Distributionally robust stochastic knapsack problem

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Quadratic knapsack problem

The *multidimensional* version of quadratic knapsack problem:

$$(KP) \quad \underset{\mathbf{x}}{\text{maximize}} \quad \mathbf{x}^T \mathbb{R} \mathbf{x} \tag{1a}$$

$$\text{subject to} \quad \mathbf{w}_j^T \mathbf{x} \leq d_j \ , \ \forall j \in \{1, 2, \ldots, M\} \tag{1b}$$

$$x_i \in \{0, 1\} \ , \forall i \in \{1, 2, \ldots, n\} \tag{1c}$$

- $\mathbf{x}$ is a vector of binary values indicating whether each item is included in the knapsack
- $\mathbb{R} \in \Re^{n \times n}$ is a matrix which (i, j)-th term describes the linear contribution to reward of holding both items i and j.
- $\mathbf{w}_j \in \Re^n$ is a vector of attributes (typically weights) which total amount must satisfy the knapsack capacity d_j.

Stochastic quadratic knapsack problem

$$(SKP) \quad \underset{\mathbf{x}}{\text{maximize}} \quad \mathbf{x}^T \mathbb{R} \mathbf{x} \tag{2a}$$

$$\text{subject to} \quad \mathbb{P}_F(\tilde{\mathbf{w}}_j^T \mathbf{x} \leq d_j, j = 1, \ldots, M) \geq 1 - \eta \tag{2b}$$

$$x_i \in \{0,1\}, \forall i \in \{1, 2, ..., n\} \tag{2c}$$

for some $0 < \eta < 1$.

Stochastic quadratic knapsack problem

$$
\begin{aligned}
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What happens if the probability distribution is unknown ?

Stochastic quadratic knapsack problem

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What happens if the probability distribution is unknown ?

- An answer can be given by **Distributionally robust optimization**

Distributionally robust problem

$$
\begin{aligned}
(DRSKP) \quad & \underset{\mathbf{x}}{\text{maximize}} && \mathbf{x}^T \mathbb{R} \mathbf{x} \\
& \text{subject to} && \inf_{F \in \mathcal{D}} \mathbb{P}_F(\tilde{\mathbf{w}}_j^T \mathbf{x} \leq d_j, j = 1, \ldots, M) \geq 1 - \eta \\
& && x_i \in \{0, 1\} \ , \forall\, i \in \{1, 2, ..., n\} \ .
\end{aligned}
$$

where $\mathcal{D}$ is a set of distributions.

Distributionally Robust SKP

Semidefinite programming approximation

We relax the nonconvex constraint $\mathbf{X} = \mathbf{x}\mathbf{x}^T$ to $\mathbf{X} \succeq \mathbf{x}\mathbf{x}^T$ which can take the form of a linear matrix inequality.

$$\begin{bmatrix} \mathbf{X} & \mathbf{x} \\ \mathbf{x}^T & 1 \end{bmatrix} \succeq 0 \, .$$

Distributionally Robust SKP

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$$\begin{bmatrix} \mathbf{X} & \mathbf{x} \\ \mathbf{x}^T & 1 \end{bmatrix} \succeq 0 \,.$$

Accordingly, we have the following semidefinite program (SDP) approximation relaxation problem:

$$\begin{aligned} (DRSKP - SDP) \quad & \underset{\mathbf{x},\mathbf{X}}{\text{maximize}} && \mathbb{R} \bullet \mathbf{X} && \text{(4a)} \\ & \text{subject to} && \inf_{F \in \mathcal{D}} \mathbb{P}_F(\tilde{\mathbf{w}}_j^T \mathbf{x} \le d_j, j = 1, \ldots, M) \ge 1 - \eta && \text{(4b)} \\ & && \begin{bmatrix} \mathbf{X} & \mathbf{x} \\ \mathbf{x}^T & 1 \end{bmatrix} \succeq 0 && \text{(4c)} \\ & && \mathbf{X}_i = x_i \,, \forall i, \quad \mathbf{X}_{i,j} \ge 0 \,, \forall i,j && \text{(4d)} \end{aligned}$$

Distributionally Robust SKP

Definition

Without loss of generality, for all $j = 1, \ldots, M$, let ξ_j be a random vector in $\Re^m$ from which $\tilde{\mathbf{w}}_j$ depends linearly and

$$\tilde{\mathbf{w}}_j = \mathbf{A}_j^{\tilde{\mathbf{w}}_j} \xi_j, \;\; j = 1, \ldots, M \; .$$

where $\mathbf{A}_j^{\tilde{\mathbf{w}}_j}$ is a deterministic matrix.

