{T}_3$, $\mathcal{T}_4$,... are constructed as solutions to (linear) generalized Lyapunov equations (GLE).

HJB equation

First step: **formal** derivation of the **Hamilton-Jacobi-Bellman** equation, by dynamic programming.

Proposition

Assume that there exists a neighborhood Y_0 of 0 such that

1. *Problem $P(y_0)$ has a continuous solution u, $\forall y_0 \in \mathcal{D}(A) \cap Y_0$*
2. *The value function is continuously differentiable on Y_0.*

Then, for all $y_0 \in \mathcal{D}(A) \cap Y_0$,

$$D\mathcal{V}(y_0)Ay_0 + \frac{1}{2}\|y_0\|_Y^2 - \frac{1}{2\beta}\big(D\mathcal{V}(y_0)(Ny_0 + B)\big)^2 = 0. \qquad \text{(HJB)}$$

Moreover, for all continuous solutions u to problem $P(y_0)$,

$$u(0) = -\frac{1}{\beta} D\mathcal{V}(y_0)(Ny + B).$$

Taylor expansion

The equations characterizing $(\mathcal{T}_k)_{k=2,3,\ldots}$ are then obtained by **formal differentiation** of the HJB equation and identification with $(D^k\mathcal{V}(0))_{k=2,3,\ldots}$.

Remark: $\mathcal{V}(0) = 0,\ D\mathcal{V}(0) = 0.$

Differentiating the HJB equation a first time w.r.t. y in the direction $z_1 \in \mathcal{D}(A)$ yields

$$D^2\mathcal{V}(y)(Ay, z_1) + D\mathcal{V}(y)Az_1 + \langle y, z_1\rangle_Y \\ - \frac{1}{\beta}\big(D^2\mathcal{V}(y)(Ny + B, z_1) + D\mathcal{V}(y)Nz_1\big)\big(D\mathcal{V}(y)(Ny + B)\big) = 0.$$

Taylor expansion

Differentiating a second time, we obtain

$$
\begin{aligned}
& D^3\mathcal{V}(y)(Ay, z_1, z_2) + D^2\mathcal{V}(y)(Az_2, z_1) + D^2\mathcal{V}(y)(Az_1, z_2) + \langle z_1, z_2 \rangle_Y \\
& - \frac{1}{\beta}\big(D^2\mathcal{V}(y)(Ny + B, z_1) + D\mathcal{V}(y)Nz_1\big)\big(D^2\mathcal{V}(y)(Ny + B, z_2) + D\mathcal{V}(y)Nz_2\big) \\
& - \frac{1}{\beta}\big(D^3\mathcal{V}(y)(Ny + B, z_1, z_2)\big)\big(D\mathcal{V}(y)(Ny + B)\big) \\
& - \frac{1}{\beta}\big(D^2\mathcal{V}(y)(Nz_2, z_1) + D^2\mathcal{V}(y)(Nz_1, z_2)\big)\big(D\mathcal{V}(y)(Ny + B)\big) = 0.
\end{aligned}
$$

For $y = 0$, using the representation $D^2\mathcal{V}(0)(z_1, z_2) = \langle z_1, \Pi z_2 \rangle$, where $\Pi\colon Y \to Y$, we obtain an algebraic Riccati equation:

$$A^*\Pi + \Pi A + \mathrm{Id} - \tfrac{1}{\beta}\Pi BB^*\Pi = 0. \qquad \text{(ARE)}$$

It has a unique self-adjoint and non-negative solution.

Taylor expansion

Differentiating a third time, we obtain for $y = 0$:

$$
\begin{aligned}
&D^3\mathcal{V}(0)(Az_3, z_1, z_2) + D^3\mathcal{V}(0)(Az_2, z_1, z_3) + D^3\mathcal{V}(0)(Az_1, z_2, z_3)\\
&- \frac{1}{\beta}\big(D^3V(0)(B, z_1, z_3) + D^2\mathcal{V}(0)(Nz_3, z_1) + D^2\mathcal{V}(0)(Nz_1, z_3)\big)D^2\mathcal{V}(0)(B, z_2)\\
&- \frac{1}{\beta}\big(D^3\mathcal{V}(0)(B, z_2, z_3) + D^2\mathcal{V}(0)(Nz_3, z_2) + D^2\mathcal{V}(0)(Nz_2, z_3)\big)D^2\mathcal{V}(0)(B, z_1)\\
&- \frac{1}{\beta}\big(D^3\mathcal{V}(0)(B, z_1, z_2) + D^2\mathcal{V}(0)(Nz_2, z_1) + D^2\mathcal{V}(0)(Nz_1, z_2)\big)D^2\mathcal{V}(0)(B, z_3) = 0.
\end{aligned}
$$