Distributionally Robust SKP

Assumption

- *The distributional uncertainty set accounts for information about the support $\mathcal{S}$, mean μ_j, and an upper bound Σ_j on the covariance matrix of the random vector $\xi_j, j = 1, \ldots, M$*

$$\mathcal{D}(\mathcal{S}, \mu_j, \Sigma_j) = \left\{ F_j \left| \begin{array}{l} \mathbb{P}(\xi_j \in \mathcal{S}) = 1 \\ \mathbb{E}_F[\xi_j] = \mu_j \\ \mathbb{E}_F[(\xi_j - \mu_j)(\xi_j - \mu_j)^T] \preceq \Sigma_j \end{array} \right. \right\}.$$

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- *The random vectors ξ_i and ξ_j are independent when $i \neq j$. The support of F_j is unbounded, i.e. $\mathcal{S} = \Re^m$.*

Solution Methods

Bonferroni's conservative bound

- A popular approximation for joint chance-constrained problems is based on Bonferroni's inequality, which decomposes the joint constraint into M individual constraints. When $\sum_{j=1}^{M} \eta_k = \eta$, we have

$$\mathbb{P}_F(\tilde{\mathbf{w}}_j^T \mathbf{x} \leq d_j) \geq 1 - \eta_j, j = 1, \cdots, M$$

$$\Rightarrow \mathbb{P}_F(\tilde{\mathbf{w}}_j^T \mathbf{x} \leq d_j, j = 1, \ldots, M) \geq 1 - \eta$$

- Notice that it is easy to prove that the optimal values of the Bonferroni approximations are lower bounds of DRSKP-SDP.

Solution Methods

Zymler et al. (2013) conservative bound

- Zymler et al. address the distributionally robust chance constraint by introducing a scaling parameter $\alpha \in \mathcal{A} = \{\alpha \in \Re^M : \alpha > 0\}$ and reformulating it as a distributionally robust conditional value at risk constraint. For any $\alpha \in \mathcal{A}$

$$
\begin{aligned}
& \inf_{F \in \mathcal{D}} \mathbb{P}_F(\tilde{\mathbf{w}}_j^T \mathbf{x} \leq d_j, j = 1, \ldots, M) \geq 1 - \eta \\
\Leftrightarrow \quad & \inf_{F \in \mathcal{D}} \mathbb{P}_F(\max_j \{\alpha_j(\tilde{\mathbf{w}}_j^T \mathbf{x} - d_j)\} \leq 0) \geq 1 - \eta \\
\Leftrightarrow \quad & \inf_{F \in \mathcal{D}} \text{CVaR-}\eta_F(\max_j \{\alpha_j(\tilde{\mathbf{w}}_j^T \mathbf{x} - d_j)\}) \leq 0
\end{aligned}
$$

- The next step involves replacing the value at risk operator by an upper bounding CVAR operator to make the both convex and tractable.
- For any $\alpha > 0$, this leads to the following conservative approximation of the DRSKP-SDP.

Solution Methods

Theorem

The DRSKP-SDP problem with constraint (4b) *replaced by,*

$$\beta + \frac{1}{\eta}\begin{bmatrix} \Sigma + \mu\mu^T & \frac{1}{2}\mu \\ \frac{1}{2}\mu^T & 1 \end{bmatrix} \bullet \mathcal{M} \leq 0$$

$$\mathcal{M} - \begin{bmatrix} \mathbf{0}_{mM,mM} & \frac{1}{2}\alpha_j \mathbf{y}_j \\ \frac{1}{2}\alpha_j \mathbf{y}_j^T & -\alpha_j d_j - \beta \end{bmatrix} \succeq 0, \ \forall j = 1, \ldots, M$$

$$\mathbf{y}_j = \begin{bmatrix} \mathbf{0}_{(j-1)m}{}^T & \mathbf{x}^T \mathbf{A_j}^{\tilde{\mathbf{w}}_j} & \mathbf{0}_{(M-j)m}{}^T \end{bmatrix}^T$$

$$\mathcal{M} \succeq 0 \ ,$$

where $\mathbf{0}_k$ *and* $\mathbf{0}_{k,k}$ *are all zeros vector in* $\Re^k$ *and matrix in* $\Re^{k\times k}$ *respectively,* $\beta \in \Re$, $\mathcal{M} \in \Re^{(Mm+1)\times(Mm+1)}$, *and* $y_j \in \mathbb{R}^{mM}$ *are auxiliary decision variables, which is an SDP problem for any* $\alpha > 0$. *The optimal solution of this SDP is feasible according to the original DRSKP-SDP and its optimal value provides a lower bound on the value of the original problem.*

Solution Methods

Deterministic formulation

- Under Assumption 1, one can actually show that the robust chance constraint (4b) is equivalent to:

$$\prod_{j=1}^{M} \inf_{F \in \mathcal{D}(\Re^m, \mu_j, \Sigma_j)} \mathbb{P}_F((\mathbf{A}_j^{\tilde{\mathbf{w}}_j} \xi)^T \mathbf{x} \leq d_j) \geq 1 - \eta \,.$$

- This constraint is therefore satisfied if and only if there exists a vector $\mathbf{y} \in \Re^M$, such that $\mathbf{y} \geq 0$, $\sum_{j=1}^M y_j = 1$, and

$$\inf_{F \in \mathcal{D}(\Re^m, \mu_j, \Sigma_j)} \mathbb{P}_F((\mathbf{A}_j^{\tilde{\mathbf{w}}_j} \xi)^T \mathbf{x} \leq d_j) \geq (1 - \eta)^{y_j} \,,\ \forall j \in \{1, 2, \ldots, M\}.$$

Solution Methods

Deterministic formulation

Problem DRSKP-SDP is equivalent to the following deterministic problem

$$\underset{\mathbf{x},\mathbf{y}}{\text{maximize}} \quad \mathbb{R} \bullet \mathbf{X} \tag{5a}$$

$$\text{subject to} \quad \mu_j^T \mathbf{A}_j^{\tilde{\mathbf{w}}_j} \mathbf{x} + \sqrt{\frac{p^{y_j}}{1-p^{y_j}}} \|\Sigma_j^{1/2} \mathbf{A_j}^{\tilde{\mathbf{w}}_j\,T} \mathbf{x}\|_2 \leq d_j \tag{5b}$$

$$\sum_{j=1}^{M} y_j = 1, \quad y_j \geq 0 \tag{5c}$$

$$\begin{bmatrix} \mathbf{X} & \mathbf{x} \\ \mathbf{x}^T & 1 \end{bmatrix} \succeq 0 \tag{5d}$$

$$\mathbf{X}_{i,i} = x_i \;, \forall\, i, \quad \mathbf{X}_{i,j} \geq 0 \;, \forall\, i,j \tag{5e}$$

Solution Methods

Remark

- For the chance constraint (5b), if we take y_j as fixed parameters and transform it into the equivalent SDP constraint, the dimension of the corresponding linear matrix inequality is $(m+1) \times (m+1)$, compared to an $(Mm+1) \times (Mm+1)$ linear matrix inequality in the bound based on Zymler et al.([Zymler et al 2013]).

Solution Methods

Sequential approximations

- When we consider the variables $y_j, j = 1, ...M$ to be fixed, the problem DRSKP-SDP becomes an SDP problem.
- We propose a **sequential approximation method** that **iteratively adjusts the parameters** y and solves the obtained SDP until no further improvement is achieved.

Solution Methods

Sequential approximations

Algorithm 1: *Sequential Approximation Procedure*

Initialization: Let $\mathbf{y}^1 \in R_+^M$ be scaling parameters, i.e., $\sum_{j=1}^M \mathbf{y}^1(j) = 1, \mathbf{y}^1 \geq 0$. Set the iteration counter to $t := 1$.

Update: Solve problem (5) with $\mathbf{y}$ fixed to $\mathbf{y}^t$ and let $\mathbf{x}^t$ and f^t denote an optimal solution and the optimal value, respectively. Let $z_j = \frac{(d_j - \mu^T \mathbf{A}_j^{\tilde{\mathbf{w}}_j} \mathbf{x}^t)^2}{(\|\Sigma_j^{1/2} \mathbf{A_j}^{\tilde{\mathbf{w}}_j\,T} \mathbf{x}^t\|_2)^2}$ and $\tilde{y}_j^t = \log_p \frac{z_j}{z_j+1}$. Set $\mathbf{y}^{t+1} \leftarrow U(\mathbf{y}; \tilde{\mathbf{y}})$ for some

$$U : \Delta^M \times [0,1]^M \to \Delta^M \ ,$$

where $\Delta^M = \{\, \mathbf{y} \in \Re^M \,|\, \mathbf{y} \geq 0 \ \& \ \sum_j y_j = 1 \,\}$ is the probability simplex, and $\times_{j=1}^M [\tilde{y}_j^t, 1]$ is the cartesian product of each $[\tilde{y}_j^t, 1]$ interval.