We set: $A_\Pi = A - \frac{1}{\beta}BB^*\Pi$, we obtain:

$$
\begin{aligned}
&D^3\mathcal{V}(0)(A_\Pi z_1, z_2, z_3) + D^3\mathcal{V}(0)(z_1, A_\Pi z_2, z_3) + D^3\mathcal{V}(0)(z_1, z_2, A_\Pi z_3)\\
&\quad = \frac{1}{2\beta}\mathcal{R}_3(z_1, z_2, z_3), \quad \forall (z_1, z_2, z_3) \in \mathcal{D}(A)^3,
\end{aligned}
$$

where the trilinear form $\mathcal{R}_3 \colon Y^3 \to \mathbb{R}$ is determined by Π, N, and B.

Taylor expansion

Further differentiation provides a linear equation, called **generalized Lyapunov equation**, with the following structure:

$$D^k\mathcal{V}(0)(A_\Pi z_1, z_2, \dots, z_k) + \dots + D^k\mathcal{V}(0)(z_1, \dots, z_{k-1}, A_\Pi z_k)$$
$$= \frac{1}{2\beta}\mathcal{R}_k(z_1, \dots, z_k), \quad \forall (z_1, \dots, z_k) \in \mathcal{D}(A)^k \qquad (\text{GLE}(k))$$

where the multilinear form $\mathcal{R}_k : Y^k \to \mathbb{R}$ is determined by Π, $D^3\mathcal{V}(0)$,...,$D^{k-1}\mathcal{V}(0)$, N, and B.

Remark: end of the formal computations.

Theorem

There exists a unique sequence $(\mathcal{T}_k)_{k=3,4,\dots}$ of symmetric bounded multilinear forms such that $\mathcal{T}_k : Y^k \to \mathbb{R}$ is a solution to GLE(k).

Feedback law

Polynomial $\mathcal{V}_p$ of degree p:

$$\mathcal{V}_p(y) = \sum_{k=2}^{p} \frac{1}{k!} \mathcal{T}_k(y, ..., y).$$

Feedback law $\mathbf{u}_p$ of order p:

$$\mathbf{u}_p : y \in Y \mapsto \mathbf{u}_p(y) = -\frac{1}{\beta} D\mathcal{V}_p(y)(Ny + B).$$

Closed-loop system of order p:

$$\dot{y}_p(t) = Ay_p(t) + (Ny_p(t) + B)\mathbf{u}_p(y_p(t)), \quad y(0) = y_0.$$

Open-loop control $\mathbf{U}_p(y_0)$ generated by the feedback $\mathbf{u}_p$ and y_0:

$$\mathbf{U}_p(y_0; t) = \mathbf{u}_p(y_p(t)).$$

Theoretical results

Theorem

For all $p \geq 2$, there exist two constants $\delta > 0$ and $C > 0$ such that for all $y_0 \in Y$ with $\|y_0\|_Y \leq \delta$:

1. *The closed-loop system is well-posed and generates an open-loop control in $L^2(0, \infty)$ such that:*

$$\mathcal{J}(\mathbf{U}_p(y_0), y_0) \leq \mathcal{V}(y_0) + C\|y_0\|_Y^{p+1}.$$

2. *It holds: $|\mathcal{V}(y_0) - \mathcal{V}_p(y_0)| \leq C\|y_0\|^{p+1}$.*

Remark: **local result**, δ decreases as β decreases and p increases.

1 Theory

2 Numerics

Numerical approach

1. **Discretize** the operators A, N, and B in such a way that the bilinear structure is preserved (e.g. with finite differences)
2. Find a **reduced-order** model with a generalization of the balanced truncation method:

$$\inf_{u \in L^2(0,\infty)} J(u, y_0) := \int_0^\infty \frac{1}{2} \|C_r y_r(t)\|^2_{\mathbb{R}^n} + \frac{\beta}{2} |u(t)|^2 \mathrm{d}t,$$

$$\text{where: } \begin{cases} \dot{y}_r(t) = A_r y_r(t) + N_r y_r(t) u(t) + B_r u(t), \\ y_r(0) = y_{0,r} \in Y. \end{cases}$$

3. Solve the reduced GLE with a **tensor-calculus technique**.

Lyapunov equations

The associated reduced GLE of order k:

$$T_{k,r}(A_{\Pi,r}z_1, z_2, \ldots, z_k) + \ldots + T_{k,r}(z_1, \ldots, z_{k-1}, A_{\Pi,r}z_k) = \frac{1}{2\beta} R_{k,r}(z_1, \ldots, z_k)$$

is equivalent to a **linear system with** r^k variables. Solution:

$$T_{k,r}(z_1, \ldots, z_k) = -\int_0^\infty R_{k,r}(e^{A_{\Pi,r}t}z_1, \ldots, e^{A_{\Pi,r}t}z_k)\mathrm{d}t.$$

An approximation is given by:

$$\sum_{i=-\ell}^{\ell} w_i R_{k,r}(e^{A_{\Pi,r}t_i}z_1, \ldots, e^{A_{\Pi,r}t_i}z_k),$$

for an appropriate choice of points t_i and weights w_i.