Stopping criterion: if $f^t - f^{t-1}$ is small enough, stop and return $\mathbf{x}^t$, f^t, and $\mathbf{y}^t$, otherwise, set $t := t+1$ and go to the **Update**.

Sequential approximations

Theorem

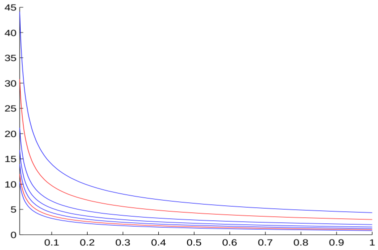
If the problem DRSKP-SDP is bounded and has a feasible solution for the initial values of $\mathbf{y}^1$*, and given that* $U(\mathbf{y}; \tilde{\mathbf{y}}) \geq \tilde{\mathbf{y}}$ *for all* $\mathbf{y} \geq \tilde{\mathbf{y}}$*, then Algorithm 1 terminates in a finite number of steps and the returned value* f^t *is a lower bound for DRSKP-SDP.*

Sequential approximations

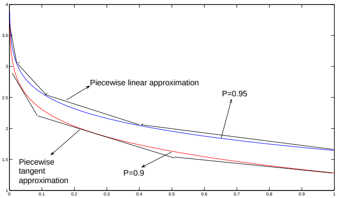
Remark

- In our implementation, for the initial parameter y^1, all the elements are set to $\frac{1}{M}$.
- We choose the following update policy: $U(\mathbf{y};\tilde{\mathbf{y}}) = \tilde{\mathbf{y}} + \alpha(1-(\mathbf{y}-\tilde{\mathbf{y}}))$, where $\alpha = \frac{\sum_j (y_j - \tilde{y}_j)}{\sum_j (1-(\mathbf{y}_j - \tilde{y}_j))}$.
- The idea behind this adjustment policy is that, under the current solution $\mathbf{x}^t$, it encourages giving more margin to the chance constraints that are the tightest at $\mathbf{x}^t$, effectively using $y_j^t - \tilde{y}_j^t$ as a measure of tightness.
- One can easily verify that it is indeed the case that this adjustment policy satisfies the required properties.

Upper bound through tangent approximation



(a)



(b)

Figure : Function $f(y) = \sqrt{\frac{p^y}{1-p^y}}$ is convex and decreasing in y for all $p \in]0,1[$. (a) presents $f(y)$ for $p = 0.5, 0.55, 0.6, 0.7, 0.8, 0.9, 0.95$. (b) presents how $f(y)$ can be lower bounded by a piecewise linear convex function that is tangent to $f(y)$ at a number of points.

Piecewise tangent approximation

Lemma

Function $\sqrt{\frac{p^y}{1-p^y}}$ *is convex and decreasing in the region* $(0, 1]$*, when* $p \in (0, 1)$.

$$\underset{\mathbf{x},\mathbf{y}}{\text{maximize}} \quad \mathbb{R} \bullet \mathbf{X} \tag{6}$$

$$\text{subject to} \quad \begin{bmatrix} \mathbf{0}_{m,m} & \Sigma_j^{1/2}\mathbf{A_j}^{\tilde{\mathbf{w}}_j\,T}\mathbf{z_j} \\ (\Sigma_j^{1/2}\mathbf{A_j}^{\tilde{\mathbf{w}}_j\,T}\mathbf{z_j})^T & 0 \end{bmatrix} \succeq (\mu_j^T\mathbf{A_j}^{\tilde{\mathbf{w}}_j\,T}\mathbf{x} - d_j)\mathbf{I} \tag{7}$$

$$\mathbf{z_j} \geq \hat{a}_l\bar{\mathbf{x}}_j + \hat{b}_l\mathbf{x}, \forall\, l \in \{1,2,...,L\} \forall\, j \in \{1,2,...,M\} \tag{8}$$

$$\sum_{j=1}^{M} \bar{\mathbf{x}}_j = \mathbf{x}\ , \forall\, i \tag{9}$$

$$0 \leq \bar{\mathbf{x}}_j \leq \mathbf{x}, \quad y_j - (1-\mathbf{x}) \leq \bar{\mathbf{x}}_j \leq y_j, \forall\, j \in \{1,2,...,M\} \tag{10}$$

$$\sum_{j=1}^{M} y_j = 1, \quad y_j \geq 0, \forall\, j \in \{1,2,...,M\} \tag{11}$$

$$\begin{bmatrix} \mathbf{X} & \mathbf{x} \\ \mathbf{x}^T & 1 \end{bmatrix} \succeq 0, \quad \mathbf{X}_{i,i} = x_i\ , \forall\, i, \quad \mathbf{X}_{i,j} \geq 0\ , \forall\, i,j\ , \tag{12}$$

Theorem

Problem DRSKP-SDP1 is a relaxed approximation of Problem DRSKP-SDP, i.e., the optimal value of DRSKP-SDP1 is an upper bound of DRSKP-SDP.

Numerical Results

Details of experiments

- We study the computational performance of our proposed methods for solving the DRSKP-SDP.
- The sequential approximation and the tangent approximation are compared with Bonferroni's approximation and Approximation by Zymler et al..
- All the considered models are generated using MATLAB environment and solved by Sedumi with default parameters on an Intel(R)D @ 2.00 GHz with 4.0 GB RAM.

η	V^I	V^Z	V^B	V^U	$\frac{V^U - V^I}{V^I}$	$\frac{V^U - V^Z}{V^Z}$	$\frac{V^U - V^B}{V^B}$	CPU^I	CPU^Z	CPU^B	CPU^U
1%	46.67	46.65	36.04	46.76	0.19%	0.24%	29.74%	5.60	393.40	0.28	1.31
2%	54.78	54.74	43.44	54.94	0.29%	0.37%	26.47%	5.80	390.72	0.24	1.13
3%	59.26	59.20	47.80	59.43	0.29%	0.39%	24.33%	5.45	496.26	0.26	1.09
4%	62.25	62.18	50.83	62.47	0.35%	0.47%	22.90%	5.62	509.21	0.25	1.10
5%	64.46	64.38	53.16	64.70	0.37%	0.50%	21.71%	5.83	862.02	0.26	1.14
6%	66.19	66.10	55.02	66.39	0.30%	0.44%	20.67%	5.86	497.16	0.23	1.05
7%	67.59	67.50	56.56	67.77	0.27%	0.40%	19.82%	5.74	627.58	0.26	1.17
8%	68.76	68.66	57.88	68.93	0.25%	0.39%	19.09%	5.75	518.17	0.29	1.15
9%	69.76	69.66	59.02	69.93	0.24%	0.39%	18.49%	5.82	506.97	0.29	1.21
10%	70.63	70.53	60.02	70.81	0.25%	0.40%	17.98%	6.21	623.61	0.25	1.53

Table : Computational results of DRSKP-SDP when $n = 10, m = 5, M = 4$.

η	V^I	V^Z	V^B	V^U	$\frac{V^U - V^I}{V^I}$	$\frac{V^U - V^Z}{V^Z}$	$\frac{V^U - V^B}{V^B}$	CPU^I	CPU^Z	CPU^B	CPU^U
1%	60.96	60.94	46.57	61.19	0.38%	0.41%	31.39%	14.70	9311.99	0.61	5.40
2%	71.61	71.54	55.96	71.83	0.31%	0.41%	28.36%	17.32	8918.62	0.59	4.82
3%	77.39	77.29	61.50	77.65	0.34%	0.47%	26.26%	17.78	6831.69	0.58	5.18
4%	81.20	81.08	65.37	81.45	0.31%	0.46%	24.60%	18.52	12461.91	0.63	5.54
5%	83.95	83.83	68.31	84.23	0.33%	0.48%	23.31%	16.39	9536.62	0.71	5.41
6%	86.07	85.94	70.66	86.38	0.36%	0.51%	22.25%	17.12	9206.91	0.75	5.41
7%	87.78	87.64	72.61	88.11	0.38%	0.54%	21.35%	17.18	19749.16	0.72	5.52
8%	89.18	89.04	74.27	89.51	0.37%	0.53%	20.52%	17.30	9370.42	0.67	5.37
9%	90.37	90.22	75.70	90.70	0.37%	0.53%	19.82%	16.55	15440.99	0.65	5.22
10%	91.39	91.24	76.95	91.73	0.37%	0.54%	19.21%	16.98	15970.64	0.68	5.02

Table : Computational results of DRSKP-SDP when $n = 20, m = 6, M = 5$

Conclusions

- We solve the multidimensional quadratic optimization problem with uncertainty and joint probabilistic constraints.
- We consider that the uncertainty is defined by an uncertain set and solve the corresponding problem using a disributionally robust approach.
- We show that our approach outperforms the existing methods.
- Our approach can be easily extended to several stochastic combinatorial optimization problems.