Fokker-Planck equation

Controlled **Fokker-Planck equation**:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \nu \Delta \rho + \nabla \cdot (\rho \nabla G) + u \nabla \cdot (\rho \nabla \alpha_j) && \text{in } \Omega \times (0, \infty), \\ 0 &= (\nu \nabla \rho + \rho \nabla G) \cdot \vec{n} && \text{on } \Gamma \times (0, \infty), \\ \rho(x, 0) &= \rho_0(x) && \text{in } \Gamma, \end{aligned}$$

where $\Omega \in \mathbb{R}^d$ denotes a bounded domain with smooth boundary Γ.

For all t, $\rho(\cdot, t)$ is the probability density function of X_t, sol. to

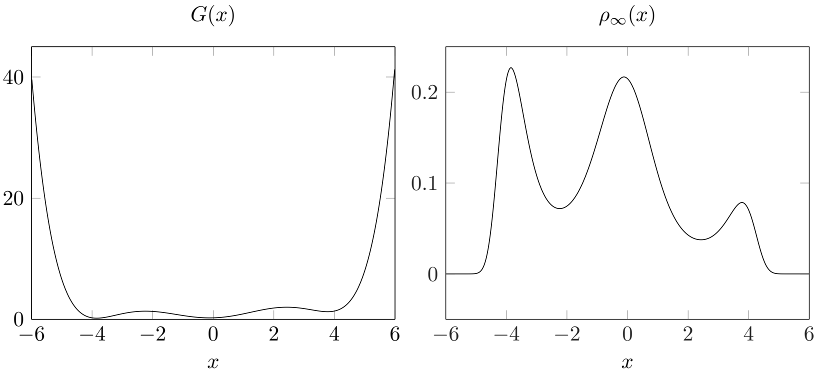
$$dX(t) = -\nabla_x V(X(t), t)dt + \sqrt{2\nu} dW_t,$$

where the **potential** V is controlled by u:

$$V(x, t) = G(x) + u(t)\alpha(x), \quad \forall x \in \Omega, \ \forall t \geq 0.$$

Fokker-Planck equation

The uncontrolled Fokker-Planck equation is known to converge to its stationary distribution ρ_∞.



(a) Ground potential (b) Stationary distribution

Fokker-Planck equation

Optimal control problem:

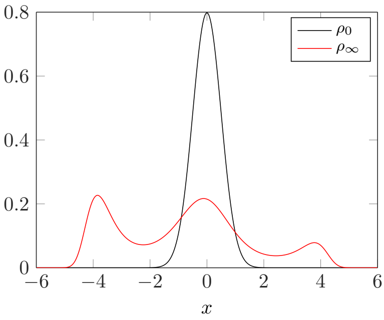
$$\inf_{u \in L^2(0,\infty)} \int_0^\infty \frac{1}{2} \|\rho(\cdot,t) - \rho_\infty(\cdot)\|^2_{L^2(\Omega)} + \beta |u(t)|^2 \mathrm{d}t,$$

where ρ satisfies the Fokker-Planck equation.

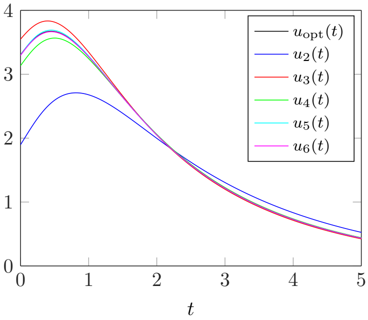
Under regularity assumptions on G and α, the problem can be reformulated, so that it falls in the abstract framework.

- Control shape function $\alpha(x) \approx x/12$.
- Discretization of $\Omega = (-6,6)$: $n = 100$.
- Reduction: $r = 21$ (selection of singular values above 10^{-6}).
- Results for two initial values (a close one/a further one), different values of β.

Numerical results (test case 1)

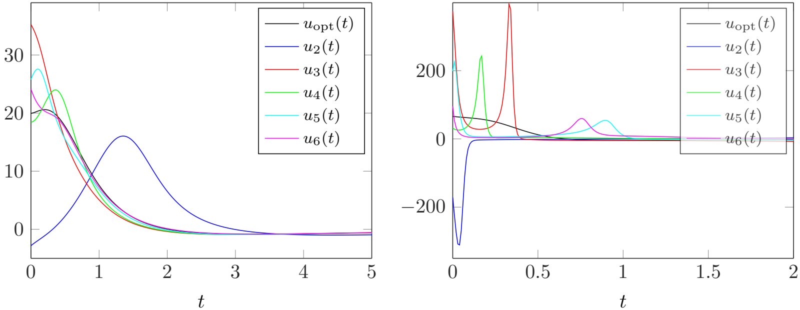


(a) Initial/stationary distributions



(b) Controls for $\beta = 10^{-3}$

Numerical results (test case 1)



(a) Controls for $\beta = 10^{-4}$ (b) Controls for $\beta = 10^{-5}$

Numerical results (test case 1)

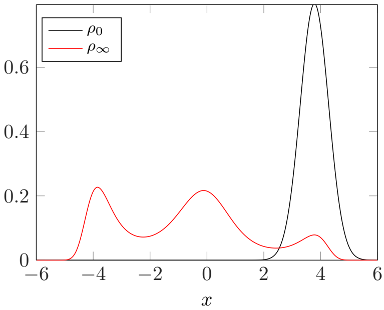
β	$J(u_2)$	$J(u_3)$	$J(u_4)$	$J(u_5)$	$J(u_6)$	$J(u_{opt})$
$1e^{-3}$	0.156	0.155	0.155	0.155	0.155	0.154
$1e^{-4}$	0.138	0.122	0.120	0.120	0.120	0.119
$1e^{-5}$	0.205	0.194	0.104	0.111	0.113	0.095

(a) Cost of the controls u_p

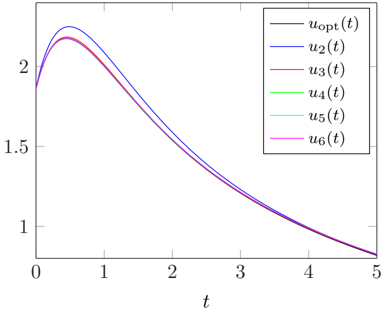
β	$\|u_p - u_{opt}\|_{L^2(0,T)}$				
	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$1e^{-3}$	1.149	0.169	0.119	0.034	0.031
$1e^{-4}$	18.50	7.02	3.16	4.01	1.52
$1e^{-5}$	90.5	78.0	39.0	42.6	34.3

(b) L^2-distance between the controls u_p and the optimal control u_{opt}

Numerical results (test case 2)

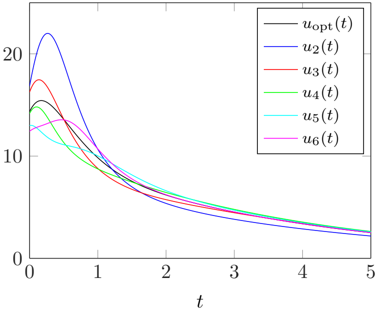


(a) Initial/stationary distributions

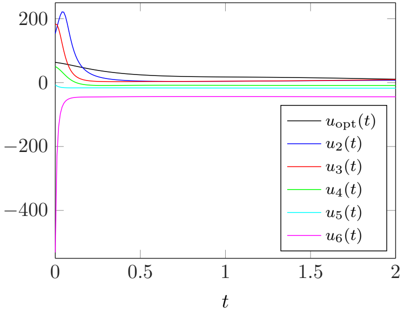


(b) Controls for $\beta = 10^{-2}$

Numerical results (test case 2)



(a) Controls for $\beta = 10^{-3}$



(b) Controls for $\beta = 10^{-4}$

Numerical results

β	$J(u_2)$	$J(u_3)$	$J(u_4)$	$J(u_5)$	$J(u_6)$	$J(u_{opt})$
$1e^{-2}$	0.788	0.788	0.788	0.788	0.788	0.787
$1e^{-3}$	0.525	0.511	0.511	0.512	0.510	0.507
$1e^{-4}$	0.381	0.368	2.689	∞	∞	0.246

(a) Cost of the controls u_p

β	$\|u_p - u_{opt}\|_{L^2(0,T)}$				
	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
$1e^{-2}$	0.19	0.15	0.15	0.15	0.15
$1e^{-3}$	4.88	1.50	1.77	2.31	1.52
$1e^{-4}$	46.34	35.36	57.08	∞	∞

(b) L^2-distance between the controls u_p and the optimal control u_{opt}

Conclusion

Summary:

- General method for deriving polynomial feedback laws
- Implementation for an infinite-dimensional problem thanks to model reduction
- Good results, but only locally.

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